Stochastic Riemannian Geometry — Summer 2022

1. Introduction & Review

Situation: • M diffb. mf, ∇ torsion-free connection on M
• (M,g) Riem. mf, ∇ Levi-Civitata connection

$$P = \begin{cases} L(M) & \text{frame bundle, resp.} \\ O(M) & \text{orthonormal frame bundle} \end{cases}$$

i.e. $u \in P_x$, $x \in M$ $\iff$ $u: \mathbb{R}^n \to T_xM$ lin. isomorphism, $n = \dim M$
resp. isometry

Identify: $u \in P_x \;\hat{=}\; (u^1, \dots, u^n) = (ue_1, \dots, ue_n)$ frame

$\pi: P \to M$ principal G-bundle ($G = Gl(n;\mathbb{R})$, resp. $G = O(n)$)

Note connection ∇ on M induces a G-connection on P, i.e.
G-invariant splitting h of the following exact sequence of vb /P

$$(*) \quad 0 \to \ker d\pi \longrightarrow TP \xrightarrow{d\pi} \pi^*TM \longrightarrow 0$$

h, $d\pi \circ h = id$

Then • $TP = V \oplus H = \ker d\pi \oplus h(\pi^*M)$ "vertical/horiz." subbundle
• $h: \pi^*TM \xrightarrow{\sim} H$ bundle isom. "horizontal lift" of the G-connection

$$h_u: T_{\pi(u)}M \xrightarrow{\sim} H_u, \quad u \in P$$

Note $u \in P$: $G \xrightarrow{I_u} P$, $g \mapsto ug$

$$K_u = (dI_u)_e : \underbrace{T_eG}_{=\mathfrak{g}} \longrightarrow T_uP$$

gives identification

$$K_u: \mathfrak{g} \xrightarrow{\sim} V_u$$

$X \in \Gamma(TP)$: $X = \text{vert}X + \text{hor}X$

Def (1) <u>connection form</u>

$$\omega \in \Gamma(T^*P \otimes \mathfrak{g}), \quad \omega_u(X_u) := \kappa_u^{-1}(\mathrm{vert}\, X)_u, \quad X \in \Gamma(TP), u \in P$$

(2) <u>canonical 1-form of P</u>

$$\vartheta \in \Gamma(T^*P \otimes \mathbb{R}^n), \quad \vartheta_u(X_u) = u^{-1}\big((d\pi)_u X_u\big), \quad X \in \Gamma(TP), u \in P$$

Splitting of (*)

$$\begin{tikzcd} & P \times \mathfrak{g} \arrow[d, equal] & & & \\ 0 \arrow[r] & \ker d\pi \arrow[r, "\iota"] & TP \arrow[r, "\pi_*"] \arrow[l, bend left, "\omega"] \arrow[d, equal] & \pi^* TM \arrow[r] \arrow[l, bend left, "h"] & 0 \\ & & V \oplus H & & \end{tikzcd}$$

- horizontal lift h $\qquad \pi_* \circ h = \mathrm{id}$
- $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ $\qquad \omega \circ \iota = \mathrm{id}$
- $TP = V \oplus H$

Def <u>Standard horizontal</u> v.f.s on P

$$L_i \in \Gamma(TP), \quad L_i(u) := h_u(ue_i), \quad i = 1, \dots, n$$

$$\Delta^{hor} := \sum_{i=1}^n L_i^2 \quad \text{horizontal Laplacian on } P = L(M), \text{ resp } P = O(M)$$

For $P = O(M)$:

$$\Delta^{hor}(f \circ \pi) = (\Delta f) \circ \pi \qquad \forall f \in C^\infty(M)$$

Δf = trace ∇df <u>Laplace - Beltrami operator</u> on M

X M-valued semimartingale, i.e. $f(X)$ real semimartingale $\forall f \in C^\infty(M)$

(1) To $u \in P$ $\exists_1$ horizontal lift U of X to P s.th. $U_0 = u$,
i.e. $\pi \circ U = X$ & $\int_U \omega \equiv \int \omega(\circ dU) = 0$

Then: $//_{s,t} : T_{X_s}M \longrightarrow T_{X_t}M$, $//_{s,t} := U_t \circ U_s^{-1}$, $s \leq t$

$$\begin{tikzcd} T_{X_s}M \arrow[rr, dashed, "//_{s,t}"] & & T_{X_t}M \\ & \mathbb{R}^n \arrow[ul, "U_s"', "\cong"] \arrow[ur, "\cong", "U_t"'] & \end{tikzcd}$$

(2) $Z := \int_U \vartheta = \int \vartheta(\circ dU)$ anti-development of X in $\mathbb{R}^n$

$\mathcal{A}(X) := U_0 \int_U \vartheta$ <u>antidevelopment</u> of X (taking values in $T_{X_0}M$)
indep. of the choice of U_0

<u>Note</u> X can be recovered from Z

$Z \longmapsto U$ $\quad dU = \sum_{i=1}^n L_i(U) \circ dZ^i$, $U_0 = u$

$U \longmapsto X$ $\quad X = \pi(U)$

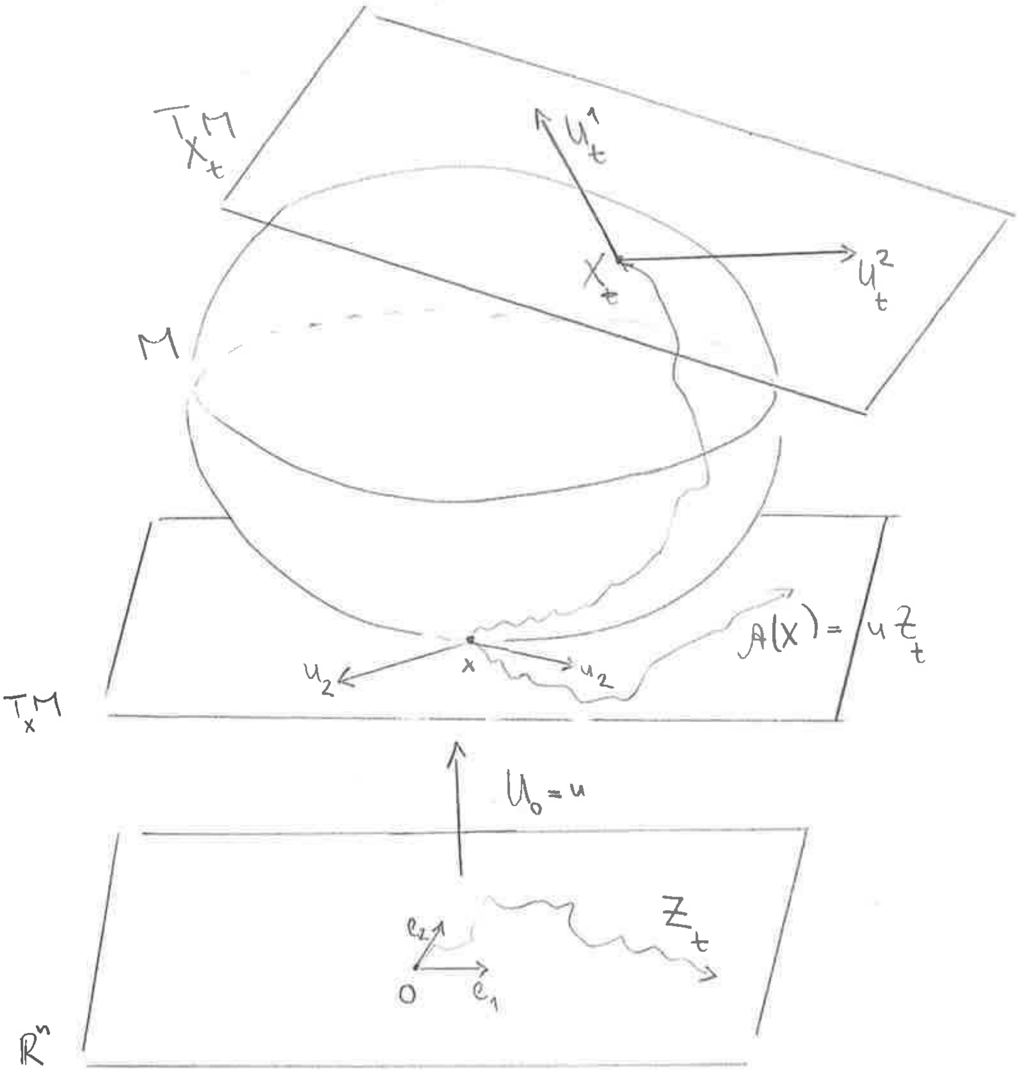
$X \longmapsto Z$ $\quad Z = \int_U \vartheta$ "X <u>stochastic development</u> of Z"

X ∇-martingale on M $\iff$ Z local martingale on $\mathbb{R}^n$

X $BM(M,g)$ $\iff$ Z $BM(\mathbb{R}^n)$

Geometric picture $\qquad P = L(M),\ O(M)$

$u \in P: \quad \mathbb{R}^n \xrightarrow[\sim]{u} T_{\pi(u)}M$



Note $\quad dU = \sum_{i=1}^{n} L_i(U) \circ dZ^i \quad \underset{X=\pi(U)}{\Longrightarrow} \quad dX = \sum_{i=1}^{n} \underbrace{d\pi\, L_i(U)}_{= Ue_i} \circ dZ^i$

i.e. $\quad dX = U \circ dZ$, or

$$\boxed{dX = /\!/_{0,t} \circ dA(X)}$$

X M-valued semimartingale

$\longrightarrow$ U horizontal lift of X to $P = L(M)$, $O(M)$

$$Z = \int_U \vartheta \quad , \text{ resp. } \quad \mathcal{A}(X) = U_0 \int_U \vartheta$$

Note $$dU = \sum_{i=1}^n L_i(U) \circ dZ^i \underset{X = \pi(U)}{\Longrightarrow} dX = \sum_{i=1}^n \underbrace{d\pi\, L_i(U)}_{= Ue_i} \circ dZ^i$$

i.e. $dX = U \circ dZ$ or

$$\boxed{dX = /\!/_{0,t} \circ d\mathcal{A}(X)}$$

Geometric Itô formula

X M-valued semimartingale $\longrightarrow$ U horizontal lift to $P = L(M)$, $O(M)$

$Z = \int_U \vartheta$, resp. $A(X) = U_0 \int_U \vartheta$

$\forall f \in \mathcal{C}^\infty(M)$,

$$d(f(X)) = \underbrace{\sum_{i=1}^n (df)_X (Ue_i)\, dZ^i}_{\equiv (df)_X (U dZ)} + \frac{1}{2} \sum_{i,j=1}^n (\nabla df)_X (Ue_i, Ue_j)\, dZ^i dZ^j$$

Intrinsically,

$$\boxed{d(f(X)) = df\left(/\!/_{0,t}\, dA(X)\right) + \frac{1}{2} \nabla df\,(dX, dX)}$$

More generally,

M, N diffb. mfs, each with a torsion-free connection

$f : M \to N$ $\mathcal{C}^\infty$

How to describe $f(X)$ for a semimartingale X on M ?

Induced connections

(1) E vb$/M$ with a connection ∇

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E) \,, \quad (\nabla a)v \equiv \nabla_v a \in \Gamma(E) \,, \quad v \in TM$$

E, E' vb$/M$ with $\nabla^E, \nabla^{E'}$

$\rightsquigarrow \nabla^{E^*}, \nabla^{E\oplus E'}, \nabla^{E\otimes E'}$ etc

(2) pullback of connections

$f : M \to N$ & F vb$/N$

$\rightsquigarrow f^*F$ vb$/M$ where $(f^*F)_x = F_{f(x)}$

$$\begin{tikzcd} f^*F \arrow[d] & F \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

∇^{f^*F} induced connection on f^*F

$$\nabla_v^{f^*F}(f^*a) = \nabla^F_{f_*v} a \,, \qquad f^*a = a\circ f \in \Gamma(f^*F), \; a \in \Gamma(F)$$

Now $f : M \to N$

$df_x : T_xM \to T_{f(x)}N$, i.e. $df \in \Gamma(\underbrace{T^*M \otimes f^*TN}_{=:E})$

$\nabla df \in \Gamma(T^*M \otimes T^*M \otimes f^*TN)$, i.e.

$(\nabla df)_x : T_xM \times T_xM \to T_{f(x)}N$ bilinear, symmetr.

Ex M Riem mf, ∇ LC-connection, $f \in \mathcal{C}^\infty(M,N)$

$\underbrace{\Delta f}_{\equiv \tau(f)} := \operatorname{trace} \nabla df \in \Gamma(f^*TN)$ tension field

$$\Delta f(x) = \sum_i (\nabla df)_x(e_i, e_i) \in T_{f(x)}N \,, \quad (e_i) \text{ onb of } T_xM$$

Composition formula

$M \xrightarrow[f]{} N \xrightarrow[\varphi]{} N'$ $\mathcal{C}^\infty$-maps between diffb mfs, each with a torsion-free connection

$$\boxed{\nabla d(\varphi \circ f) = \varphi_* \nabla df + f^* \nabla d\varphi}$$

For Riem mfs

$$\boxed{\Delta(\varphi \circ f) = \varphi_* \Delta f + \mathrm{tr}\,(f^* \nabla d\varphi)}$$

Thm M, N diffb. mfs with torsion-free connections
$f \in \mathcal{C}^\infty(M,N)$

X M-valued semimartingale $\longrightarrow$ $\tilde{X} := f(X)$ N-valued semimartingale

$/\!/_{0,t}$ $/\!/$-transport on M along X

$\tilde{/\!/}_{0,t}$ $/\!/$-transport on N along $\tilde{X}$

Then

$$\boxed{d\,\mathcal{A}(\tilde{X}) = \tilde{/\!/}_{0,\cdot}^{-1} (df)_X /\!/_{0,\cdot}\, d\mathcal{A}(X) + \tfrac{1}{2} \tilde{/\!/}_{0,\cdot}^{-1} \nabla df(dX, dX)}$$

In particular,
(M,g), (N,h) Riem mfs with LC-connection, $f \in \mathcal{C}^\infty(M,N)$
and X $BM(M,g)$ with $X_0 = x \in M$.

Then $\mathcal{A}(X)$ $BM(T_xM)$ and

$$\boxed{d\,\mathcal{A}(\tilde{X}) = \tilde{/\!/}_{0,\cdot}^{-1} (df)_X /\!/_{0,\cdot}\, d\mathcal{A}(X) + \tfrac{1}{2} \tilde{/\!/}_{0,t}^{-1} \Delta f(X)\, dt}$$

In addition

$$\boxed{h(d\tilde{X}, d\tilde{X}) = |df|^2(X)\, dt}$$

Pr Let $\varphi \in \mathcal{C}^\infty(N)$

$$\underset{\text{Itô formula}}{\Longrightarrow} \quad d(\varphi(\tilde X)) = \varphi_* \tilde{/\!/}_{0,\cdot}\, dA(\tilde X) + \frac{1}{2} \underbrace{\nabla^N d\varphi (d\tilde X, d\tilde X)}_{\substack{= (f^* \nabla^N d\varphi)(dX, dX) \\ \text{pullback formula} \quad \tilde X = f(X)}}$$

$$\|$$

$$d(\varphi\circ f(X)) = (\varphi\circ f)_* /\!/_{0,\cdot}\, dA(X) + \frac{1}{2} \underbrace{\nabla^M d(\varphi\circ f)(dX, dX)}_{= (\varphi_* \nabla df + f^* \nabla d\varphi)(dX, dX)}$$

$$\Longrightarrow \quad \varphi_* \tilde{/\!/}_{0,\cdot}\, dA(\tilde X) = \varphi_* f_* /\!/_{0,\cdot}\, dA(X) + \frac{1}{2}(\varphi_* \nabla df)(dX, dX) \qquad \forall \varphi \in \mathcal{C}^\infty(N)$$

$$\Longrightarrow \quad \tilde{/\!/}_{0,\cdot}\, dA(\tilde X) = f_* /\!/_{0,\cdot}\, dA(X) + \frac{1}{2} \nabla df (dX, dX)$$

Riem. case (M,g), (N,h) & X $BM(M,g)$, i.e. Z $BM(\mathbb{R}^n)$

$$h(d\tilde X, d\tilde X) \underset{\text{pullback formula}}{=} f^*h(dX, dX) = \sum_{i,j} (f^*h)_X (Ue_i, Ue_j)\, dZ^i dZ^j$$

$$\underset{Z\ BM(\mathbb{R}^n)}{=} \sum_i h_{f(X)}(f_* Ue_i, f_* Ue_i)\, dt$$

$$= |df|^2(X)\, dt$$

Situation $M \xrightarrow{f} N$ $\quad T_{X_0}M \quad T_{\tilde{X}_0}N$

$X \longmapsto \tilde{X} = f(X) \quad \mathcal{A}(X) \longmapsto \tilde{\mathcal{A}}(\tilde{X})$

$$\boxed{d\tilde{\mathcal{A}}(\tilde{X}) = \tilde{/\!/}_{0,\cdot}^{-1} (df)_X \, /\!/_{0,\cdot} \, d\mathcal{A}(X) + \frac{1}{2} \tilde{/\!/}_{0,\cdot}^{-1} \nabla df (dX, dX)}$$

Def M,N diffb mfs with torsionfree connections, $f \in \mathcal{C}^\infty(M,N)$

f affine $:\Longleftrightarrow \nabla df = 0$

Cor f affine $\Longleftrightarrow$ f maps $\mathrm{Mart}(M, \nabla^M)$ to $\mathrm{Mart}(N, \nabla^N)$

Def (M,g), (N,h) Riem mfs with LC-connections, $f \in \mathcal{C}^\infty(M,N)$

$\mathrm{tr}\, f^*h = |df|^2 \in \mathcal{C}^\infty(M)$ energy density of f

$\mathrm{tr}\, \nabla df = \Delta f \equiv \tau(f) \in \Gamma(f^*TN)$ tension field of f

f harmonic $:\Longleftrightarrow \Delta f = 0$

f harmonic morphism $\Longleftrightarrow$ f harmonic & horizontally conformal

Recall f horizontally conformal means

i) $x \in M$, $df_x \neq 0 \Rightarrow df_x : T_xM \to T_{f(x)}N$ surj.

ii) $\exists \lambda : M \to \mathbb{R}_+$ s.th. $f^*h = \lambda^2 g$ on $\underbrace{(\ker df)^\perp}_{\subset TM}$ "horizontal directions"

& $\lambda(x) = 0$ if $df_x = 0$.

In other words: $h_{f(x)}(f_*v, f_*w) = \lambda^2(x)\, g_x(v,w) \quad \forall v,w \in (\ker df_x)^\perp$

λ is called dilatation of f $\left(\text{necessarily: } \lambda^2 = |df|^2_{op} = \frac{1}{k}|df|^2,\right.$ where $\left. k = \dim N\right)$

Remark For $\lambda : M \to \mathbb{R}_+$ are equivalent

i) f harm. morphism

ii) $df \circ (df)^{ad} = \lambda^2 \, \mathrm{id}|_{f^*TM}$

iii) $\Delta_M(\varphi \circ f) = \lambda^2 (\Delta_N \varphi \circ f) \quad \forall \varphi \in \mathcal{C}^\infty(N)$

where $(df)^{ad} : f^*TN \to TM$ (defined fiberwise as adjoint to df, i.e. $g_x((df)^{ad}_x u, v) = h_{f(x)}(u, (df)_x v)$ $\forall u \in T_{f(x)}N, v \in T_xM$)

Pr Exercise

Proposition (M,g), (N,h) Riem mfs & $f \in \mathcal{C}^\infty(M,N)$. Then

i) f harmonic $\Leftrightarrow$ f maps $BM(M,g)$ to $Mart(N,h)$

ii) f harmonic morphism $\Leftrightarrow$ f map $BM(M,g)$ to $BM(N,h)$ modulo time change

i.e. X $BM(M,g) \Rightarrow \exists \tilde{X}$ $BM(N,h)$ s.th. $f(X_t) = \tilde{X}_{T_t}$ $\forall t$

where $T_t = \int_0^t \lambda^2(X_s)\,ds$

Pr f harmonic $\Leftrightarrow$ $\tilde{X} = f(X)$ $Mart(N,h)$ $\Leftrightarrow$ $\tilde{Z} = \tilde{U}_0^{-1} A(\tilde{X})$ loc. mart on $\mathbb{R}^{\dim N}$

To show: $d\tilde{Z}^k d\tilde{Z}^\ell = \lambda^2(X)\,\delta_{k\ell}\,dt$ $\forall k,\ell$ $\Leftrightarrow$ f horizontally conformal

But $d\tilde{Z}^k d\tilde{Z}^\ell = \left(\tilde{U}_0 \tilde{/\!/}_{0,\cdot}^{-1} (df)_X /\!/_{0,\cdot}\, dA(X)\right)^k \left(\tilde{U}_0 \tilde{/\!/}_{0,\cdot}^{-1} (df)_X /\!/_{0,\cdot}\, dA(X)\right)^\ell$

Note that $\tilde{U}_0 \tilde{/\!/}_{0,t}^{-1} (df)_{X_t} /\!/_{0,t}\, dA(X)_t = \tilde{U}_t^{-1} (df)_{X_t} U_t\, dZ_t$

$$= \sum_{i=1}^n \tilde{U}_t^{-1} (df)_{X_t} U_t e_i\, dZ_t^i \qquad , \quad Z\ BM(\mathbb{R}^n),\ dZ^i dZ^j = \delta_{ij}\,dt$$

$$\Rightarrow d\tilde{Z}^k d\tilde{Z}^\ell = \sum_{i=1}^n \langle \tilde{U}^{-1} (df)_X U e_i, e_k\rangle \langle \tilde{U}^{-1} (df)_X U e_i, e_\ell \rangle\, dt$$

$$= \sum_{i=1}^n h\left((df)_X U e_i, \tilde{U} e_k\right) h\left((df)_X U e_i, \tilde{U} e_\ell\right) dt$$

$$= \sum_{i=1}^n g\left(U e_i, (df)_X^{ad} \tilde{U} e_k\right) g\left(U e_i, (df)_X^{ad} \tilde{U} e_\ell\right) dt$$

$$= g\left((df)_X^{ad} \tilde{U} e_k, (df)_X^{ad} \tilde{U} e_\ell\right) dt$$

$$= h\left((df)_X (df)_X^{ad} \tilde{U} e_k, \tilde{U} e_\ell\right) dt$$

$$\underset{\substack{\uparrow \\ f \text{ horizontally conformal}}}{=} \lambda^2(X)\, \underbrace{h(\tilde{U} e_k, \tilde{U} e_\ell)}_{=\delta_{k\ell}}\, dt$$

∎

2. Brownian motion & curvature

(M,g) Riem mf, $\dim M \geq 2$

$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ Levi-Civita connection

$(X,Y) \longmapsto \nabla_X Y$

Def ① Riemann curvature tensor

$R \in \Gamma(T^*M \otimes T^*M \otimes T^*M \otimes TM)$ defined by

$$R(X,Y,Z) \equiv R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z$$

read as $\Gamma(TM)^3 \xrightarrow{R} \Gamma(TM)$ $\mathcal{C}^\infty(M)$ trilinear

or $\Gamma(TM\otimes TM) \longrightarrow \mathrm{Hom}_{\mathcal{C}^\infty(M)}(\Gamma(TM), \Gamma(TM))$

② Riemannian sectional curvature

Let $G_2TM \to M$ be the Grassmannian 2-bundle defined (as set) by

$G_2TM = \dot\bigcup_{x\in M} G_2T_xM$ where $G_2T_xM := \{E \subset T_xM \mid E \text{ 2-dim. linear subspace}\}$

$$\mathrm{Riem}^M : G_2TM \longrightarrow \mathbb{R}$$
$$E = \mathrm{span}(u,v) \longmapsto \frac{\langle R(u,v)v,u\rangle}{|u\wedge v|^2} = \frac{\langle R(u,v)v,u\rangle}{|u|^2|v|^2 - \langle u,v\rangle^2}$$

(well-defined, i.e. indep of the choice of u,v)

③ Ricci curvature

$\mathrm{Ric}^M \in \Gamma(T^*M\otimes T^*M)$, $\mathrm{Ric}^M_x(u,v) := \mathrm{trace}\,\underbrace{(w \longmapsto R(w,u,v))}_{\in \mathrm{Hom}(T_xM,T_xM)}$ where $u,v\in TM$, $\pi(u)=\pi(v)=x$,

$$= \sum_i \langle R(e_i,u)v, e_i\rangle \quad \text{with } (e_i) \text{ onb of } T_xM$$

$\mathrm{Ric}^M_x : T_xM \times T_xM \longrightarrow \mathbb{R}$ symmetric bilinear form

④ Scalar curvature

$k^M \in \mathcal{C}^\infty(M)$: $k^M(x) := \mathrm{trace}\,\mathrm{Ric}^M_x = \sum_j \mathrm{Ric}^M_x(e_j,e_j) = \sum_{i,j}\langle R(e_i,e_j)e_j, e_i\rangle$

with (e_i) onb of T_xM

Curvature idendities

$\forall X, Y, Z, U \in \Gamma(TM)$

i) $\langle R(X,Y)Z, U\rangle = -\langle R(Y,X)Z, U\rangle = -\langle R(X,Y)U, Z\rangle$

ii) $\langle R(X,Y)Z, U\rangle = \langle R(Z,U)X, Y\rangle$

iii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (Bianchi idendity)

In particular,

$\langle R(X,U)V, X\rangle \overset{ii)}{=} \langle R(V,X)X, U\rangle \overset{i)}{=} -\langle R(X,V)X, U\rangle = \langle R(X,V)U, X\rangle$

$\Longrightarrow \ Ric^M(X,Y) = Ric^M(Y,X)$

Notation

i) (M,g) constant (positive, negative) curvature

$\iff Riem^M$ constant (positive, negative)

ii) (M,g) flat $\iff Riem^M \equiv 0 \iff R \equiv 0$

iii) $Ric^M \geqslant k$ $[\leqslant]$ $\iff Ric^M(X,X) \geqslant k\, g(X,X)$ $[\leqslant]$ $\forall X \in \Gamma(TM)$

iv) (M,g) Einstein mf $\iff Ric^M = c g$ for some constant c.

Interpretation of curvature

$x \in M$ & $u, v, w \in T_xM \rightsquigarrow R(u,v)w \in T_xM$?

Choose $\alpha :]-\varepsilon, \varepsilon[\times]-\varepsilon, \varepsilon[\longrightarrow M$, $(s,t) \longmapsto \alpha(s,t)$ s.th. $\alpha(0,0) = x$

and $\frac{\partial}{\partial s}\alpha(0,0) = u$ & $\frac{\partial}{\partial t}\alpha(0,0) = w$

e.g. $\alpha(s,t) := \exp_x(su+tv)$

$$\begin{tikzcd}
(0,t) \arrow[d] & (s,t) \arrow[l] \\
(0,0) \arrow[r] & (s,0) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
T_{\alpha(0,t)}M \arrow[d] & T_{\alpha(s,t)}M \arrow[l] \\
T_{\alpha(0,0)}M \arrow[r] & T_{\alpha(s,0)}M \arrow[u]
\end{tikzcd}$$

$s,t \in]-\varepsilon,\varepsilon[\rightsquigarrow w_{s,t} \in T_xM$ defined by $\parallel$- transport of $w \in T_xM = T_{\alpha(0,0)}M$ along the outlined curve

Then

$$R(u,v)w = \lim_{s,t \to 0} \frac{w_{s,t} - w}{st}$$

curvature $\hat{=}$ "infinitesimal measure" of the deviation of the $\parallel$- transport from being path-independent

$\parallel$- transport on M path-independent $\Longleftrightarrow$ M flat (i.e. $R \equiv 0$)

Heat equation on sections of vector bundles

M Riem. mf., $\dim M = n$

∇ LC-connection on M

$P = O(M) \xrightarrow{\pi} M$ orthonormal frame bundle

$E \to M$ an associated vector bundle with fiber V and structure group $G = O(n)$, e.g.

$$E = T^r M := \otimes^r TM \quad \text{with } V = \otimes^r \mathbb{R}^n$$

$$E = T^{*r} M := \otimes^r T^*M \quad \text{with } V = \otimes^r \mathbb{R}^n$$

$$E = T^r M \otimes T^{*s} M \quad \text{with } V = \mathbb{R}^{n(r+s)}$$

$$E = \Lambda^p TM := \mathrm{Alt}^p TM \quad \text{with } V = \Lambda^p \mathbb{R}^n$$

$$E = \Lambda T^*M = \bigoplus_{p \geq 0} \Lambda^p T^*M \quad \text{with } V = \Lambda \mathbb{R}^n = \bigoplus_{p=1}^{n} \Lambda^p \mathbb{R}^n \quad \text{etc.}$$

The covariant derivative ∇ (induced by ∇ on M)

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

defines a canonical differential operator $\square$ on $\Gamma(E)$:

$$\square a = \mathrm{tr}\, \nabla^2 a \qquad (\text{"connection" or "rough" Laplacian})$$

$$\nabla^2 a \in \Gamma(T^*M \otimes T^*M \otimes E) \longrightarrow (\square a)_x = \sum_{i=1}^n \nabla^2 a(v_i, v_i), \quad (v_i) \text{ o.n.b. of } T_x M$$

Remark $L_1, \ldots, L_n$ standard-horizontal vfs on $O(M)$

$$L_i(u) := h_u(ue_i), \quad u \in O(M)$$

where $h : \pi^* TM \xrightarrow{\sim} H$, $h_u : T_{\pi(u)} M \xrightarrow{\sim} H_u \subset T_u O(M)$ hor lift

$$\Delta^{hor} := \sum_{i=1}^n L_i^2 \qquad \text{Bochner's horizontal Laplacian}$$

Δ^{hor} operates on $C^\infty(O(M))$

Relation between $\square$ and Δ^{hor}?

Remark. Write sections $a \in \Gamma(E)$ as "equivariant" maps on $O(M)$:

$$\mathcal{F}_a : O(M) \longrightarrow V \,, \quad \mathcal{F}_a(u) := u^{-1} a_{\pi(u)}$$

where we read $u \in O(M)$ as isomorphism $V \xrightarrow{\sim} E_{\pi(u)}$.

"equivariant" $\hat{=}$ $\mathcal{F}(ug) = g^{-1}\mathcal{F}(u)$, $u \in O(M)$, $g \in G = O(n)$

Lemma $a \in \Gamma(E) \rightsquigarrow \mathcal{F}_a : O(M) \longrightarrow V$, $\mathcal{F}_a(u) = u^{-1} a_{\pi(u)}$

Then $\Delta^{hor} \mathcal{F}_a = \mathcal{F}_{\Box a}$

where $\Box = \operatorname{tr} \nabla^2$

$$\Box : \Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M \otimes E) \xrightarrow{\operatorname{tr}} \Gamma(E)$$

Pr. We show that

$$(L_i \mathcal{F}_a)(u) = \mathcal{F}_{\nabla_{ue_i} a}(u) \,, \quad u \in O(M)$$

Fix $u \in O(M)$ & let γ be a curve on M s.th.

$$\gamma(0) = \pi(u) \quad \& \quad \dot\gamma(0) = ue_i$$

Let $t \mapsto u(t)$ be the horizontal lift of γ to $O(M)$ s.th. $u(0) = u$

$$\Rightarrow \dot u(t) = h_{u(t)}(\dot\gamma(t)) \Rightarrow \dot u(0) = h_u(ue_i) = L_i(u)$$

$$\Rightarrow \mathcal{F}_{\nabla_{ue_i} a}(u) = u^{-1}(\nabla_{ue_i} a)_{\pi(u)} = u^{-1} \lim_{\varepsilon \downarrow 0} \frac{/\!/_{0,\varepsilon}^{-1} a_{\gamma(\varepsilon)} - a_{\gamma(0)}}{\varepsilon}$$

$$\underset{\substack{\uparrow \\ /\!/_{0,\varepsilon}^{-1} = u(0)u(\varepsilon)^{-1}}}{=} \lim_{\varepsilon \downarrow 0} \frac{u(\varepsilon)^{-1} a_{\gamma(\varepsilon)} - u(0)^{-1} \gamma(0)}{\varepsilon}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{\mathcal{F}_a(u(\varepsilon)) - \mathcal{F}_a(u(0))}{\varepsilon} = (L_i)_u \mathcal{F}_a = (L_i \mathcal{F}_a)(u)$$

Given $a \in \Gamma(E)$. Consider

$$\begin{cases} \frac{\partial}{\partial t} a_t = \frac{1}{2} \Box a_t \\ a_t \big|_{t=0} = a \end{cases}$$

"heat equation" to $\Box$

Want to show that

$$a_t(x) = \mathbb{E}\left[/\!/_{0,t}^{-1} a(X_t(x)) \right]$$

where X BM(M,g)

& $/\!/_{0,t} : E_{X_0} \to E_{X_t}$ $/\!/$-transport along X

(induced by

$/\!/_{0,t} := u_t u_0^{-1} : T_{X_0}M \to T_{X_t}M$

(1) $T > 0$. Show that

$$N_t := /\!/_{0,t}^{-1} a_{T-t}(X_t(x))$$

is a (local) martingale

(2) Then $\underbrace{\mathbb{E}[N_0]}_{= a_T(x)} = \underbrace{\mathbb{E}[N_T]}_{= \mathbb{E}\left[/\!/_{0,T}^{-1} a(X_T(x)) \right]}$

E associated vb to $O(M)$, e.g. $E = T^r M \otimes T^{*s} M$ or $E = \Lambda T^* M = \bigoplus_{p \geq 0} \Lambda^p T^* M$

For $a \in \Gamma(E)$ want to study

$$/\!/_{0,t}^{-1} a(X_t(x)) \quad \text{where } X(x) \ BM(M,g) \text{ with } X_0(x) = x$$

<u>Idea</u>: Write a as equivariant function F_a on $O(M)$:

$$F_a(u) = u^{-1} \underbrace{a_{\pi(u)}}_{\in E_{\pi(u)}} \in V \quad \text{with } V = \mathbb{R}^{n(r+s)}, \text{ resp. } V = \Lambda \mathbb{R}^n = \bigoplus_{p \geq 0} \Lambda^p \mathbb{R}^n$$

$$X \ BM(M,g) \rightsquigarrow Z = \int_U \vartheta \quad BM(\mathbb{R}^n) \text{ anti-development}$$

$$dU = \sum_{i=1}^n L_i(U) \circ dZ^i$$

$$\underset{\text{Itô}}{\Longrightarrow} \quad dF(U) = L_i F(U) dZ^i + \tfrac{1}{2} \Delta^{hor} F(U)\, dt$$

$$\underset{F = F^a}{\Longrightarrow} \quad L_i F_a = F_{\nabla_{ue_i} a} \quad \& \quad \Delta^{hor} F_a = F_{\Box a}.$$

$$\text{where } \Box a : \Gamma(E) \xrightarrow{\nabla^E} \Gamma(T^*M \otimes E) \xrightarrow{\nabla^{T^*M \otimes E}} \Gamma(T^*M \otimes T^*M \otimes E) \xrightarrow{\text{trace}} \Gamma(E)$$

$$/\!/_t^{-1} a(X_t) = u_0 \underbrace{u_t^{-1} a(X_t)}_{F_a(U_t)}$$

$$\Rightarrow d\left(/\!/_{0,t}^{-1} a(X_t)\right) = u_0\, dF_a(U_t) = u_0 \Big(\sum_{i=1}^n \underbrace{(F_{\nabla_{ue_i} a})}_{= u^{-1}(\nabla_{ue_i} a)(X_t)} dZ^i + \tfrac{1}{2} \underbrace{F_{\Box a}(U)}_{u^{-1}(\Box a)(X_t)} dt \Big)$$

$$= \sum_i /\!/_{0,t}^{-1} (\nabla_{ue_i} a)\, dZ^i + \tfrac{1}{2} /\!/_{0,t}^{-1} (\Box a)(X_t)\, dt$$

Now fix $T > 0$ & let a_t be the solution to

$$\begin{cases} \frac{\partial}{\partial t} a_t = \frac{1}{2} \Box a_t & \text{on } [0,T] \times M \\ a_t|_{t=0} = a \end{cases}$$

Consider $N_t = /\!/_{0,t}^{-1} a_{T-t}(X_t(x))$

$$\Rightarrow dN_t \overset{m}{=} /\!/_{0,t}^{-1} \underbrace{\left(\tfrac{1}{2} \Box a_{T-t} + \partial_t a_{T-t} \right)}_{\equiv 0} (X_t(x))\, dt = 0, \text{ i.e. } N_t \text{ is a local martingale}$$

In particular, if N is a true martingale then

$$\underbrace{E[N_0]}_{a_T(x)} = \underbrace{E[N_T]}_{= E\left[//_{0,T}^{-1} a(X_T(x)) \right]}$$

<u>Philosophy</u>: Interesting operators $L^{\mathcal{R}}$ on $\Gamma(E)$ differ from $\square$ by a 0-order term, i.e.

$$\square - L^{\mathcal{R}} = \mathcal{R} \quad \text{where } \mathcal{R} \in \Gamma(\mathrm{End}\, E),$$

$$\mathcal{R}_x \in \mathrm{Hom}(E_x, E_x) \quad \forall x \in M$$

<u>Ex</u>. $E = \Lambda^p T^*M$ & $A^p(M) = \Gamma(\Lambda^p T^*M)$

Then de Rham-Hodge Laplacian

$$\Delta^p = -(d^*d + dd^*) : A^p(M) \longrightarrow A^p(M)$$

$d: A^p(M) \to A^{p+1}(M)$ exterior derivative & $d^*: A^p(M) \to A^{p-1}(M)$ co-differential (formal adjoint to d)

takes the form

$$\Delta^p a = \square a - \underbrace{\mathcal{R}a}_{\text{defined in terms of the curvature tensor}}$$ (Weitzenböck decomposition),

i.e. $p=1$, $\mathcal{R}a = \mathrm{Ric}(\cdot, a^\#)$

$$\flat : T_xM \xrightarrow{\sim} T_x^*M, \quad v \longmapsto \langle \cdot, v \rangle$$
$$\# = \flat^{-1}$$

Consider now an operator of the form $L = \square - \mathcal{R}$ with $\mathcal{R} \in \Gamma(\mathrm{End}\, E)$ & look for stoch. representations of the solutions to

$$(*) \quad \begin{cases} \partial_t a_t = \frac{1}{2}(\square - \mathcal{R}) a_t \\ a_t|_{t=0} = a \in \Gamma(E) \end{cases}$$

<u>Def</u> For $x \in M$ & X $BM(M,g)$, starting at x, define an $\mathrm{Aut}(E_x)$-valued process by the following pathwise linear differential equation

$$\frac{d}{dt} Q_t = -\frac{1}{2} Q_t \mathcal{R}_{//_t}, \quad Q_0 = \mathrm{id}_{E_x}$$

where $\mathcal{R}_{//_t} := //_t^{-1} \circ \mathcal{R}_{X_t} \circ //_t \in \mathrm{End}(E_x)$

with $//_t$ //-transport in E.

$$\begin{tikzcd}
E_x \arrow[r, dashed, "R_{/\!/_t}"] \arrow[d, "/\!/_t"'] & E_x \\
E_{X_t} \arrow[r, "R_{X_t}"'] & E_{X_t} \arrow[u, "/\!/_t^{-1}"']
\end{tikzcd}$$

Then $d\Big(Q_t \underbrace{/\!/_{0,t}^{-1} a(X_t)}_{= n_t}\Big) = \underbrace{dQ_t}\, n_t + Q_t\, dn_t$

$= -\frac{1}{2} Q_t /\!/_{0,t}^{-1} \mathcal{R}_{X_t} /\!/_{0,t} n_t\, dt = -\frac{1}{2} Q_t /\!/_{0,t}^{-1} \mathcal{R}_{X_t} a(X_t)\, dt$

But $dn_t = \sum_i /\!/_{0,t}^{-1} (\nabla_{u_t e_i} a)\, dz^i + \frac{1}{2} /\!/_{0,t}^{-1} (\Box a)(X_t)\, dt$

$\Rightarrow d\big(Q_t /\!/_{0,t}^{-1} a(X_t)\big) = \sum_i Q_t /\!/_{0,t}^{-1} (\nabla_{u_t e_i} a)\, dz^i + \frac{1}{2} Q_t /\!/_{0,t}^{-1} (\Box a - \mathcal{R} a)(X_t)\, dt$

Let a_t be defined by $(*)$

$\Rightarrow d\big(Q_t /\!/_{0,t}^{-1} a_{T-t}(X_t)\big) \overset{(m)}{=} Q_t /\!/_{0,t}^{-1} \underbrace{\Big(\frac{1}{2}(\Box - \mathcal{R}) a_{T-t} + \partial_t a_{T-t}\Big)}_{\equiv 0} (X_t)\, dt = 0$

$\Rightarrow Q_t /\!/_{0,t}^{-1} a_{T-t}(X_t)$ local martingale

If a true martingale, then

$$\alpha_T(x) = \mathbb{E}\big[Q_T /\!/_{0,T}^{-1} a(X_T)\big]$$

<u>Notation</u> E/M Riem v.b.

$$\left.\begin{array}{ll} \rightsquigarrow \Gamma(E) & \mathcal{C}^\infty \text{ sections} \\ L^2\text{-}\Gamma(E) & L^2 \cap \mathcal{C}^\infty \text{ ---''---} \\ L^2(E) & L^2 \text{ sections} \end{array}\right\} \text{ of } E$$

where $\langle a, b\rangle_{L^2(M)} := \int_M \langle a(x), b(x)\rangle_{E_x} \, vol(dx)$

Def M Riem mf & $X \in \Gamma(TM)$

$\operatorname{div} X \in \mathcal{C}^\infty(M)$, $\operatorname{div} X = \operatorname{trace}(v \mapsto \nabla_v X)$

i.e. $(\operatorname{div} X)(x) = \sum_i \langle \nabla_{e_i} X, e_i \rangle$ with (e_i) onb of $T_x M$

Note To $x \in M$

$\exists$ (e_i) local on system at x s.th. $(\nabla e_i)_x = 0$

Then $\operatorname{div}(X) = \sum_i \langle \nabla_{e_i} X, e_i \rangle (x)$

$$= \sum_i \left(e_i \langle X, e_i \rangle - \langle X, \nabla_{e_i} e_i \rangle \right)(x)$$

$\uparrow$ ∇ Riemannian

$$= \left(\sum_i e_i \langle X, e_i \rangle \right)(x)$$

Cor $\Box = \operatorname{tr} \nabla^2 : \Gamma(E) \to \Gamma(E)$ negative & formally s.a.

$\langle \Box a, b \rangle_{L^2(E)} = -\langle \nabla a, \nabla b \rangle_{L^2(T^*M \otimes E)}$ $\forall\, a, b \in \Gamma(E)$ with a or b of compact support

In this sense: $\Box = \nabla^* \nabla$

Pr $x \in M$, (e_i) as above

$$\langle \underbrace{\operatorname{tr} \nabla^2 a}_{\Box a}, b \rangle = \sum_i \langle \nabla_{e_i} \nabla_{e_i} a, b \rangle$$

$$= \sum_i \left(e_i \langle \nabla_{e_i} a, b \rangle - \langle \nabla_{e_i} a, \nabla_{e_i} b \rangle \right)$$

$$= \operatorname{div}(X) - \langle \nabla a, \nabla b \rangle$$

$\uparrow$ determined by $\langle X, v \rangle = \langle \nabla_v a, b \rangle$, $v \in TM$

$$\Rightarrow \langle \Box a, b \rangle_{L^2(E)} = \underbrace{\int_M \operatorname{div}(X)}_{=0} - \langle \nabla a, \nabla b \rangle_{L^2(T^*M \otimes E)}$$

Self-adjoint extension of $L = \square - \mathcal{R}$ where $\mathcal{R} \in \Gamma(\mathrm{End}\, E)$ is assumed to be symmetric, i.e. $\mathcal{R}_x : E_x \to E_x$ symmetric $\forall x \in M$.

Suppose that

$$(\square - \mathcal{R}) \,|\, \Gamma_c(E) \quad \text{is bdd above},$$

i.e. $\lambda_0(\mathcal{R}) := \sup \left\{ \dfrac{\langle (\square - \mathcal{R}) a, a \rangle_{L^2}}{\langle a, a \rangle_{L^2}} : 0 \neq a \in \Gamma_c(E) \right\} < \infty$

$$\mathcal{E}(a,b) := -\langle \nabla a, \nabla b \rangle_{L^2} - \langle \mathcal{R} a, b \rangle_{L^2}, \quad a, b \in \mathcal{D}(\mathcal{E}) = \Gamma_c(E)$$

Let $q(a,b) := -\mathcal{E}(a,b) + c \langle a, b \rangle_{L^2}$, $c > \lambda_0(\mathcal{R})$

positive quadratic form on $\mathcal{D}(E)$;

complete $\mathcal{D}(E)$ to $\overline{\mathcal{D}}(E)$ in the q-norm

& extend $\mathcal{E}$ by continuity to a closed form $\overline{\mathcal{E}}$ on $\overline{\mathcal{D}}(\mathcal{E})$

$\Longrightarrow \overline{\mathcal{E}}(a,b) = \langle (\square - \mathcal{R})^{\sim} a, b \rangle_{L^2}$

for some s.a. operator $\underbrace{(\square - \mathcal{R})^{\sim}}_{\text{Friedrichs extension of } (\square - \mathcal{R}) \Gamma_c(E)}$ with domain $\overline{\mathcal{D}}(\mathcal{E}) \subset L^2(E)$

Define $P_t a = e^{t(\square - \mathcal{R})^{\sim}/2} a$, $a \in L^2(E)$, by spectral theorem

Then $(t,x) \mapsto (P_t a)(x)$ is smooth on $]0,\infty[\times M$ &

$\exists$ kernel $(t,x,y) \mapsto p(t,x,y) \in \mathrm{Hom}(E_y, E_x)$ smooth on $]0,\infty[\times M \times M$

s.th. $P_t a(x) = \int_M p(t,x,y)\, a(y)\, \mathrm{vol}(dy)$ for the $\mathcal{C}^\infty$ version of $P_t a$.

<u>Def</u> For $g : M \to \mathbb{R}$ cont. & f measurable on M, let

$$P_t^g f(x) := \mathbb{E}\left[\exp\left(\int_0^t g(X_s(x))\, ds \right) f(X_t(x))\, 1_{\{t < \zeta(x)\}} \right]$$

if the RHS is well-defined.

Let $\underline{\mathcal{R}}(x) := \min \{ \langle \mathcal{R}_x v, v \rangle : v \in E_x, |v| = 1 \}$, $\underline{\mathcal{R}}$ continuous

If $\frac{d}{dt} Q_t = -\frac{1}{2} Q_t \mathcal{R}_{/\!/t}$ with $Q_0 = \mathrm{id}_{E_x}$,

then $|Q_t|_{op} \leq \exp\left(-\frac{1}{2} \int_0^t \underline{\mathcal{R}}(X_s(x))\, ds \right)$

<u>Proposition</u> Let $\square - \mathcal{R}$ be as above with $\mathcal{R} \in \Gamma(\mathrm{End}\, E)$ symmetric. Suppose that $(\square - \mathcal{R})\Gamma_c(E)$ is bdd from above and for $a \in L^2(E)$ let

$$P_t a = e^{\frac{t}{2}(\Delta - \mathcal{R})^{\sim}} a$$

be the C^∞-version of the L^2-semigroup.

Then

$$(P_t a)(x) = \mathbb{E}\left[Q_t \, /\!/_t^{-1} a(X_t(x)) \, 1_{\{t < \zeta(x)\}} \right]$$

holds for all $a \in L^2(E)$ with $P_t^{\underline{\mathcal{R}}} |a|(x) < \infty$

<u>Pr</u> Without restriction

$$a \in L^2(E) \cap \Gamma_c(E)$$

since $P_t a$ has an integral kernel.

$D_n \uparrow M$ exhaustion of M by relatively compact open domains with smooth boundary.

Let $\tilde{L}_n$ denote the Friedrichs extension on $(\square - \mathcal{R})|\Gamma_c(E|D_n)$

Then $P_t^{(n)} a = e^{t\tilde{L}_n} a \to P_t a$ in L^2, $\operatorname{supp} a \subset D_n$

We use i) $(t,x) \longmapsto P_t^{(n)} a(x)$ smooth on $[0,T] \times D_n$ ($\Rightarrow$ in particular bounded)

ii) $P_t^{(n)} a \,|\, \partial D_n = 0$

Know $N_s^{(n)} = Q_s \, /\!/_s^{-1} P_{t-s}^{(n)} a(X_s(x))$ is a local martingale with lifetime $t \wedge \tau_{D_n}(x)$, $N_t^{(n)}$ bdd !

first exit time of $X(x)$ from D_n

$$\Rightarrow \underbrace{\mathbb{E}[N_0^{(n)}]}_{= P_t^{(n)} a} = \mathbb{E}\left[N^{(n)}_{t \wedge \tau_{D_n}(x)}\right] = \mathbb{E}\left[N_t^{(n)} 1_{\{t < \tau_{D_n}(x)\}}\right] + \underbrace{\mathbb{E}\left[N^{(n)}_{\tau_{D_n}(x)} 1_{\{t \geq \tau_{D_n}(x)\}}\right]}_{=0}$$

$$\left| N_t^{(n)} 1_{\{t < \tau_{D_n}(x)\}} \right| = \left| Q_t \, /\!/_t^{-1} a(X_t(x)) 1_{\{t < \tau_{D_n}(x)\}} \right| \leq \exp\left(-\tfrac{1}{2}\int_0^t \underline{\mathcal{R}}(X_s(x))\, ds\right) |a|(X_t(x)) \, 1_{\{t < \zeta(x)\}} \in L^1$$

$$\underset{\substack{\uparrow \\ \text{dominated convergence}}}{\Rightarrow} P_t^{(n)} a(x) = \mathbb{E}[N_t^{(n)}] \longrightarrow \mathbb{E}\left[Q_t \, /\!/_t^{-1} a(X_t(x)) 1_{\{t < \zeta(x)\}}\right]$$

and in addition

$$|P_t a|(x) \leq P_t^{\underline{\mathcal{R}}} |a|(x)$$

<u>(semigroup domination)</u>

E Riem v.b., $\mathcal{R} \in \Gamma(\operatorname{End} E)$ pointwise symmetr.

Consider operators L on $\Gamma(E)$ of the form

$$L = \Box - \mathcal{R} \quad , \quad \Box = \operatorname{tr} \nabla^2$$

Ex $\quad E = \Lambda T^* M$, $\Gamma(E) = A(M) = \bigoplus_p A^p(M)$ where $A^p(M) = \Gamma(\Lambda^p T^* M)$

$L = -\{d, d^*\} = -(dd^* + d^* d)$, $\quad L = \Box - \mathcal{R} \equiv \Delta$

$\underline{\mathcal{R}}(x) = \inf \{ \langle \mathcal{R}_x v, v \rangle : v \in E_x, |v| = 1 \}$

Observation (Bochner)

(1) (Vanishing theorem)

$\underline{\mathcal{R}} \geq 0$ & $\underline{\mathcal{R}}(x_0) > 0$ for one $x_0 \in M \Longrightarrow \ker \underbrace{\Delta}_{= \Box - \mathcal{R}} = \{0\}$

(2) (Estimation theorem)

$\underline{\mathcal{R}} \geq 0 \Longrightarrow |a|$ const. $\forall a \in \ker \Delta$ & $\dim \ker \Delta \leq \min_{x \in M} \dim \ker \mathcal{R}_x \leq \dim E$

Pr $\quad \underline{\mathcal{R}} \geq 0$, $\Delta a = 0$

$$\Longrightarrow 0 = \langle \Delta a, a \rangle_{L^2(E)} = -\underbrace{\langle \nabla^* \nabla a, a \rangle}_{\langle \nabla a, \nabla a \rangle_{L^2}} - \langle \mathcal{R} a, a \rangle_{L^2}$$

$\Longrightarrow \underbrace{\langle \mathcal{R} a, a \rangle = 0}_{\text{i.e. } a(x) \in \ker \mathcal{R}_x \ \forall x \in M}$ & $\underbrace{\langle \nabla a, \nabla a \rangle = 0}_{\text{i.e. } \nabla a = 0}$, i.e. a parallel

$\Longrightarrow$ a parallel along arbitrary curves

$\Longrightarrow$ a already determined by its value at one point

in particular, if $\underline{\mathcal{R}}(x_0) > 0$ & $a(x_0) = 0$ then $a \equiv 0$

$$\Gamma(\Lambda T^*M) = A(M) = \bigoplus_p A^p(M)$$

$$\Delta = -\{d, d^*\} = -(dd^* + d^*d) \quad : \quad A(M) \to A(M)$$

$$\Delta = \Box - \mathcal{R}$$

$$\mathcal{H}(M) = \{ a \in L^2 : \Delta a = 0 \} = \{ a \in L^2(E) \cap \Gamma(E) : \Delta a = 0 \}$$

$$= \bigoplus_p \mathcal{H}^p(M) \quad \text{with} \quad \mathcal{H}^p(M) = \{ a \in A^p(M) \cap L^2 : \Delta a = 0 \}$$

$$H^p(M;\mathbb{R}) \cong H^p_{DR}(M) = \{ a \in A^p(M) : da = 0 \} / d A^{p-1}(M)$$

Hodge $\quad H^p_{DR}(M) = \mathcal{H}^p(M) \quad (M \text{ cpt})$; More precisely

Abstract Hodge decomposition

$$A^p(M) = \mathcal{H}^p(M) \oplus d A^{p-1}(M) \oplus d^* A^{p+1}(M) \qquad L^2\text{-orth. decomposition}$$

Note $\quad a = Ha + d\beta + d^*\gamma \quad \& \quad da = 0$

$$\Rightarrow a = Ha + d\beta$$

$$\langle a, d^*\gamma \rangle = \langle da, \gamma \rangle = 0$$

$$b_p(M) := \dim H^p(M;\mathbb{R}) \qquad p. \text{ Betti number}$$

Remark $\quad a \in A^p(M)$

(1) Suppose $\exists\ L^2\text{-}\lim_{t\to\infty} P_t a =: Ha \in A^p(M) \cap L^2$ (spectral theorem)

$$\Big(\Rightarrow Ha \text{ harmonic} : \quad P_t(Ha) = \lim_{s\to\infty} P_t(P_s a) = \lim_{s\to\infty} P_{t+s} a = Ha$$

$$\text{i.e. } \Delta Ha = \Delta P_t(Ha) = 2\frac{d}{dt} P_t(Ha) = 0 \Big)$$

(2) Suppose $P_t a \to Ha$ "sufficiently fast"

s.th. $\quad Ga := \frac{1}{2}\int_0^\infty (Ha - P_t a)\,dt \quad$ is well-defined

$$\Rightarrow \Delta Ga = -\frac{1}{2}\int_0^\infty \Delta P_t a\,dt = -\int_0^\infty \left(\frac{d}{dt} P_t a\right) dt = -P_t a \Big|_{t=0}^{t=\infty} = a - Ha$$

i.e. $\exists G : A^\cdot(M) \to A^\cdot(M)$ homogenous of degree 0 (Green operator)

$$\mathrm{id}_{A^\cdot(M)} = H + \Delta G = H - dd^*G - d^*dG$$

$$A^p(M) = \mathcal{H}^p(M) \oplus d A^{p-1}(M) \oplus d^* A^{p+1}(M) \quad , \quad p = 0, 1, 2, \ldots$$

Problem : Make (2) rigorous, e.g. if M is cpt.

Stochastic approach to vanishing theorems

$$P_t a(x) = \mathbb{E}\left[Q_t \, /\!/_t^{-1} a(X_t(x)) \, 1_{\{t<\zeta(x)\}} \right]$$

$$|P_t a|(x) \leq P_t^{\underline{\mathcal{R}}} |a|(x) = \mathbb{E}\left[\exp\left(-\int_0^t \underline{\mathcal{R}}(X_s(x))\, ds\right) |a|(X_s(x)) \, 1_{\{t<\zeta(x)\}} \right]$$

semigroup domination

$$\begin{cases} \partial_t a_t^{(n)} = (\square - \mathcal{R}) a_t^{(n)} \\ a_t^{(n)}|\partial D_n = 0 \\ a_t^{(n)}\big|_{t=0} = a \end{cases} \qquad D_n \uparrow M$$

$$a_t^{(n)}(x) = \mathbb{E}\left[Q_t \, /\!/_t^{-1} a(X_t(x)) \, 1_{\{t<\tau_{D_n}(x)\}} \right], \quad x \in D_n$$

$$\downarrow \qquad |a_t^n|(x) \leq \mathbb{E}\left[\exp\left(-\int_0^t \underline{\mathcal{R}}(X_s(x))\, ds\right) |a|(X_s(x)) \, 1_{\{t<\tau_{D_n}(x)\}} \right]$$

$$a_t(x)$$

Suppose that $\underline{\mathcal{R}}$ is in the Green class, i.e.

$$G^{\underline{\mathcal{R}}} 1_K := \int_0^\infty P_t^{\underline{\mathcal{R}}} 1_K \, dt < \infty \quad \forall K \subset M \text{ cpt.}$$

Then $\ker(\square - \mathcal{R}) = \{0\}$

Pr Let $a \in \ker(\square - \mathcal{R})$

$$|a_t^{(n)}| \leq \| a|_{K_n} \|_\infty \, P_t^{\underline{\mathcal{R}}} 1_{K_n} \quad \forall n$$

$$K_n = \overline{D}_n$$

$$\Rightarrow \int_0^\infty |a_t^{(n)}| \, dt < \infty \quad \forall n,$$

but $a_t^{(n)} \longrightarrow P_t a = a$

if $(\square - \mathcal{R}) a = a$

Gradient formulas & Bismut type derivative formulas

$f \in \mathcal{C}_b(M)$, e.g. $f \in \mathcal{C}_0(M)$

Heat flow: $\begin{cases} \partial_t u = \frac{1}{2}\Delta u \\ u|_{t=0} = f \end{cases}$, $u(t,\cdot) = P_t f$

Want to understand (& estimate)

$$\nabla P_t f$$

Note that $dP_t f$ solves the heat equation on 1-forms (since $d\Delta = \Delta d$)

$$\partial_t dP_t f = d\big(\underbrace{\partial_t P_t f}_{=\frac{1}{2}\Delta P_t f}\big) = \frac{1}{2}\Delta(dP_t f)$$

Know that

$$N_t := Q_t \, /\!/_{0,t}^{-1} (dP_{T-t} f)_{X_t(x)} \qquad \text{local martingale}$$

$v \in T_x M$, $N_t(v) = (N_t, v) = \big((dP_{T-t}f)_{X_t(x)}, /\!/_{0,t} Q_t^{tr} v\big)$ local martingale

↑ pairing between $T_x^* M$ and $T_x M$

ℓ adapted T_xM-valued process s.th. $\int_0^T |\dot{\ell}_s|^2 ds \in L^2$

$$\Rightarrow \quad n_t := (N_t, \ell_t) - \int_0^t (N_s, d\ell_s)$$

$$= \big((dP_{T-t}f)_{X_t(x)}, /\!/_{0,t} Q_t^{tr} \ell_t\big) - \int_0^t \big((dP_{T-s}f)_{X_s(x)}, /\!/_{0,s} Q_s^{tr} \dot{\ell}_s\big) ds \qquad \text{local martingale}$$

$$\Rightarrow \quad \tilde{n}_t = \big((dP_{T-t}f)_{X_t(x)}, /\!/_{0,t} Q_t^{tr} \ell_t\big) - \underbrace{\int_0^t \big((dP_{T-s}f)_{X_s(x)}, /\!/_{0,s} dZ_s\big)}_{=(P_{T-t}f)_{X_t(x)}} \cdot \int_0^t \langle Q_s^{tr} \dot{\ell}_s, dZ_s \rangle \qquad \text{local martingale}$$

, Z BM(T_xM)

Idea: Choose ℓ_t s.th.

1. $\tilde{n}_t$ is a true martingale
2. $\ell_0 = v$ & $\ell_T = 0$

without restrictions: $\ell_t = \tilde{\ell}_t \cdot v$ where $\tilde{\ell}_t$ $\mathbb{R}$-valued , $\tilde{\ell}_0 = 1$ & $\tilde{\ell}_T = 0$

Thm Assume that the BM $X_\cdot(x)$ has infinite lifetime.

Then $\forall\, v \in T_xM$

$$\underbrace{(d P_T f)_x v}_{=\langle (\nabla P_T f)_x,\, v\rangle} = \mathbb{E}\left[f(X_T(x)) \int_0^{\tau_D \wedge T} \langle \Theta_s \dot{\ell}_s, dZ_s\rangle \right]$$

where

- Θ_t takes values in $\mathrm{End}(T_xM)$

$$\frac{d}{dt}\Theta_t = -\frac{1}{2}\mathrm{Ric}_{//_t}(\cdot, \Theta_t)^{\#}, \quad \Theta_0 = \mathrm{id}_{T_xM}$$

- τ_D is the first exit time of $X_\cdot(x)$ from some rel. cpt neighbourhood D of x, e.g. $D = B(x,R)$
- Z $BM(T_xM)$
- ℓ_t is any adapted process in T_xM with abs. cont. paths s.th. $\ell_0 = v$, $\ell_{\tau_D} = 0$ & $\left(\int_0^{\tau_D \wedge T} |\dot{\ell}_t|^2\, dt \right)^{1/2} \in L^1$

Assume that $\mathrm{Ric} \geq K$ $\left(\Rightarrow |\Theta_t| \leq e^{-\frac{K}{2}t} \right)$

and let M be complete $\left(\Rightarrow dP_t f = P_t df \right)$

Then $\forall f \in C_0^\infty(M)$ $\quad dP_t f = P_t \underbrace{df}_{\text{cpt support}}$

& $\tilde{n}_t$ is also a true martingale

for $\ell_t = v$ on $[0,T]$ (1)

or $\ell_t = \left(1 - \frac{t}{T}\right) v$ on $[0,T]$ (2)

This leads to the following two consequences:

(1) Derivative formula $\qquad f \in C_0^\infty(M)$

$$dP_t f(v) = \mathbb{E}\left[Q_t \, //_{0,t} \, (df)_{X_t(x)} \right](v)$$

$$\Rightarrow \; |\nabla P_t f|(x) \le e^{-\frac{K}{2}t} \underbrace{\mathbb{E}\left[|\nabla f|(X_t(x)) \right]}_{P_t|\nabla f|(x)}$$

(2) Bismut formula

$$dP_t f(v) = -\mathbb{E}\left[f(X_T(x)) \, \frac{1}{T} \int_0^T \langle \Theta_s v, dZ_s \rangle \right]$$

Two places where curvature showed up

(1) Heat semigroup on $\Gamma(E)$

$P_t a = \mathbb{E}\left[Q_t \, /\!/_t^{-1} a(X_t)\right]$ semigroup generated by $L = \square - \mathcal{R}$ (for simplicity $BM(M,g)$ infinite lifetime)

$a \in \Gamma(E)$ bdd

$\Rightarrow |P_t a| \leq \mathbb{E}\left[|Q_t| \cdot \|a\|_\infty\right]$ when $|Q_t| \leq \exp\left(-\frac{1}{2}\int_0^t \underline{\mathcal{R}}(X_s)\,ds\right)$

e.g. $\Gamma(E) = \Gamma(T^*M)$. $\Delta = \square - \mathrm{Ric}$

a harmonic, i.e. $P_t a = a$

For vanishing of a, enough positive curvature is needed,

s.th. $\mathbb{E}|Q_t| \to 0$ for $t \to \infty$

e.g. $\mathrm{Ric} \geq 0$ & $\mathrm{Ric}_{x_0} > 0$ at one point

& $BM(M,g)$ recurrent

(2) Gradient formula for scalar heat equation M

$u(t,x) = P_t f = \mathbb{E}[f(X_t(x))]$, $f \in C_0^\infty(M)$

$\Rightarrow dP_t f = \mathbb{E}\left[Q_t \, /\!/_t^{-1} (df)_{X_t}\right]$

In particular,

$$|\nabla P_t f| \leq \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^t \underline{\mathrm{Ric}}(X_s)\,ds\right) |\nabla f|(X_s)\right]$$

or

$$|\nabla P_t f|^2 \leq \mathbb{E}\left[\exp\left(-\int_0^t \underline{\mathrm{Ric}}(X_s)\,ds\right) |\nabla f|^2(X_s)\right]$$

__Theorem__ Let $K \in \mathcal{C}(M)$. Then equivalent

i) $\mathrm{Ric} \geq K$

ii) $|\nabla P_t f|^2 \leq \mathbb{E}\left[\exp\left(-\int_0^t K(X_s)\,ds\right) |\nabla f|^2(X_t)\right]$ $\forall f \in C_c^\infty(M)$

In particular, for $K \in \mathbb{R}$. Equivalent

i) $\mathrm{Ric} \geq K$

ii) $|\nabla P_t f|^2 \leq e^{-tK} P_t |\nabla f|^2$ $\forall f \in C_c^\infty(M)$

Pr of the Thm   Recall

gradient of f: $\nabla f = \operatorname{grad} f$ $\langle \nabla f, A\rangle = Af$

Hessian of f: $\operatorname{Hess}(f) = \nabla df$ $\nabla df(A,B) = \langle \nabla_A \nabla f, B\rangle$ $A, B \in \Gamma(TM)$

$= ABf - (\nabla_A B) f$, since $A\langle \nabla f, B\rangle = \langle \nabla_A \nabla f, B\rangle + \langle \nabla f, \nabla_A B\rangle$

∇ Riemannian

① Bochner-Lichnerowicz formula

$\forall f \in \mathcal{C}^\infty(M)$,

$$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla \Delta f, \nabla f\rangle = |\nabla df|^2_{HS} + \operatorname{Ric}(\nabla f, \nabla f)$$

where $|\ |_{HS}$ is the Hilbert-Schmidt norm, i.e.

$$|\nabla df|^2_{HS}(x) = \sum_{i,j} \langle \nabla_{v_i} \nabla f, v_j\rangle^2 \quad , (v_i) \text{ onb of } T_xM$$

Pr. Check the formula at every $p \in M$.

To this end, choose geodesic normal coordinates at p:

$$\begin{tikzcd}(\mathbb{R}^n, 0) \arrow[r, "\sim", "u\in O(M)_p"'] & (T_pM, 0) \arrow[r, "\exp_p"] & (M, p) \arrow[ll, bend right, "\varphi"'] \\ & v \arrow[r, mapsto] & \gamma_v(1)\end{tikzcd}$$

φ chart locally about p

where γ_v is the geodesic determined by $\gamma_v(0) = p$ & $\dot\gamma_v(0) = v$

If $X_i = \frac{\partial}{\partial \varphi_i}$ then $\langle X_i, X_j\rangle_p = \delta_{ij}$

$(\nabla_{X_i} X_j)_p = 0$

Note that

$$(d\exp_p)_0 v = \frac{d}{dt}\Big|_{t=0} \exp_p(tv) = \frac{d}{dt}\Big|_{t=0} \gamma_{tv}(1) = \frac{d}{dt}\Big|_{t=0} \gamma_v(t) = v$$

Computation at p:

$$\frac{1}{2}\Delta|\nabla f|^2 = \frac{1}{2}\sum_i \Big(X_i X_i |\nabla f|^2 - \underbrace{(\nabla_{X_i} X_i) f}_{=0 \text{ at } p}\Big)$$

$$= \sum_i X_i \langle X_i \nabla f, \nabla f\rangle$$

$$= \sum_i X_i \underbrace{\operatorname{Hess}(f)(X_i, \nabla f)}_{\operatorname{Hess}(f)(\nabla f, X_i)} \qquad \operatorname{Hess}(f) \text{ symmetric}$$

$$= \sum_i X_i \langle \nabla_{\nabla f} \nabla f, X_i\rangle$$

$$= \sum_i \langle \nabla_{X_i} \nabla_{\nabla f} \nabla f, X_i\rangle + \langle \nabla_{\nabla f}\nabla f, \underbrace{\nabla_{X_i} X_i}_{\text{vanishes at } p}\rangle$$

$$= \underbrace{\sum_i \langle R(X_i, \nabla f)\nabla f, X_i\rangle}_{=:(a)} + \underbrace{\sum_i \langle \nabla_{\nabla f} \nabla_{X_i} \nabla f, X_i\rangle}_{=:(b)} + \underbrace{\sum_i \langle \nabla_{[X_i, \nabla f]} \nabla f, X_i\rangle}_{=:(c)}$$

$$\nabla_{X_i}\nabla_{\nabla f} A - \nabla_{\nabla f}\nabla_{X_i} A = R(X_i, \nabla f)A + \nabla_{[X_i, \nabla f]} A$$

(a) $= \mathrm{Ric}(\nabla f, \nabla f)$

(b) $= \sum_i \nabla f \langle \nabla_{X_i} \nabla f, X_i \rangle - \langle \nabla_{X_i} \nabla f, \underbrace{\nabla_{\nabla f} X_i}_{=0 \text{ at } p} \rangle$

$= \sum_i \nabla f \underbrace{\Delta f}_{=h} = \langle \nabla f, \nabla h \rangle = \langle \nabla f, \nabla \Delta f \rangle$

(c) $= \sum_i \mathrm{Hess}(f)([X_i, \nabla f], X_i)$

$= \sum_i \mathrm{Hess}(f)(\nabla_{X_i} \nabla f - \nabla_{\nabla f} X_i, X_i)$

$= \sum_i \mathrm{Hess}(f)(\nabla_{X_i} \nabla f, X_i) - \underbrace{\mathrm{Hess}(f)(\nabla_{\nabla f} X_i, X_i)}_{=0 \text{ at } p}$

$= \sum_i \mathrm{Hess}(f)(X_i, \nabla_{X_i} \nabla f)$

$= \sum_i \langle \nabla_{X_i} \nabla f, \nabla_{X_i} \nabla f \rangle = |\mathrm{Hess}(f)|^2$

(2) Thm $x \in M$ & $X \in T_x M$ with $|X| = 1$

Let $f \in \mathcal{C}_c^\infty(M)$ s.th. $\nabla f(x) = X$ & $\mathrm{Hess}(f)(x) = 0$

Then $\mathrm{Ric}_x(X,X) = \lim_{t \downarrow 0} \dfrac{P_t |\nabla f|^2(x) - |\nabla P_t f|^2(x)}{t}$

Pr Note if $h \in \mathcal{C}^2(M)$ with Δh bdd, then with $L = \frac{1}{2}\Delta$

$$h(X_t(x)) = h(x) + \int_0^t Lh(X_s(x))\,ds + \mathrm{mart}_t$$

$\Rightarrow P_t h(x) = h(x) + \int_0^t P_s Lh(x)\,ds$ (Dynkin formula)

$\Rightarrow \frac{d}{dt}\Big|_{t=0} P_t h(x) = Lh(x)$, $L = \frac{1}{2}\Delta$

Hence, $P_t |\nabla f|^2 = |\nabla f|^2 + t \underbrace{\frac{d}{dt}\Big|_{t=0} P_t |\nabla f|^2}_{=\frac{1}{2}\Delta |\nabla f|^2} + o(t)$

$$|\nabla P_t f|^2 = |\nabla f|^2 + t \underbrace{\frac{d}{dt}\Big|_{t=0} \langle \nabla P_t f, \nabla P_t f \rangle}_{= 2\langle \nabla \frac{d}{dt}|_{t=0} P_t f, \nabla f \rangle = 2\langle \nabla Lf, \nabla f \rangle = \langle \nabla \Delta f, \nabla f \rangle} + o(t)$$

$\Rightarrow \lim_{t \downarrow 0} \dfrac{P_t |\nabla f|^2(x) - |\nabla P_t f|^2(x)}{t} = \frac{1}{2}\Delta |\nabla f|^2 - \langle \nabla Lf, \nabla f \rangle = \underbrace{|\nabla df(x)|^2}_{=0} + \mathrm{Ric}(\underbrace{\nabla f}_{X}, \underbrace{\nabla f}_{X})$, $|X| = 1$

(3) Suppose that

$$|\nabla P_t f|^2 \leq \mathbb{E}\left[\exp\left(-\int_0^t K(X_s)\,ds\right) |\nabla f|^2(X_t)\right]$$

$$\Longrightarrow \quad \frac{P_t |\nabla f|^2(x) - |\nabla P_t f|^2(x)}{t} \geq \mathbb{E}\left[\frac{\left(1-\exp\left(-\int_0^t K(X_s(x))\,ds\right)\right)}{t} |\nabla f|^2(X_t(x))\right]$$

$\uparrow$

! $v \in T_x M$, $\nabla f(x) = v$, $|v| = 1$

$\mathrm{Hess}(f)(x) = 0$

$$\underset{t \downarrow 0}{\Longrightarrow} \quad \mathrm{Ric}_x(v,v) \geq K(x), \quad \text{i.e.} \quad \underline{\mathrm{Ric}}(x) \geq K(x)$$

Fermionic calculus on the exterior algebra and the Weitzenböck decomposition

$(V, \langle , \rangle)$ Euclidean vector space ($\dim V = n$)

and $\Lambda V = \bigoplus_{p \geq 0} \Lambda^p V$ the exterior algebra with the induced inner product :

on $\Lambda^p V$: $\langle v_1 \wedge \ldots \wedge v_p , w_1, \ldots, w_p \rangle = \det(\langle v_i, w_j \rangle)_{1 \leq i,j \leq p}$

Identify V et V^* via the metric : $\Lambda V^* = \bigoplus_{p \geq 0} \Lambda^p V^*$

Notation Let $v \in V$

C_v linear operation on ΛV^* : $C_v \alpha = (v, \cdot) \wedge \alpha$

creation (exterior product)

$A_v := C_v^*$ annihilation (interior product)

Note that

$C : V^* \otimes \Lambda V^* \longrightarrow \Lambda V^*$, $v \otimes w \longmapsto v \wedge w$

$A : V^* \otimes \Lambda V^* \longrightarrow \Lambda V^*$, $v \otimes w \longmapsto v \lrcorner w$

Remark $A_v(v_1 \wedge \ldots \wedge v_p) = \sum_{j=1}^{p} (-1)^{j+1} \langle v_j, v \rangle \, v_1 \wedge \ldots \underset{\uparrow \text{ omission}}{\hat{v}_j} \wedge \ldots \wedge v_p$

Indeed : Calculate the determinant

$A_v(v_1 \wedge \ldots \wedge v_p)(w_1 \wedge \ldots \wedge w_{p+1})$

by the Laplace expansion formula

Remark $\mathrm{Alt}\, T^*M = \Lambda T^*M = \bigoplus_{p \geq 0} \Lambda^p T^*M$

$A^p(M) = \Gamma(\Lambda^p T^*M)$ & $A^\cdot(M) = \Gamma(\Lambda T^*M)$

$d : A^\cdot(M) \longrightarrow A^\cdot(M)$ exterior differential & adjoint w/r to $\langle , \rangle_{L^2}$

the codifferential

$d^* : A^\cdot(M) \longrightarrow A^\cdot(M)$

We have

$d : A^p(M) \longrightarrow A^{p+1}(M)$, $\alpha \longmapsto d\alpha$

$d^* : A^p(M) \longrightarrow A^{p-1}(M)$, $\alpha \longrightarrow d^*\alpha$

Classical formulas:

$$d\alpha(X_1,\dots,X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}(\nabla_{X_i}\alpha)(X_1,\dots,\hat{X}_i,\dots,X_{p+1})$$

$$d^*\alpha(X_1,\dots,X_{p-1}) = -\sum_{i=1}^{n}\nabla_{e_i}\alpha(e_i,X_1,\dots,X_{p-1})$$

where (e_i) is an onb for T_xM, $X_1,\dots,X_{p+1}\in T_xM$

Proposition

$$d:\ \Gamma(\Lambda T^*M)\xrightarrow{\nabla}\Gamma(T^*M\otimes\Lambda T^*M)\xrightarrow{C}\Gamma(\Lambda T^*M)$$

$$d^*:\ \Gamma(\Lambda T^*M)\xrightarrow{\nabla}\Gamma(T^*M\otimes\Lambda T^*M)\xrightarrow{-A}\Gamma(\Lambda T^*M)$$

Note that if $\alpha\in\Gamma(T^*M\otimes\Lambda T^*M)$ then

$$\alpha(x)\in T_x^*M\otimes\Lambda T_x^*M\cong \mathrm{Hom}(T_xM,\Lambda T_x^*M),$$

hence $\alpha=\sum_{i=1}^n \underbrace{\langle e_i,\cdot\rangle}_{=:\varepsilon^i}\otimes\alpha(e_i)$ with (e_i) onb for T_xM

Hence

$$d=C\nabla=\sum_{i=1}^n C_{e_i}\nabla_{e_i}$$

$$d^*=-A\nabla=-\sum_{i=1}^n A_{e_i}\nabla_{e_i}$$

where $A_{e_i}=A(\varepsilon^i\otimes\cdot)=:a^i$

$C_{e_i}=C(\varepsilon^i\otimes\cdot)=:(a^i)^*$

Pr (1) $\displaystyle\sum_{i=1}^n (a^i)^*\nabla_{e_i}\alpha(X_1,\dots,X_{p+1}) = \sum_{i=1}^n \varepsilon^i\wedge\nabla_{e_i}\alpha(X_1,\dots,X_{p+1})$

$$=\sum_{i=1}^n\sum_{k=1}^{p+1}(-1)^{k+1}\underbrace{\varepsilon^i(X_k)}_{\langle e_i,X_k\rangle}(\nabla_{e_i}\alpha)(X_1,\dots,\hat{X}_k,\dots,X_{p+1})$$

$$=\sum_{k=1}^{p+1}(-1)^{k+1}\nabla_{X_k}\alpha(X_1,\dots,\hat{X}_k,\dots,X_{p+1}) = d\alpha(X_1,\dots,X_{p+1})$$

(2) On the other hand,

$$\langle C\nabla\alpha,\beta\rangle_{L^2(\Lambda^pT^*M)} = -\langle\alpha,A\nabla\beta\rangle_{L^2(\Lambda T^*M)}\quad\forall\ \alpha\in A^p(M)\ \&\ \beta\in A^{p+1}(M),$$

α or β of cpt. support

<u>Pr</u> Define $X \in \Gamma(TM)$ by

$$\langle C_v \alpha, \beta \rangle_{\Lambda^p T^* M} = \langle X, v \rangle_{TM}, \quad v \in TM$$

Then $\forall$ orthonormal local frame (e_i) on M

$$e_i \langle X, e_i \rangle = e_i \langle C_{e_i} \alpha, \beta \rangle$$
$$= \langle \nabla_{e_i}(C_{e_i}\alpha), \beta \rangle + \langle C_{e_i}\alpha, \nabla_{e_i}\beta \rangle$$
$$= \langle C_{\nabla_{e_i} e_i}\alpha, \beta \rangle + \langle C_{e_i}\nabla_{e_i}\alpha, \beta \rangle + \langle \alpha, \underbrace{C^*_{e_i}}_{A_{e_i}} \nabla_{e_i}\beta \rangle$$

$$\Rightarrow \operatorname{div} X = \sum_{i=1}^n \langle \nabla_{e_i} X, e_i \rangle_{TM}$$
$$= \sum_{i=1}^n e_i \langle X, e_i \rangle_{TM} - \langle X, \nabla_{e_i} e_i \rangle_{TM}$$
$$= \sum_{i=1}^n \langle C_{e_i} \nabla_{e_i}\alpha, \beta \rangle + \langle \alpha, A_{e_i} \nabla_{e_i} \beta \rangle$$
$$= \langle C\nabla\alpha, \beta \rangle_{\Lambda^p T^* M} + \langle \alpha, A\nabla\beta \rangle_{\Lambda^p T^* M}$$

$\Rightarrow$ claim, since $\int_M \operatorname{div} X \, d\mathrm{vol} = 0$

<u>Weitzenböck decomposition</u>

For $\alpha \in A^p(M)$ calculate

$$-(dd^*\alpha + d^*d\alpha) = \Box\alpha - \mathcal{R}$$

(e_i) onb of T_xM & (ε^i) corresponding dual basis for T_x^*M

$a^i := A_{e_i} =: a^i$

$(a^i)^* = C_{e_i} = (a^i)^*$

Know. $(d\alpha)_x = \sum_{i=1}^n (a^i)^* \nabla_{e_i}\alpha$

$(d^*\alpha)_x = \sum_{i=1}^n a^i \nabla_{e_i}\alpha$

∇ is compatible with C & A: $\nabla C = C\nabla$ & $\nabla A = A\nabla$

$$\Rightarrow (dd^*\alpha)_x = -\sum_{j,k} (a^j)^* \nabla_{e_j} a^k \nabla_{e_k}\alpha = -\sum_{j,k} (a^j)^* a^k (\nabla_{e_j}\nabla_{e_k}\alpha)_x$$

$$(d^*d\alpha)_x = -\sum_{j,k} a^k (a^j)^* (\nabla_{e_k}\nabla_{e_j}\alpha)_x$$

$$\Rightarrow -(dd^*\alpha + d^*d\alpha)_x = \sum_{j,k} (a^j)^* a^k (\nabla_{e_j}\nabla_{e_k}\alpha)_x + \sum_{j,k} a^k (a^j)^* (\nabla_{e_j}\nabla_{e_k}\alpha)_x$$

$$= \sum_{j,k} \underbrace{\{(a^j)^*, a^k\}}_{=\delta_{ij}} (\nabla_{e_j}\nabla_{e_k}\alpha)_x + \sum_{j,k} a^k (a^j)^* \left([\nabla_{e_k}, \nabla_{e_j}]\alpha\right)_x$$

$$= \underbrace{\mathrm{trace}\,(\nabla^2\alpha)_x}_{=(\Box\alpha)_x} + \sum_{j,k} a^k (a^j)^* R(e_k, e_j)\alpha \quad \text{since } [e_k, e_j] = 0$$

Note that $R(e_k, e_l) = [\nabla_{e_k}, \nabla_{e_l}] - \nabla_{[e_k, e_l]} \in \mathrm{Hom}(T_xM)$.

How to describe the action of $R(e_k, e_k)$ on $A^p(M)$?

<u>Def.</u> $V = T_xM$ & $(e_1, \dots, e_n)$ onb of V $\rightsquigarrow$ $(\varepsilon_1, \dots, \varepsilon_n)$ dual basis to $(e_1, \dots, e_n)$

$\rightsquigarrow a_i = a(e_i)$

For $A \in \mathrm{End}(V)$ consider

$$DA \in \mathrm{End}(\Lambda V) \quad DA = \bigoplus_{p \geq 0} D^p A \quad \text{where}$$

$$D^pA(v_1 \wedge \dots \wedge v_p) = \sum_{i=1}^{p} v_1 \wedge \dots \wedge Av_i \wedge \dots \wedge v_p$$

(By convention $\Lambda^0 A = \mathrm{id}$ on $\Lambda^0 V = \mathbb{R}$)

<u>Claim</u> $DA = \sum_{i,j} A_{ij} a_i^* a_j$ where $A_{ij} = \langle Ae_j, e_i \rangle$

Let $B = \sum_{i,j} A_{ij} a_i^* a_j$. To show: i) $B|_{\Lambda^1 V} = A$

ii) $B(u \wedge v) = (Bu) \wedge v + u \wedge (Bv)$, $u, v \in \Lambda V$

i) clear: $Be_k = \sum_{i,j} A_{ij} a_i^* a_j \underbrace{a_k^* 1}_{e_k} = \sum_i A_{ik} a_i^* 1 = \sum_i A_{ik} e_i = Ae_k$

ii) It's enough to check ii) for $B = a_i^* a_j$ (using linearity)

In the same way, for $A \in \mathrm{End}(V)$ (& $A^* \in \mathrm{End}(V^*)$)

consider

$$A^\wedge := DA^* \in \mathrm{End}(\Lambda V^*) = \mathrm{End}(\mathrm{Alt}\, V)$$

Thus $A^\wedge = \sum_{i,j} A^*_{ij} (a^i)^* a^j$ where $(a^i)^* = a(\varepsilon^i)^*$, $a^j = a(\varepsilon^j)$

$A^*_{ij} = \langle Ae_i, A_j \rangle$

$R : \Gamma(TM\otimes TM) \longrightarrow \mathrm{Hom}_{C^\infty(M)}(\Gamma TM, \Gamma TM)$ $\quad$ ($C^\infty(M)$-linear)

$X\otimes Y \longmapsto R(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]}$

$R_x(X\otimes Y) \equiv R(X(x)\otimes Y(x)) \in \mathrm{End}\, T_xM$

$R(X(x)\otimes Y(x))(Z(x)) = (R(X,Y)Z)_x \qquad (X,Y,Z\in\Gamma(TM))$

$R(X,Y)$ operates on $A^\cdot(M)$:

$R(X,Y)\in \mathrm{Hom}_{C^\infty(M)}(A^\cdot(M), A^\cdot(M))$

Claim $(R(X\otimes Y)\alpha)(x) = -R(X(x)\otimes Y(x))^\wedge \alpha(x)$ where

$$-R(X(x)\otimes Y(x))^\wedge\alpha(x) = -\sum_{k,l}\langle R(X(x),Y(x))e_k, e_l\rangle (a^k)^* a^l$$

(e_i) onb for T_xM

Pr $\nabla_X(\alpha\wedge\beta) = (\nabla_X\alpha)\wedge\beta + \alpha\wedge(\nabla_X\beta)$

$\Rightarrow \nabla_X\nabla_Y(\alpha\wedge\beta) = (\nabla_Y\nabla_X\alpha)\wedge\beta + (\nabla_Y\alpha)\wedge(\nabla_X\beta) + (\nabla_X\alpha)\wedge(\nabla_Y\beta) + \alpha\wedge(\nabla_X\nabla_Y\beta)$

$\Rightarrow \nabla_X\nabla_Y(\alpha\wedge\beta) = [\nabla_X,\nabla_Y]\alpha\wedge\beta + \alpha\wedge([\nabla_X,\nabla_Y]\beta)$

i.e. $R(X\otimes Y)$ operates on $A^p(M)$ as $D^p\underbrace{R(X\otimes Y)}_{\text{operating on } A^1(M)}$

Hence it is enough to show the claim for $A^1(M)$.

Fix $x\in M \leadsto (e_i)_{i=1,\dots,n}$ on vectorfields defined locally about x

$\leadsto (\varepsilon^i)_{i=1,\dots,n}$ dual frame

Then $Y(\alpha(Z)) = (\nabla_Y\alpha)(Z) + \alpha(\nabla_Y Z) \qquad (\alpha\in A^1(M)\ \&\ Z\in\Gamma(TM))$

$\Rightarrow XY(\alpha(Z)) = (\nabla_X\nabla_Y\alpha)(Z) + (\nabla_Y\alpha)(\nabla_X Z) + (\nabla_X\alpha)(\nabla_Y Z) + \alpha(\nabla_X\nabla_Y Z)$

$\Rightarrow XY(\alpha(Z)) = ([\nabla_X,\nabla_Y]\alpha)(Z) + \alpha([\nabla_X,\nabla_Y]Z)$

Hence, with $\alpha=\varepsilon^k$, $Z=e_l$, we have locally about x,

$$\underbrace{([\nabla_X,\nabla_Y]\varepsilon^k)(e_l)}_{\langle[\nabla_X,\nabla_Y]\varepsilon^k,\,\varepsilon^l\rangle} = -\underbrace{\varepsilon^k([\nabla_X,\nabla_Y]e_l)}_{=\langle[\nabla_X,\nabla_Y]e_l,\,e_k\rangle}$$

also $\langle\nabla_{[X,Y]}\varepsilon^k,\varepsilon^l\rangle = -\langle\nabla_{[X,Y]}e_l,e_k\rangle$

Hence $\langle R(X,Y)\varepsilon^k,\varepsilon^l\rangle = -\langle R(X,Y)e_l,e_k\rangle$ ∎

Cor (Weitzenböck formula)

$\Delta = -\{d, d^*\}$ on $A^\cdot(M)$

$\Rightarrow \Delta = \square - K$ where $\square = \operatorname{trace} \nabla^2$ & $K \in \Gamma(\operatorname{End} \Lambda T^*M)$

Know: $K = \sum_{i,j} a^i (a^j)^* R(e_i, e_j)^\Lambda$

$$= \sum_{i,j,k,l} a^i (a^j)^* \langle R(e_i, e_j) e_k, e_l \rangle (a^k)^* a^l$$

$$\overset{\uparrow}{=} - \sum_{i,j,k,l} \underbrace{\langle R(e_i, e_j) e_k, e_l \rangle}_{= -\langle R(e_j, e_i) e_k, e_l \rangle} (a^j)^* a^i (a^k)^* a^l$$

$\{a^i, (a^j)^*\} = \delta_{ij}\, \mathrm{id}$
& $R(e_i, e_i) = 0$

$$= \sum_{i,j,k,l} \langle R(e_i, e_j) e_k, e_l \rangle (a^i)^* a^j (a^k)^* a^l$$

Special case $p = 1$

$\alpha \in A^1(M) \Rightarrow \Delta\alpha = \underbrace{\operatorname{trace} \nabla^2 \alpha}_{\square\alpha} - \operatorname{Ric}(\cdot, \alpha^\#)$

Indeed: $K\alpha = \sum_{i,j} a^i (a^j)^* R(e_i, e_j)^* \alpha$

$$= \sum_{i,j} a^i \varepsilon^j \wedge (\alpha \circ R(e_i, e_j)(\cdot))$$

$$= \sum_{i,j} \underbrace{\langle \varepsilon^i, \varepsilon^j \rangle}_{= \delta_{ij}} \alpha \circ \underbrace{R(e_i, e_j)}_{=0 \text{ for } i=j}(\cdot) - \sum_{i,j} \langle \alpha \circ R(e_i, e_j)(\cdot), \varepsilon^i \rangle \varepsilon^j$$

$\Rightarrow$ for $v \in T_xM$,

$$(K\alpha)(v) = -\sum_{i,j} (\alpha \circ R(e_i, e_j))(e_i)\, \varepsilon^j(v)$$

$$= -\sum_i (\alpha \circ R(e_i, v))(e_i)$$

$$= -\sum_i \langle R(e_i, v) e_i, \alpha^\# \rangle$$

$$= \sum_i \langle R(e_i, v) \alpha^\#, e^i \rangle = \operatorname{Ric}(v, \alpha^\#).$$

Heat equation approach to index theorems

General setting: Let L, P, Q be linear operators on a $\mathbb{K}$-Hilbert space H;

L, Q s.a., P bounded & s.a. $\Big\}$ determined on a common core $\mathring{H}$ s.th. $L|\mathring{H}$ and $Q|\mathring{H}$ are essentially s.a.

L, P, Q define a <u>supersymmetry</u> on H if $L = -Q^2$, $P^2 = \mathrm{id}$ & $\{Q,P\} = QP + PQ = 0$

$$H = \underbrace{\{h \in H : Ph = h\}}_{= H_b} \oplus \underbrace{\{h \in H : Ph = -h\}}_{= H_f}$$

<u>bosonic</u> / <u>fermionic states</u>

$$P = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & -\mathrm{id} \end{pmatrix} : H_b \oplus H_f \longrightarrow H_b \oplus H_f$$

$$Q = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix} \qquad \text{where } B = Q|H_b$$

$$-L = \begin{pmatrix} B^*B & 0 \\ 0 & BB^* \end{pmatrix} \qquad (\text{in particular } H_{b/f} \text{ } L\text{-invariant})$$

<u>Def</u>. $\mathrm{ind}_s(L) \equiv \mathrm{ind}_s(L,P) := \dim\ker L|H_b - \dim\ker L|H_f$ (if well-defined) is called the <u>supersymmetric index</u> of L

<u>Note</u> $\mathrm{ind}_s(L) = \mathrm{ind}(B) := \dim\ker B - \dim\ker B^*$ (if well-defined)

<u>Ex</u> Let M be a compact Riem mf

$(A^{\cdot}(M), d)$ the de Rham complex; $A^{\cdot}(M) = \Gamma(\Lambda T^*M)$

$d : A^{\cdot}(M) \longrightarrow A^{\cdot+1}(M)$ exterior differential

$d^* : A^{\cdot}(M) \longrightarrow A^{\cdot-1}(M)$ codifferential (w/r to $\langle\alpha,\beta\rangle_{L^2} = \int_M \langle\alpha,\beta\rangle \, d\,\mathrm{vol}$)

$\Delta = -\{d, d^*\} = -(d+d^*)^2$

$\tau = (-1)^{\cdot} : A^{\cdot}(M) \to A^{\cdot}(M)$, $\tau\alpha = (-1)^p\alpha$ for $\alpha \in A^p(M)$

defines decomposition

$$A^{\cdot}(M) = \Gamma(\Lambda^e T^*M) \oplus \Gamma(\Lambda^o T^*M) \qquad \text{where } \Lambda^e T^*M = \bigoplus_k \Lambda^{2k} T^*M$$

$$\Lambda^o T^*M = \bigoplus_k \Lambda^{2k+1} T^*M$$

$$(d+d^*)_{e/o} : \Gamma(\Lambda^{e/o} T^*M) \longrightarrow \Gamma(\Lambda^{o/e} T^*M)$$

$\bar{A}^{\cdot}(M) = L^2(\Lambda T^* M)$

$\bar{d}$ closure of d

$\bar{d}^*$ adjoint of d

$\bar{\Delta}$ closure of $-(dd^* + d^* d)$

(write again $d, d^*, \Delta, \ldots$)

Then: $H = \bar{A}^{\cdot}(M)$

$Q = d + d^*$

$P = (-1)^{\cdot}$, $L = \Delta = -Q^2$

$(\Rightarrow P^2 = id$ & $\{P, Q\} = 0$

Well-known facts

① (Elliptic regularity)

$$\mathcal{H}^{\cdot}(M) \equiv \{\alpha \in \bar{A}^{\cdot}(M) : \Delta\alpha = 0\} = \{\alpha \in A^{\cdot}(M) : \underbrace{\Delta\alpha = 0}_{\text{ie } d\alpha = 0 \ \& \ d^*\alpha = 0}\}$$

② (Hodge)

$$H^p(M; \mathbb{R}) = H^p_{DR}(M) = \mathcal{H}^p(M)$$

$$\Rightarrow \operatorname{ind}_S(\Delta) = \operatorname{ind}_S(\Delta, \tau)$$

$$= \dim \ker \Delta|_{\Gamma(\Lambda^e T^* M)} - \dim \ker \Delta|_{\Gamma(\Lambda^o T^* M)}$$

$$= \sum_{p=0}^{n} (-1)^p \dim \underbrace{\ker \Delta|_{A^p(M)}}_{= \mathcal{H}^p(M)}$$

$$= \sum_{p=0}^{n} (-1)^p \underbrace{\dim H^p(M; \mathbb{R})}_{:= b_p(M)} = \chi(M)$$

Euler - Poincaré characteristic

Note $\operatorname{ind}_S(\Delta) = \operatorname{ind}(d + d^*)_e$

Ex M oriented & $2|n$ $(n = \dim M)$

$A^\cdot(M;\mathbb{C}) := \Gamma(\Lambda T^*M \otimes \mathbb{C})$ $\mathbb{C}$-valued differential forms

σ canonical involution on $A^\cdot(M;\mathbb{C})$

$$\sigma|_{A^p(M;\mathbb{C})} = i^{p(p-1)+\frac{n}{2}} *_p \quad , \quad \sigma^2 = \mathrm{id}$$

where $*_p : A^p(M;\mathbb{C}) \to A^{n-p}(M;\mathbb{C})$ defined by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle_{\Lambda^p T^*M \otimes \mathbb{C}} \, \omega_M \quad , \quad \alpha, \beta \in A^p(M;\mathbb{C})$$

ω_M canonical volume form

σ gives a decomposition

$$A^\cdot(M;\mathbb{C}) = \Gamma(\Lambda^+ T^*M \otimes \mathbb{C}) \oplus \Gamma(\Lambda^- T^*M \otimes \mathbb{C});$$

$$(d+d^*)_{+/-} : \Gamma(\Lambda^{+/-} T^*M \otimes \mathbb{C}) \to \Gamma(\Lambda^{-/+} T^*M \otimes \mathbb{C})$$

where $\Gamma(\Lambda^{+/-} T^*M \otimes \mathbb{C}) = \{ \alpha \in A^\cdot(M;\mathbb{C}) : \sigma\alpha = {+/-}\alpha \}$

Let $H = \bar{A}^\cdot(M;\mathbb{C}) \equiv L^2(\Lambda T^*M \otimes \mathbb{C})$

$Q = d + d^*$; $P = \sigma$

Then $\mathrm{ind}_S(\Delta, \sigma) = \dim \{+1 \text{ eigenspace of } \sigma \text{ on } \mathcal{H}^\cdot(M;\mathbb{C})\}$
$\qquad - \dim \{-1 \quad \text{—— " ——} \}$

$\qquad =: \text{signature}(M)$

Note $\mathrm{ind}_S(\Delta, \sigma) = \mathrm{ind}(d+d^*)_+$

Remark $n = 2 \pmod 4$: signature$(M) = 0$
$n = 0 \pmod 4$: signature$(M) = \dim \{+1 \text{ eigenspace of } * \text{ on } \mathcal{H}^{n/2}(M;\mathbb{C})\}$
$\qquad - \dim\{-1 \quad \text{—— " ——}\}$

<u>Situation</u>: $H = L^2(E)$ where $E = \Lambda T^*M$, $\Lambda T^*M \otimes \mathbb{C}$, .. (M compact)

$L = \Delta : \Gamma(E) \to \Gamma(E)$ elliptic, formally s.a. and ≤ 0

<u>Elliptic theory</u>: $L^2(E)$ has an o.n.b. $(\varphi_i)_{i\in\mathbb{N}}$ of $\mathcal{C}^\infty$-eigensections of L and all eigenspaces are finite-dim.

Same for $L|H_{b/g}$ where $H_{b/g} = L^2(E_{b/g})$

Consider the heat equation on $L^2(E)$:

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}L\right)u = 0 \quad \text{where } u|_{t=0} = \alpha$$

Then: $u(t,x) = (\underbrace{e^{\frac{1}{2}tL}}_{\text{integral operator}}\alpha)(x) = \int P_t(x,y)\alpha(y)\,\mathrm{vol}(dy)$

integral operator with $\mathcal{C}^\infty$-kernel P_t: $P_t(x,y) \in \mathrm{Hom}(E_y, E_x)$

$$P_t(x,y) = \sum_{i=0}^\infty e^{\frac{1}{2}t\lambda_i}\Big(\sum_{\substack{\phi \text{ onb}\\ \text{in } H_{\lambda_i}(L)}} \phi(x)\otimes\phi(y)\Big)$$

where $0 = \lambda_0 > \lambda_1 > \lambda_2 > \dots$ are the eigenvalues of L & $H_{\lambda_i}(L)$ the corresponding eigenspaces

For $C \in \mathrm{Hom}_{\mathcal{C}^\infty(M)}\Gamma(E)$ which leaves $\Gamma(E_{b/g})$ invariant

define

$$\mathrm{str}\, C := \mathrm{trace}(P\circ C) : M \to \mathbb{R}$$
$$x \mapsto \mathrm{trace}\, C|_{E_{b,x}} - \mathrm{trace}\, C|_{E_{g,x}}$$

$\Rightarrow \forall t > 0$

$$\int \mathrm{str}\, P_t(x,x)\,\mathrm{vol}(dx) = \int\left[\mathrm{trace}\, P_t(x,x)|_{E_{b,x}} - \mathrm{trace}\, P_t(x,x)|_{E_{g,x}}\right]\mathrm{vol}(dx)$$

$$= \sum_{i=0}^\infty\Bigg[e^{\frac{1}{2}t\lambda_i}\sum_{\substack{\phi:\text{ onb of}\\ H_{\lambda_i}(L|H_b)}}\|\phi\|^2_{L^2} - e^{\frac{1}{2}t\mu_i}\sum_{\substack{\phi:\text{ onb of}\\ H_{\mu_i}(L|H_g)}}\|\phi\|^2_{L^2}\Bigg]$$

$$= \sum_{i=0}^\infty\left(e^{\frac{1}{2}t\lambda_i}\dim H_{\lambda_i}(L|H_b) - e^{\frac{1}{2}t\mu_i}\dim H_{\mu_i}(L|H_g)\right)$$

$$= \dim\ker(L|H_b) - \dim\ker(L|H_g) = \mathrm{ind}_S L$$

↑ for $\lambda_i \neq 0$, Q maps $H_{\lambda_i}(L|H_b)$ isomorphically to $H_{\lambda_i}(L|H_g)$; in particular, same eigenvalues & all terms except the ones with $\lambda_0 = \mu_0 = 0$ cancel out

Result: $\underbrace{\int \mathrm{str}\, P_t(x,x)\, \mathrm{vol}(dx)}_{=\mathrm{ind}_S L} \in \mathbb{Z}$ (indep. of t)

Idea: Calculate

$$\mathrm{ind}_S L \text{ as } \lim_{t\downarrow 0} \int \mathrm{str}\, P_t(x,x)\, \mathrm{vol}(dx) = \int \lim_{t\downarrow 0} \mathrm{str}\, P_t(x,x)\, \mathrm{vol}(dx)$$

Show: $\mathrm{str}\, P_t(x,x) = C(x) + o(t)$ as $t\downarrow 0$ uniformly in x,

then $\mathrm{ind}_S L = \int C(x)\, \mathrm{vol}(dx)$

In the examples:

for $\mathrm{ind}_S(\Delta,\tau) = \mathrm{ind}\,(d+d^*)_e$: Theorem of Gauß-Bonnet-Chern

for $\mathrm{ind}_S(\Delta,\sigma) = \mathrm{ind}\,(d+d^*)_+$: Hirzebruch's signature theorem

Gauß-Bonnet-Chern

M cpt oriented Riem mf of even dimension ($n=2m$)

$$\chi(M) = \sum_{p=0}^{n} (-1)^p \dim H^p(M) = \int_M \underbrace{e(\Omega,\dots,\Omega)}_{\text{Euler form } (n\text{-form})}$$

Classical Thm of Gauß-Bonnet ($n=2$)

$$\chi(M) = 2-2g = \int E(x)\, \mathrm{vol}(dx) \quad \text{where } E = \frac{K}{4\pi} \quad (K \text{ scalar curvature of } M)$$

& $g :=$ genus of M

Know already: $\chi(M) = \int_M C(x)\, \mathrm{vol}(dx)$ with $\underbrace{C(x)}$ as above

polynomial expression in the components of the Riemann tensor

To show: $C(x)\, \mathrm{vol}(dx) = e(\Omega,\dots,\Omega)$

A divergence formula and integrated logarithmic gradient estimates for the heat kernel

(M,g) Riem. mf & X $BM(M,g)$

$$\mathbb{P}\{X_t(x)\in A\} = \int \underbrace{p(t,x,y)}_{\mathcal{C}^\infty\text{- transition kernel}} 1_A(y)\, \mathrm{vol}(dy)$$

Lemma M cpt mf

For each $q\in[1,\infty[$ $\exists\, C_q<0$ s. th

$$\underbrace{\left(\int_M \left|\nabla_z \log p(t,x,z)\right|^q p(t,x,z)\, \mathrm{vol}(dz)\right)^{1/q}}_{=\mathbb{E}\left[\left|\nabla_z \log p(t,X_t(x),z)\right|^q\right]^{1/q}} \leq C_q\, e^{K_- t}\frac{1}{\sqrt{t}}$$

where K is a lower bound for Ricci, i.e. $\mathrm{Ric}^M \geq K$, $K_- = :-K\vee 0$

Notation heat flow on 1-forms

$$\begin{cases} \partial_t\alpha = \frac{1}{2}\Delta\alpha \\ \alpha|_{t=0} = \alpha_0 \in \Gamma(T^*M) \end{cases}$$

$F_\alpha : O(M) \longrightarrow \mathbb{R}^n$, $u \longmapsto u^{-1}\alpha_{\pi(u)}$

$\mathrm{div}\,\alpha = \mathrm{trace}(v\longmapsto \nabla_v\alpha) = \sum_{i=1}^n \langle\nabla_{e_i}\alpha, e_i\rangle$ ($e_1,\dots,e_n$ onb of T_xM)

Know: $N_t := Q_t\, /\!/_{0,t}^{-1}\,\alpha_{T-t}$ local martingale $(0\leq t\leq T)$

More precisely,

$$dN_t = \sum_i /\!/_{0,t}^{-1}\left(\nabla_{u_t e_i}\alpha_{T-t}\right) dZ^i, \quad Z = A(X)\ BM(T_xM)$$

anti-development of X

Observation $\mathrm{div}\,\alpha_{T-t}(X_t)$ $(0\leq t\leq T)$ local martingale

Pr
$$d\left(\mathrm{div}\,\alpha_{T-t}(X_t)\right) \overset{m}{=} \frac{1}{2}\Delta\, \mathrm{div}\,\alpha_{T-t}(X_t)dt + \partial_t\, \mathrm{div}\,\alpha_{T-t}(X_t)dt$$

$$\underset{\Delta\mathrm{div} = \mathrm{div}\,\Delta}{=} \mathrm{div}\Big(\underbrace{\tfrac{1}{2}\Delta\alpha_{T-t} + \partial_t\alpha_{T-t}}_{=0}\Big)(X_t)dt = 0$$

Laplacian on 1-forms

<u>Notation</u> We identify differential forms $\alpha \in \Gamma(T^*M)$ and vector fields $Y \in \Gamma(TM)$ via the metric

$$\alpha \longleftrightarrow \alpha^\#, \quad Y \longleftrightarrow Y^b$$

In particular, then

$$\operatorname{div} Y \triangleq \operatorname{div} \alpha, \ \alpha = Y^b$$

and $(\operatorname{div} Y)(x) = \operatorname{trace}\left(T_xM \xrightarrow{\nabla\alpha} T_x^*M \xrightarrow{\#} T_xM\right)$

Let $Y_t = \alpha_t^\#$. Then

$m_t := Q_t \, /\!/_t^{-1} Y_{T-t}$ local martingale where $\frac{d}{dt} Q_t = -\frac{1}{2} Q_t \operatorname{Ric}_{/\!/_t}$, $Q_0 = id_{T_xM}$

& $\operatorname{Ric}_{/\!/_t} = /\!/_t^{-1} \circ \operatorname{Ric}_{X_t} \circ /\!/_t \in \operatorname{End}(T_xM)$

<u>Proposition</u> (divergence formula)

M cpt Riem mf, $T > 0$ & $(\ell_t)_{0 \le t \le T}$ adapted real process with abs. cont. paths s.th. $\left(\int_0^T |\dot{\ell}_s|^2 ds\right)^{1/2} \in L^1$ & $\ell_0 = 0$, $\ell_T = 1$.

Then $\forall \ Y \in \Gamma(TM)$

$$E\left[(\operatorname{div} Y)(X_T)\right] = E\left[\left\langle Q_T \, /\!/_T^{-1} Y(X_T), \int_0^T \dot{\ell}_t (Q_t^{-1})^* dZ_t \right\rangle\right]$$

where $X_t = X_t(x)$ & $Z_t = A(X_t)$ anti-development of X_t in T_xM

<u>Pr</u> (B. Driver & A.Th. JFA 2001)

1. Recall that

$n_t := \operatorname{div} Y_{T-t}(X_t) \quad (0 \le t \le T)$ local martingale (1)

$\Rightarrow \tilde{n}_t := \underbrace{n_t \ell_t - \int n_s \, d\ell_s}_{= \int \ell_s \, dn_s + n_0 \ell_0}$ local martingale (2)

M cpt $\Rightarrow$ both (1) & (2) true martingales

(1) $\Rightarrow \operatorname{div} Y_T(x) = n_0 = E[n_T] = E[\operatorname{div} Y(X_T)]$

$$(2) \Rightarrow (\operatorname{div} Y_T)(x) = -\mathbb{E}\left[\int_0^T n_s \, d\ell_s\right]$$

$$= -\mathbb{E}\left[\int_0^T \operatorname{div} Y_{T-t}(X_t)\, d\ell_t\right]$$

$$= -\mathbb{E}\left[\sum_{i=1}^n \int_0^T \langle \nabla_{u_t e_i} Y_{T-t}, u_t e_i \rangle \, d\ell_t\right]$$

$$= -\mathbb{E}\left[\sum_{i=1}^n \int_0^T \langle /\!/_t^{-1} \nabla_{u_t e_i} Y_{T-t}, u_0 e_i \rangle \, d\ell_t\right]$$

$$= -\mathbb{E}\left[\sum_{i=1}^n \int_0^T \langle Q_t /\!/_t^{-1} \nabla_{u_t e_i} Y_{T-t}, (Q_t^{-1})^* u_0 e_i \rangle \underbrace{d\ell_t}_{=\dot{\ell}_t dt}\right]$$

Recall that $N_t = Q_t /\!/_t^{-1} Y_{T-t}(X_t)$ is a loc. martingale

and $dN_t = \sum_i /\!/_t^{-1}(\nabla_{u_t e_i} Y_{T-t})\, dz^i$, $z = A(X)$

$$\Rightarrow (\operatorname{div} Y_T)(x) = -\langle N_T, \int_0^T \dot{\ell}_t (Q_t^{-1})^* dz_t \rangle$$

$$= -\langle Q_T /\!/_T^{-1} Y(X_T), \int_0^T \dot{\ell}_t (Q_t^{-1})^* dz_t \rangle$$

<u>Cor</u> $\int_M |\nabla_z \log p(t,x,z)|^q \, p(t,x,z) \operatorname{vol}(dz) \leq C_q e^{2Kt} \frac{1}{\sqrt{t}}$

<u>Pr</u>. Without loss of generality $q \geq 2$.

$$\mathbb{E}\left[\operatorname{div} Y(X_T(x))\right] = \int_M (\operatorname{div} Y)(z)\, p(T,x,z) \operatorname{vol}(dz)$$

$$= -\int_M \langle Y(z), \log p(T,x,z) \rangle_{TM} \, p(T,x,z) \operatorname{vol}(dz)$$

$$\underset{\substack{\uparrow \\ \text{Proposition}}}{=} \mathbb{E}\left[\langle Q_T /\!/_T^{-1} Y(X_T), \int_0^T \dot{\ell}_t (Q_t^{-1})^* dz_t \rangle\right]$$

$$\Rightarrow \int_M \langle Y(z), \log p(T,x,z) \rangle \, p(T,x,z) \operatorname{vol}(dz)$$

$$\leq e^{-\frac{1}{2}KT} \mathbb{E}\left[|Y(X_T(x))| \left| \int_0^T \dot{\ell}_t (Q_t^{-1})^* dz_t \right|\right]$$

$$\underset{\text{Hölder}}{\leq} e^{-\frac{1}{2}KT} \left(\mathbb{E}|Y(X_T(x))|^p\right)^{1/p} \underbrace{\left(\mathbb{E}\left|\int_0^T \dot{\ell}_t (Q_t^{-1})^* dz_t\right|^q\right)^{1/q}}_{\leq C_q \left(\mathbb{E}\left|\int_0^T \operatorname{tr} Q_t^{-1} (Q_t^{-1})^* |\dot{\ell}_t|^2 dt\right|^{q/2}\right)^{1/q}}$$

$\frac{1}{p} + \frac{1}{q} = 1$ Hölder

Burkholder

$|Q_t^{-1}| \leq e^{\frac{1}{2}Kt} \leq e^{\frac{1}{2}KT}$

$$\leq C_q e^{\frac{1}{2}KT} \left(\mathbb{E}\left|\int_0^T |\dot{\ell}_t|^2 dt\right|^{q/2}\right)^{\frac{1}{q}}$$

$$\leq C_q e^{KT} \left(\mathbb{E} \left| Y(X_T(x)) \right|^p \right)^{1/p} \left(\mathbb{E} \left| \int_0^T |\dot{\ell}_t|^2 dt \right|^{q/2} \right)^{1/q}$$

Now let $\ell_t = \frac{t}{T}$

$$\Rightarrow \int_M \langle Y(z), \log p(T,x,z) \rangle \, p(T,x,z) \, \mathrm{vol}(dz)$$

$$\leq \frac{1}{\sqrt{T}} C_q e^{K_-T} \big(\underbrace{\mathbb{E} \left| Y(X_T(x)) \right|^p}_{= \int |Y(z)|^p p(T,x,z) \, \mathrm{vol}(dz)} \big)^{\frac{1}{p}}$$

Let $Y(z) = \left| \nabla_z \log p(T,x,z) \right|^{q-2} \nabla_z \log p(T,x,z)$

$$\Rightarrow \int_M \left| \nabla_z \log p(T,x,z) \right|^q p(T,x,z) \, \mathrm{vol}(dz)$$

$$\leq \frac{1}{\sqrt{T}} C_q e^{K_-T} \left(\int_M \left| \nabla_z \log p(T,x,z) \right|^{p(q-1)} p(T,x,z) \, \mathrm{vol}(dz) \right)^{1/p}$$

$$= \frac{1}{\sqrt{T}} C_q e^{K_-T} \left(\int_M \left| \nabla_z \log p(T,x,z) \right|^q p(T,x,z) \, \mathrm{vol}(dz) \right)^{1-\frac{1}{q}}$$

$\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow p = \frac{q}{q-1}$

Solving for $\int_M \left| \nabla_z \log p(T,x,z) \right|^q p(T,x,z) \, \mathrm{vol}(dz)$

gives the wanted estimate.

Brownian bridges and Brownian loops

(M,g) complete Riem mf, $\dim M = n$

(1) $X_\cdot(x)$ BM(M,g) starting from $x \in M$.

$$\mathbb{P}\{X_s(x) \in A\} = \int 1_A(z)\, \mathbb{P}\{X_s(x) \in dz\} = \int 1_A(z)\, p(s,x,z)\, \mathrm{vol}(dz)$$

(2) $X^t(x,y)$ Brownian bridge from x to y of lifetime $t>0$

$\triangleq$ BM X with $X_0 = x$ plus additional condition $X_t = y$

$$\mathbb{P}\{X^t_s(x,y) \in A\} = \int 1_A(z)\, \frac{p(s,x,z)\, p(t-s,z,y)}{p(t,x,y)}\, \mathrm{vol}(dz)$$

Heat kernel estimate: Fix $t>0$. $\exists$ constants $c_i, k_i > 0$ $(i=1,2)$

$$k_1 s^{-\frac{n}{2}} \exp\left(-c_1 \frac{d(x,y)^2}{s}\right) \leq p(s,x,y) \leq k_2 s^{-\frac{n}{2}} \exp\left(-c_2 \frac{d(x,y)^2}{s}\right) \quad \forall\, 0 < s \leq t$$

Construction of Br bridges

$(\mathscr{C}(\mathbb{R}_+, \mathbb{R}^n), \mathscr{F}, \mathbb{P})$ Wiener space

with $B_s = \mathrm{pr}_s : \mathscr{C}(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}^n$ standard BM, $s \geq 0$

X BM(M,g) with $X_0 = x \in M$

$X = \pi(U)$ where $dU = \sum_{i=1}^n L_i(U) \circ dB^i$, $U_0 = u \in \pi^{-1}x \in O(M)$

$\mathscr{F}^0_s = \sigma\{\mathrm{pr}_r : r \leq s\} \subset \mathscr{F}_s$

$h : [0,t[\times M \to]0,\infty[$ $\mathscr{C}^\infty$ s.th. $\left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta\right) h \equiv 0$

"h space-time harmonic"

e.g. $h(s,\cdot) = \dfrac{p(t-s,\cdot,y)}{p(t,x,y)}$

Lemma $Z_s = h(s, X_s(x))$, $0 \leq s < t$ positive martingale

& $\exists$ probability measure $\mathbb{P}^h$ on $(\mathscr{C}([0,t[, \mathbb{R}^n), \sigma\{\mathrm{pr}_s : s < t\})$

s.th. $\left.\dfrac{d\mathbb{P}^h}{d\mathbb{P}}\right|_{\mathscr{F}^0_s} = Z_s$ $\mathbb{P}$-a.s. $(0 \leq s < t)$

Cor. Let $h(s,\cdot) = \dfrac{p(t-s,\cdot,y)}{p(t,x,y)} \rightsquigarrow \mathbb{P}^t_{x,y} := \mathbb{P}^h$ on $\mathcal{C}([0,t[;\mathbb{R}^n)$ as above

(1) $\mathbb{P}^t_{x,y}\{X_s(x)\in A\} = \mathbb{E}\left[1_A(X_s(x))\, Z_s\right] = \mathbb{E}\left[1_A(X_s(x))\,\dfrac{p(t-s, X_s(x), y)}{p(t,x,y)}\right]$

(2) $X(x)|[0,t[$ $\mathbb{P}^t_{x,y}$-semimartingale

(3) $\left(\dfrac{1}{h(s,X_s(x))}\right)_{0\le s<t}$ positive $\mathbb{P}^t_{x,y}$-martingale

$\Rightarrow$ has a $\mathbb{P}^t_{x,y}$-a.s. limit as $s\uparrow t$

But $\dfrac{1}{p(t-s, X_s(x), y)} \geqslant \text{const}\,(t-s)^{-n/2}\exp\left(c\,\dfrac{d^2(X_s(x),y)}{t-s}\right)$

$\Rightarrow X_s(x) \to y$ as $s\uparrow t$ $\mathbb{P}^t_{x,y}$-a.s.

Notation $(X_s(x))_{0\le s\le t}$ w/r to $\mathbb{P}^t_{x,y}$ is called Brownian bridge on M ("pinned Brownian motion") from x to y with lifetime t

The law $\mu^t_{x,y}$ of $X(x)|[0,t]$ on $\{w\in\mathcal{C}([0,t],M) : w(0)=x,\ w(t)=y\}$ is called Brownian bridge measure

Remarks

(1) Br. bridge Y from x to y with lifetime t

$\hat{=}$ cont. Markov process $(Y_s)_{0\le s\le t}$ with kernel

$$\mathbb{P}\{Y_s\in dz\,|\,Y_0=x\} = \frac{p(s,x,z)\,p(t-s,z,y)}{p(t,x,y)}\,\mathrm{vol}(dz)$$

$\Rightarrow \hat{Y}_s := Y_{t-s}$ Br bridge from y to x of lifetime t

(2) $X(x)|[0,t[$ $\mathbb{P}^t_{x,y}$-semimartingale

$\Rightarrow (X_s(x))_{s<t}$ stoch. development of an $\mathbb{R}^n$-valued $\mathbb{P}^t_{x,y}$-semimartingale: namely of B ($\hat{=}$ BM$(\mathbb{R}^n)$ w/r to $\mathbb{P}$) considered as $\mathbb{P}^t_{x,y}$-semimartingale

Problem: Find $\mathbb{P}^t_{x,y}$-Doob-Meyer decomposition of $(B_s)_{s<t}$

Note that

$$Z_s = h(s, X_s(x)) \;, \quad 0 \le s < t \qquad \mathbb{P}\text{- martingale}$$

$$\underset{\text{Itô}}{\Rightarrow} dZ_s = \sum_i \langle \nabla h(s, X_s(x)), u_s e_i \rangle_{T_{X_s}M} \, dB^i \qquad \left(\text{where } \nabla h(s,z) = (\nabla h(s,\cdot))_z\right)$$

$$\Rightarrow dZ = Z \langle c, dB \rangle_{\mathbb{R}^n} \quad \text{with} \quad c_s^i := \frac{\langle \nabla h(s, X_s(x)), u e_i \rangle}{h(s, X_s(x))}$$

$$\Rightarrow c_s = \frac{u_s^{-1} \nabla h(s, X_s(x))}{h(s, X_s(x))}$$

$$\tilde{B}_s := B_s - \int_0^s c_r \, dr \qquad \mathbb{P}^t_{x,y} - BM(\mathbb{R}^n) \quad \text{on the interval } [0,t[$$

$$B_s = \underbrace{\tilde{B}_s}_{\text{has a limit for } s \uparrow t \ \mathbb{P}^t_{x,y}\text{-a.s}} + \underbrace{\int_0^s u_r^{-1} \nabla \log h(r, X_r(x)) \, dr}_{\substack{\text{converges absolutely for } s \uparrow t \ \mathbb{P}^t_{x,y}\text{-a.s} \\ \text{(by Lemma with } q=1)}} \qquad \mathbb{P}^t_{x,y}\text{-Doob-Meyer decomposition of } (B_s)_{s<t}$$

$$\Rightarrow B \quad \mathbb{P}^t_{x,y}\text{-semimartingale on } [0,t]$$

$$\Rightarrow X(x) \quad \mathbb{P}^t_{x,y}\text{-semimartingale on } [0,t] \quad \text{with anti-development}$$

$$A(X(x)) = BM(T_xM) + \int_0^{\cdot} /\!/_r^{-1} (\nabla \log h)(r, X_r(x)) \, dr$$

$$\text{where} \quad (\nabla \log h)(r, \cdot) = \frac{\nabla p(t-r, \cdot, y)}{p(t-r, \cdot, y)}$$

In particular,

$$dU = \sum_i L_i(U) \circ dB^i \quad , \quad U_0 = u$$

$$= \sum_i L_i(U) \circ \left(d\tilde{B}^i + \langle U^{-1}(\nabla \log h)(s, \pi(U_s)), e_i \rangle \, ds\right)$$

$$= \sum_i L_i(U) \circ d\tilde{B}^i + L_0(s, U_s)\, ds$$

$\tilde{B}|[0,t]$ $P^t_{x,y}$-BM$(\mathbb{R}^n)$ &

$L_0(s,\cdot) \in \Gamma(T\mathcal{O}(M))$ horizontal lift of the (time-dependent) vector field $\nabla \log h(s,\cdot) = \dfrac{\nabla p(t-s,\cdot,y)}{p(t-s,\cdot,y)} \in \Gamma(TM)$ $\quad (0 \leq s < t)$

Hence, let $(U_s)_{0 \leq s < t}$ be a solution to

$$(*) \quad dU = \sum_i L_i(U) * d\tilde{B} + L_0(s, U_s)\, ds \quad , \quad U_0 = u \in \pi^{-1}\{x\}$$

where $\tilde{B}$ is a BM$(\mathbb{R}^n)$, then

X: $X_s = \pi(U_s)$ $\quad (0 \leq s < t)$ is a Brownian bridge on M from x to y of lifetime t $\quad (\Rightarrow X_s \to y$ a.s. as $s \uparrow t)$

& $U_t := \lim_{s \uparrow t} U_s$ exists a.s. & $(U_s)_{0 \leq s \leq t}$ solves $(*)$

X flow process to $\frac{1}{2}\Delta_M + \nabla \log p(t-s,\cdot,y)$, more precisely,

$(s, X_s)_{0 \leq s \leq t}$ flow process to $\frac{\partial}{\partial s} + \frac{1}{2}\Delta_M + \nabla \log p(t-s,\cdot,y)$

Ex: $M = \mathbb{R}^n$

$$dX_s = dB_s + \frac{y - X_s}{t-s}\, ds \quad , \quad X_0 = x$$

Remark For $t>0$,

$y \mapsto P^t_{x,y}$ is a desintegration of $\mathbb{P}|_{\sigma\{X_s : s\le t\}}$ according to the values y of a BM(M,g) X at time t, i.e. $A\in\sigma\{X_s : s\le t\}$:

$$\mathbb{P}(A) = \int P^t_{x,y}(A)\,\underbrace{(\mathbb{P}\circ X_t^{-1})(dy)}_{=\,p(t,x,y)\,\mathrm{vol}(dy)}$$

(well-defined $\forall y$, not only for $\mathbb{P}\circ X_t^{-1}$-a.a. y)

Formally we write

$$P^t_{x,y}(\ldots) = \mathbb{E}\left[\ldots \mid X_t(x) = y\right]$$

Rescaling of Brownian bridges and asymptotics for small times

Rescaling of SDEs

Let $dX = \sum_i A_i(X) \circ dB^i + A_0(X)\,ds$, $X_0 = x$

and for $\varepsilon > 0$ let

$$dX^\varepsilon = \sqrt{\varepsilon} \sum_i A_i(X^\varepsilon) \circ dB^i + \varepsilon A_0^\varepsilon(X^\varepsilon)\,ds \quad , \quad X_0^\varepsilon = x$$

Then $\{X_{s\varepsilon} : s \geq 0\}$ & $\{X_s^\varepsilon : s \geq 0\}$ have the same law , in particular $X_\varepsilon \sim X_1^\varepsilon$

Indeed: $s \mapsto \sqrt{\varepsilon} B_s$ & $s \mapsto B_{s\varepsilon}$ are equivalent

Application (rescaling of Brownian bridges of different lifetimes to a unified lifetime)

$X \equiv$ Brownian bridge of lifetime $\varepsilon > 0$ & $X_0 = X_\varepsilon = x \in M$ a.s.
determined by

$$dU = \sum_i L_i(U) \circ dB^i + \underbrace{L_0(s,U)}_{\text{horiz. lift of } \nabla \log p(\varepsilon - s, \cdot, x) \text{ at } U}\,ds \quad , \quad U_0 = u_0 \in \pi^{-1}\{x\}$$

$X^\varepsilon \equiv (X_s^\varepsilon)_{0 \leq s \leq 1}$ defined by

$$dU^\varepsilon = \sqrt{\varepsilon} \sum_i L_i(U^\varepsilon) \circ dB^i + \underbrace{L_0^\varepsilon(s, U_s^\varepsilon)}_{\text{horiz. lift of } \varepsilon \nabla \log p(\varepsilon(1-s), \cdot, x) \text{ at } U_s^\varepsilon}\,ds \quad , \quad U_0^\varepsilon = u_0$$

Let $X_{s\varepsilon} := \pi(U_{s\varepsilon})$ & $X_s^\varepsilon = \pi(U_s^\varepsilon)$

Then $(X_{s\varepsilon})_{0 \leq s \leq 1}$ & $(X_s^\varepsilon)_{0 \leq s \leq 1}$ are equivalent

$\left(\Rightarrow (X^\varepsilon_{s/\varepsilon})_{0 \leq s \leq \varepsilon} \text{ Br. bridge from } x \text{ to } x \right)$

$$(*)\quad \begin{cases} dX^\varepsilon = \sqrt{\varepsilon}\, U^\varepsilon \circ dB + \varepsilon \nabla \log p(\varepsilon(1-s), X_s^\varepsilon, x)\,ds \\ X_0^\varepsilon = x \end{cases}$$

Study the 1st order part of $(*)$ as $\varepsilon \downarrow 0$

Excursion (Euler form)

$\pi : \underbrace{SO(M)}_{=P} \longrightarrow M$ G-principal bundle of oriented orthonormal frames over M

$G = SO(n)$, $\mathfrak{g} = so(n) = o(n) = \{ A \in gl(n;\mathbb{R}) : A + A^* = 0 \}$

$\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ connection 1-form

$\Omega \in \Gamma(\Lambda^2 T^*P \otimes \mathfrak{g})$ connection form

$$\Omega(X_1, X_2) = d\omega(\mathrm{hor}X_1, \mathrm{hor}X_2), \quad X_1, X_2 \in T_u P$$

Identify

$$so(n) \cong \Lambda^2 \underbrace{\mathbb{R}^n}_{(\mathbb{R}^n)^*} : \quad A = (a_{ij}) \longmapsto \frac{1}{2} \sum_{i,j} a_{ij}\, dx^i \wedge dx^j$$

($dx^i \mathrel{\hat{=}} \varepsilon^i$ standard dual basis)

$\boxed{n = 2m}$, $n = \dim M$

$$A \in so(n) : \quad \underbrace{A \wedge \dots \wedge A}_{m \text{ times}} = m!\, \underbrace{Pf(A)}_{=:\ \text{Pfaffian of } A}\, dx^1 \wedge \dots \wedge dx^n$$

Ex. $A = \text{block-diag}(d_1, \dots, d_m) = \begin{pmatrix} \begin{matrix} 0 & d_1 \\ -d_1 & 0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 0 & d_m \\ -d_m & 0 \end{matrix} \end{pmatrix}$

$$\Rightarrow Pf(A) = d_1 \cdot \ldots \cdot d_m$$

Conversely, to $\forall$ skew-symmetric matrix A $\exists\, g \in G = SO(n)$ s.th. gAg^{-1} is block-diagonal ; $Pf(gAg^{-1}) = Pf(A)$

In particular, $\det A = Pf(A)^2$

In explicit terms,

$$Pf(A) = \frac{1}{2^m m!} \sum_{\rho \in \mathfrak{S}_n} \mathrm{sign}(\rho)\, A_{\rho(1)\rho(2)} \cdots A_{\rho(n-1)\rho(n)}, \quad A \in so(n) = \mathfrak{g}$$

($\mathfrak{S}_n$ symmetric group of n elements)

Consider

$$e : \underbrace{\mathfrak{g} \times \dots \times \mathfrak{g}}_{m \text{ times}} \longrightarrow \mathbb{R}, \quad e(A, \dots, A) = (2\pi)^{-m} \, Pf(A)$$

e defines a symmetric multilinear G-invariant function

$$e(gA_1g^{-1}, \dots, gA_mg^{-1}) = e(A_1, \dots, A_m) \quad \forall g \in G$$

Def $$\underbrace{e(\Omega, \dots, \Omega)}_{m \text{ times}} = (2\pi)^{-m} \, Pf(\Omega) \in \Gamma(\Lambda^{2m} T^*P)$$

$e(\Omega^m)$ is horizontal (since Ω is horizontal)

and lift of a $2m$-form on M, also denoted by $e(\Omega^m)$

$e(\Omega^m) \in \Gamma(\Lambda^n T^*M)$ Euler form

More precisely, we have

$$(R(X,Y)Z)_x = u\left(2\Omega_u(\bar{X}_u, \bar{Y}_u)\right)(u^{-1}Z_x) \in T_xM$$

$X, Y, Z \in \Gamma(TM)$ & $\bar{X}, \bar{Y} \in \Gamma(TP)$ horiz. lifts of X, Y

$u \in P : \pi(u) = x$

In particular,

$$\langle R(X_1, X_2)X_3, X_4 \rangle_{T_xM} = \langle 2\,\Omega(\bar{X}_1, \bar{X}_2)\, u^{-1}X_3, u^{-1}X_4 \rangle_{\mathbb{R}^n}, \quad X_1, \dots, X_4 \in \Gamma(TM)$$

Cor Let $e_1, \dots, e_n \in \Gamma_U(TM)$ local sections of TM over U s.th.

$(e_1(x), \dots, e_n(x))$ pos. oriented onb of $T_xM \quad \forall x \in U$

Let $\Omega_{ik} = \frac{1}{2}\langle R(\cdot,\cdot)e_k, e_i \rangle \in \Gamma(\Lambda^2 T^*M)$

$$Pf(\Omega) = \frac{1}{2^m m!} \sum_{\varrho \in S_n} \operatorname{sign}(\varrho)\, \Omega_{\varrho(1)\varrho(2)} \wedge \dots \wedge \Omega_{\varrho(n-1)\varrho(n)} \in \Gamma(\Lambda^n T^*M)$$

and $$\Omega_{\varrho(1)\varrho(2)} \wedge \dots \wedge \Omega_{\varrho(n-1)\varrho(n)}(e_1 \wedge \dots \wedge e_n) = \frac{(-1)^m}{2^m} \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, R_{\sigma(1)\sigma(2)\varrho(1)\varrho(2)} \cdots R_{\sigma(n-1)\sigma(n)\varrho(n-1)\varrho(n)}$$

where $R_{ijkl} = \langle R(e_i, e_j)e_k, e_l \rangle$

In summary: $e(\Omega, \dots, \Omega)(e_1 \wedge \dots \wedge e_n)$

$$= \frac{(-1)^m}{(4\pi)^{-m} 2^m m!} \sum_{\sigma, \varrho \in S_n} \operatorname{sign}(\sigma) \operatorname{sign}(\varrho)\, R_{\sigma(1)\sigma(2)\varrho(1)\varrho(2)} \cdot \dots \cdot R_{\sigma(n-1)\sigma(n)\varrho(n-1)\varrho(n)}$$

Thm of Gauss-Bonnet-Chern (Chern 1944)

M cpt oriented mf, $\dim M = n$. Then

$$\chi(M) = \sum_{p=0}^{n} (-1)^p \dim H^p(M;\mathbb{R}) = \begin{cases} \int_M e(\Omega,\dots,\Omega) & 2\mid n \\ 0 & 2\nmid n \end{cases}$$

where $e(\underbrace{\Omega,\dots,\Omega}_{n/2 \text{ times}})$ is the Euler form

(1) Know already

$$\chi(M) = \int_M \mathrm{str}\, P_t(x,x)\, \mathrm{vol}(dx)$$

where $P_t(x,y) \in \mathrm{Hom}(E_y, E_x)$ is the C^∞-integral kernel of $P_t = e^{t/2 \Delta}$ on $A^\cdot(M) = \Gamma(E)$ with $E = \Lambda T^* M$

We want to show

$$\lim_{t \downarrow 0} \mathrm{str}\, P_t(x,x) = E(x) \quad \text{where } E(x)\,\mathrm{vol}(dx) = \begin{cases} \text{Euler form} & 2\mid n \\ 0 & 2\nmid n \end{cases}$$

(local theorem of G-B-C, McKean-Singer 1967, Patodi 1971)

(2) Brownian bridge measure

$$\mathbb{P}^t_{x,y}(\dots) = \mathbb{E}[\dots \mid X_t(x) = y]$$

$$\mathbb{P}(\dots) = \int \mathbb{P}^t_{x,y}(\dots)\, \mathbb{P}\circ X_t^{-1}(dy) = \int \mathbb{P}^t_{x,y}(\dots)\, p(t,x,y)\, dy$$

Know $$(e^{t/2\Delta})\alpha = \int P_t(x,y)\,\alpha(y)\,\mathrm{vol}(dy)$$
$$= \mathbb{E}\left[Q_t \,/\!/_{0,t}^{-1}\, \alpha\right], \quad \alpha \in \Gamma(T^*M)$$

where $$\frac{d}{dt} Q_t = -\frac{1}{2} Q_t \,/\!/_{0,t}^{-1}\, \mathcal{R}_{X_t} \,/\!/_{0,t}, \quad Q_0 = \mathrm{id}$$

$$\Big(\Delta = \underbrace{\mathrm{tr}\,\nabla^2}_{=\Box} - \mathcal{R}\ ; \text{ here } \mathcal{R} = \mathrm{Ric}\Big)$$

In particular,

$$P_t(x,y) = \mathbb{E}^t_{x,y}\left[Q_t \,/\!/_{0,t}^{-1}\right] p(t,x,y)$$
$$= \mathbb{E}\left[Q_t \,/\!/_{0,t}^{-1} \mid X_t(x) = y\right] p(t,x,y)$$

$$\Rightarrow \mathrm{str}\, P_t(x,x) = \mathbb{E}^t_{x,x}\left[\mathrm{str}\left(Q_t \,/\!/_{0,t}^{-1}\right)\right] p(t,x,x)$$

(3) Rescaling of the Brownian bridge

X Brownian bridge of lifetime $\varepsilon > 0$. $X_0 = x$ & $X_\varepsilon = y$

$\rightsquigarrow X^\varepsilon$. $X^\varepsilon_s \sim X_{s\varepsilon}$ $\quad (0 \leq s \leq 1)$

$p(\varepsilon, x, y) = p^\varepsilon(1, x, y)$

$\Rightarrow \operatorname{str} P_\varepsilon(x,x) = \hat{\mathbb{E}}\left[\operatorname{str}\left(Q^\varepsilon_1 /\!/^\varepsilon_{1,0}\right)\right] \underbrace{p^\varepsilon(1,x,x)}_{\sim (2\pi\varepsilon)^{-n/2}}$

where i) $\hat{\mathbb{E}}[\ldots] = \mathbb{E}[\ldots | X^\varepsilon_1(x) = x]$

ii) $/\!/^\varepsilon_{0,s}$ $/\!/$-transport along X^ε_s

iii) $\frac{d}{ds} Q^\varepsilon_s = -\frac{\varepsilon}{2} Q^\varepsilon_s \underbrace{/\!/^\varepsilon_{s,0} \mathcal{R}_{X^\varepsilon_s} /\!/^\varepsilon_{0,s}}_{=: \mathcal{R}^\varepsilon_s}$, $\quad Q^\varepsilon_0 = \mathrm{id}$

Recall $/\!/^\varepsilon_{0,s} = (U^\varepsilon_0)^{-1} U^\varepsilon_s = U_0^{-1} U^\varepsilon_s$ $\quad /\!/$-transport along X^ε

$\underbrace{O(T_x M)}_{\cong O(n)}$ - valued random variable

Crucial lemma:

- $\exists$ r.v. $K \in L^p$ $\forall p \geq 1$

s.th. $\left| /\!/^\varepsilon_{0,1} - \mathrm{id} \right| \leq K \varepsilon$ a.s.

Back to fermionic calculus

$(V, \langle , \rangle)$ Euclidean vector space

$e_1, \dots, e_n$ onb of V $\leadsto$ $a_i = a(e_i)$, $\{a_i, a_j\} = a_i a_j + a_j a_i = 0$

$I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$ $\quad (i_1 < i_2 < \dots < i_r)$

let $a_I = a_{i_1} \cdots a_{i_r}$ & $a_I^* = (a_I)^*$

(Convention $a_\emptyset = \mathrm{id}$ as empty product)

Def For $B \in \mathrm{End}(\Lambda V)$ let

$$\operatorname{str} B := \operatorname{trace}\left((-1)^{\cdot} B\right) \qquad \text{supertrace of } B$$

where $(-1)^{\cdot}\big|_{\Lambda^p V} := (-1)^p \, \mathrm{id}_{\Lambda^p V}$

Elementary lemma

a) Each $B \in \mathrm{End}(\Lambda V)$ has a unique representation as

$$B = \sum_{I,J} \beta_{I,J} \, a_I^* a_J$$

- (Note: $\dim \Lambda V = \sum_{p=0}^n \dim \Lambda^p V = \sum_{p=0}^n \binom{n}{p} = 2^n$

$\Rightarrow \dim \mathrm{End}(\Lambda V) = (2^n)^2 = \# \{ a_I^* a_J : I, J \subset \{1, \dots, n\} \}$)

b) Berezin - Patodi formula (Cycon, Froese, Kirsch, Simon: Schrödinger operators with Applications to Quantum Mechanics and Global Geometry)

$$\operatorname{str} B = (-1)^n \beta_{\{1,\dots,n\},\{1,\dots,n\}} \qquad \forall B \in \mathrm{End}(\Lambda V)$$

Recall $T \in \mathrm{End}(V)$

$DT \in \mathrm{End}(\Lambda V)$: $DT(v_1 \wedge \dots \wedge v_n) = \sum_{i=1}^n v_1 \wedge \dots Tv_i \wedge \dots \wedge v_n$

$\Lambda T \in \mathrm{End}(\Lambda V)$: $\Lambda T(v_1 \wedge \dots \wedge v_n) = Tv_1 \wedge \dots \wedge Tv_n$

$$\Rightarrow \frac{d}{dt}\Big|_{t=0} \Lambda(e^{tT}) = DT \Rightarrow \Lambda e^{tT} = e^{tDT}$$

Apply this calculus fiberwise to $\Lambda T^* M$ with $V = T_x^* M$

a) $\hat{T} = D(T^*) \in \mathrm{End}(\Lambda V^*)$

$$\hat{T} = \sum_{i,j} \underbrace{T^*_{ij}}_{\langle Te_i, e_j\rangle} a_i^* a_j$$

b) $R \in \mathrm{Hom}(V \otimes V, \mathrm{End}\, V)$

$\hat{R} \in \mathrm{End}(\Lambda V^*)$, $\hat{R} = \sum_{i,j,k,l} \langle R(e_i, e_j)e_k, e_l\rangle \, a_i^* a_j a_k^* a_l$

Observation (1) $2 \mid n$, $n = \dim V$

$$\underbrace{\hat{R} \circ \dots \circ \hat{R}}_{n/2 \text{ times}} = \sum_{\sigma, \rho \in \mathfrak{S}_n} \mathrm{sign}(\sigma)\,\mathrm{sign}(\rho)\, R_{\sigma(1)\rho(1)\sigma(2)\rho(2)} \cdot \ldots \cdot R_{\sigma(n-1)\rho(n-1)\sigma(n)\rho(n)}$$

$$= \frac{1}{2^{n/2}} \sum_{\sigma, \rho \in \mathfrak{S}_n} \mathrm{sign}(\sigma)\,\mathrm{sign}(\rho)\, R_{\sigma(1)\sigma(2)\rho(1)\rho(2)} \cdots R_{\sigma(n-1)\sigma(n)\rho(n-1)\rho(n)}$$

if R is of curvature type, i.e. R has the symmetry behavior of the Riemann tensor, i.e.

$\forall\, x, y, z, w \in V$: i) $\langle R(x,y)z, w\rangle = -\langle R(y,x)z, w\rangle = -\langle R(x,y)w, z\rangle$

ii) $\langle R(x,y)z, w\rangle = \langle R(z,w)x, y\rangle$

iii) $R(x,y)z + R(y,z)x + R(z,x)y = 0$

(2) $\dim V = n$

$$\left.\begin{array}{l} T_1, \dots, T_i \in \mathrm{End}(V) \\ R_1, \dots, R_j \in \mathrm{Hom}(V \otimes V, \mathrm{End}\, V) \end{array}\right\} \Longrightarrow \mathrm{str}\big(\underbrace{\hat{T}_1 \circ \dots \circ \hat{T}_i \circ \hat{R}_1 \circ \dots \circ \hat{R}_j}_{\in \mathrm{End}(\Lambda V^*)}\big) = 0 \quad \text{if } i + 2j < n$$

Berezin - Patodi

Here: $\mathrm{str}(Q_1^\varepsilon \, /\!\!/_{1,0}^\varepsilon)$

$V_\varepsilon := /\!\!/_{1,0}^\varepsilon$ $\underbrace{O(T_xM)}_{O(n)}$ - valued, $V_\varepsilon = e^{v_\varepsilon}$ with v_ε $o(n)$-valued, $|v_\varepsilon| \leq k\varepsilon$, $k \in L^p$ $\forall p \geq 1$

$|/\!\!/_{0,1}^\varepsilon - \mathrm{id}| \leq K\varepsilon$, $K \in L^p$ $\forall p \geq 1$

$\Rightarrow /\!\!/_{0,1}^\varepsilon = V_\varepsilon^{-1}$

$\frac{d}{ds} Q_s^\varepsilon = -\frac{\varepsilon}{2} Q_s^\varepsilon \mathcal{R}_s^\varepsilon$, $Q_0^\varepsilon = \mathrm{id}$

$$\Rightarrow Q_1^\varepsilon = \mathrm{id} - \frac{\varepsilon}{2}\int_0^1 Q_s^\varepsilon \mathcal{R}_s^\varepsilon \, ds$$

$$= \mathrm{id} - \frac{\varepsilon}{2}\int_0^1 \mathcal{R}_s^\varepsilon \, ds + \frac{\varepsilon^2}{4}\int_0^1\int_0^s Q_r^\varepsilon \mathcal{R}_r^\varepsilon \mathcal{R}_s^\varepsilon \, dr\, ds$$

$$= \mathrm{id} + \Theta_1^{\varepsilon,1} + \Theta_1^{\varepsilon,2} + \Theta_1^{\varepsilon,3} + \dots + \Theta_1^{\varepsilon,N} + O(\varepsilon^{N+1})$$

where $\Theta_1^{\varepsilon,j} = \frac{(-1)^j}{2^j}\varepsilon^j \int_0^1 \int_0^{s_j} \dots \int_0^{s_2} \mathcal{R}_{s_1}^\varepsilon \circ \dots \circ \mathcal{R}_{s_j}^\varepsilon \, ds_1 \dots ds_j$

Hence

$$Q_1^\varepsilon /\!/_{1,0}^\varepsilon = \sum_{i,j\leq N} \Theta_1^{\varepsilon,j} \frac{(D_{v_\varepsilon})^i}{i!} + O(\varepsilon^{N+1})$$

$$\Rightarrow \mathrm{str}\left(Q_1^\varepsilon /\!/_{1,0}^\varepsilon\right) = \sum_{i,j\leq N} \mathrm{str}\underbrace{\left(\Theta_1^{\varepsilon j}\frac{(D_{v_\varepsilon})^i}{i!}\right)}_{|\;|\;\leq C(\omega)\varepsilon^{i+j}} + O(\varepsilon^{N+1}) \qquad (*)$$

$\mathrm{str}(\dots) \neq 0$ in $(*)$ only possible as $\varepsilon\downarrow 0$ if $\underbrace{i+2j \geq n \;\&\; i+j\leq \frac{n}{2}}_{\text{i.e. } 2i+2j\leq n\leq i+2j \;\Leftrightarrow\; i=0 \;\&\; n=2j}$

$$\Rightarrow \lim_{\varepsilon\downarrow 0} \mathrm{str}\left(Q_1^\varepsilon /\!/_{1,0}^\varepsilon\right)\frac{1}{(2\pi\varepsilon)^{n/2}} = \begin{cases} \lim_{\varepsilon\downarrow 0} \mathrm{str}\left(\Theta_1^{\varepsilon,n/2}\right)(2\pi\varepsilon)^{-n/2} & 2\mid n \\ 0 & 2\nmid n \end{cases}$$

Let $m = n/2$

$$\mathrm{str}\left(\Theta_1^{\varepsilon,m}\right)(2\pi\varepsilon)^{-m} = \left(\frac{-1}{4\pi}\right)^m \mathrm{str}\underbrace{\left(\int_0^1\int_0^{s_m}\dots\int_0^{s_2} \mathcal{R}_{s_1}^\varepsilon \circ\dots\circ \mathcal{R}_{s_m}^\varepsilon \, ds_1\dots ds_m\right)}$$

$$\xrightarrow{\varepsilon\downarrow 0} \int_0^1\int_0^{s_m}\dots\int_0^{s_2} \mathcal{R}_x\circ\dots\circ\mathcal{R}_x \, ds_1 \dots ds_m = \frac{1}{m!}\underbrace{\mathcal{R}_x\circ\dots\circ\mathcal{R}_x}_{m \text{ times}}$$

Hence

$$\lim_{\varepsilon\downarrow 0}\mathrm{str}\, p_\varepsilon(x,x) = \begin{cases} 0 & n \text{ odd} \\ (-1)^m (4\pi)^{-m}\frac{1}{m!}\,\mathrm{str}\,(\underbrace{\mathcal{R}_x\circ\dots\circ\mathcal{R}_x}_{m \text{ times}}) & n \text{ even}\end{cases}$$

where

$$\mathrm{str}\,(\mathcal{R}_x\circ\dots\circ\mathcal{R}_x) = \frac{1}{2^{n/2}}\sum_{\sigma,\varrho\in \mathfrak{S}_n} \mathrm{sign}(\sigma)\,\mathrm{sign}(\varrho)\, R_{\sigma(1)\sigma(2)\varrho(1)\varrho(2)} \cdots R_{\sigma(n-1)\sigma(n)\varrho(n-1)\varrho(n)}$$

with $R_{ijkl} = \langle R(e_i,e_j)e_k, e_l\rangle$, $e_1,\dots,e_n$ onb for T_xM