Stochastic Analysis on manifolds

First goal: Construction of canonical processes on a smooth mf M, e.g. martingales & Brownian motions $BM(M)$

Difficulties:

- Both concepts are not invariant under coordinate transformations
- BM have a strong rigidity, e.g.

  $\emptyset \subset U \subset \mathbb{R}^n$ open & connected, $f \in C^2(U, \mathbb{R}^n)$, $f \not\equiv$ const

  Then: f preserves BM (i.e. X $BM(U) \Rightarrow f(X)$ $BM(\mathbb{R}^n)$ mod time transformation)

  $$\Longleftrightarrow \begin{cases} f \text{ linear} & n=1 \\ f \text{ holomorphic resp. antiholomorphic} & n=2 \\ f = \lambda A + b \text{ where } \lambda > 0,\ A \in O(n),\ b \in \mathbb{R}^n & n \geq 3 \end{cases}$$

- martingale concept based on the notion of conditional expectations (requires per definition a linear structure); no straightforward generalization possible

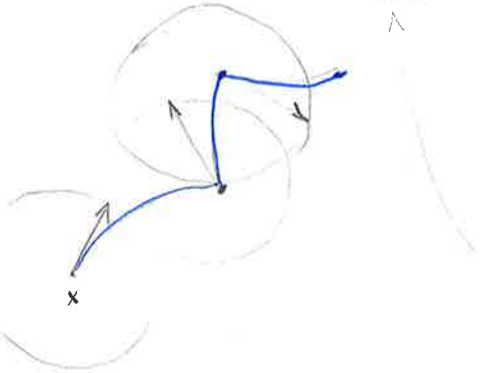
Intuitive idea

requires as additional structure a linear connection on M

martingale $\hat{=}$ "driftless" random motion

$BM(M)$ $\hat{=}$ continuous limit of a "geodesic random walk"

requires as additional structure a Riem. metric on M



go along the chosen geodesic a distance ε at a speed $1/\varepsilon$; then $\varepsilon \to 0$

Def M differentiable Mf; $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0})$ filtered prob space

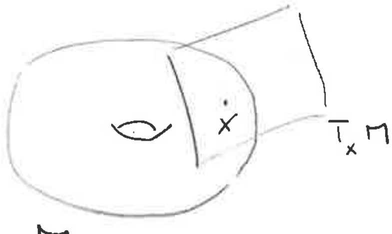
X cont. adapted process taking values in M

X is called a semimartingale if $f(X)$ real semimartingale $\forall f \in C^2(M)$

1. Stochastic flows and SDEs on a mf

M differentiable mf

$TM \xrightarrow{\pi} M$ tangent bundle of M



$TM = \dot{\bigcup}_{x \in M} T_xM$, $\pi|T_xM = x$

M

$$\Gamma(TM) = \{ A : M \to TM \ \mathcal{C}^\infty \mid \underbrace{\pi \circ A = id_M}_{\text{i.e. } A(x) \in T_xM \ \forall x \in M} \}$$

vector fields

$$\Gamma(TM) \hat{=} \left\{ A : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M) \ \mathbb{R}\text{-linear} \ \middle| \ A(fg) = f A(g) + g A(f) \ \forall f, g \in \mathcal{C}^\infty(M) \right\}$$

$\mathbb{R}$-derivations on $\mathcal{C}^\infty(M)$

$$A(f)(x) := df_x \cdot A(x) \in \mathbb{R}$$

Flow to a vector field

Let $A \in \Gamma(TM)$

$$\begin{cases} x(0) = x \in M \\ \dot{x}(t) = A(x(t)) \end{cases}$$

$t \mapsto x(t) =: \varphi_t(x) \in M$ flow curve to A starting at $x \in M$

$\forall f \in \mathcal{C}_c^\infty(M)$ this means

$$\begin{cases} \frac{d}{dt} f(\varphi_t) = A(f)(\varphi_t) \\ f(\varphi_0) = f \end{cases},$$

or equivalently

$$f(\varphi_t(x)) - f(x) - \int_0^t A(f)(\varphi_s(x)) \, ds \equiv 0 , \quad t \geq 0 , \quad x \in M$$

In particular, $\left.\frac{d}{dt}\right|_{t=0} f \circ \varphi_t = A(f)$

Flow to a 2^{nd} order PDO L, e.g. $L = A_0 + \sum_{i=1}^r A_i^2$

where $A_0, A_1, \dots, A_r \in \Gamma(TM)$

<u>Def.</u> Let $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space

An adapted continuous process

$$X(x) \equiv (X_t(x))_{t\geq 0}$$

on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ taking values in M is called

<u>flow process</u> (or <u>L-diffusion</u>) to L (with starting point $X_0(x) = x$)

if $\forall f \in \mathcal{C}_c(M)$

$$N_t^f(x) := f(X_t(x)) - f(x) - \int_0^t (Lf)(X_s(x))\,ds \quad , t\geq 0$$

is a martingale, i.e.

$$E^{\mathcal{F}_s}\Big[\underbrace{f(X_t(x)) - f(X_s(x)) - \int_s^t (Lf)(X_r(x))\,dr}_{= N_t^f(x) - N_s^f(x)}\Big] \equiv 0 \quad \forall\, s\leq t$$

<u>Note</u> $\frac{d}{dt}\Big|_{t=0} E\big[f(X_t(x))\big] = (Lf)(x)$

<u>Ex</u> $M = \mathbb{R}^n$, $L = \frac{1}{2}\Delta$ where $\Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial}{\partial x_n}\right)^2$

$X \equiv (X_t)$ BM $(\mathbb{R}^n)$

$\forall f \in \mathcal{C}_c^\infty(M)$,

$$d f(X_t) = \langle (\nabla f)(X_t), dX_t\rangle_{\mathbb{R}^n} + \frac{1}{2}(\Delta f)(X_t)\,dt$$

$\Rightarrow$ $f(X_t) - f(X_0) - \int_0^t \frac{1}{2}(\Delta f)(X_s)\,ds \quad , t\geq 0$ martingale

i.e. $X_t(x) := x + X_t$ flow process to $\frac{1}{2}\Delta$

SDEs on a mf M

Def M differentiable mf & $TM \xrightarrow{\pi} M$ its tangent bundle
E $\mathbb{R}$-vector space $(\dim E < \infty)$; without restriction $E = \mathbb{R}^r$.
An <u>SDE</u> on M is a pair (A, Z) where

1) Z is a semimartingale taking values in E
2) $A : M \times E \to TM$ is a homomorphism of vector bundles over M, i.e.

$$(x,e) \longmapsto A(x,e) =: A(x)e$$

$$\begin{tikzcd} M \times E \arrow[r, "A"'] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\cong"] & TM \arrow[d, "\pi"] \\ M \arrow[r, "id"'] & M \end{tikzcd}$$

such that $A(x) : E \to T_xM$ linear $\forall x \in M$

In other words:

$\forall x \in M$, $A(x) \in \mathrm{Hom}(E, T_xM)$

$\forall e \in E$, $A(\cdot)e \in \Gamma(TM)$

<u>Notation</u> We write formally for the SDE (A, Z) also

$$dX = A(X) \circ dZ$$

or

$$\boxed{dX = \sum_{i=1}^{r} A_i(X) \circ dZ^i}$$

where $A_i := A(\cdot)e_i \in \Gamma(TM)$ and $e_1, \dots, e_r$ basis of E

<u>Def</u> An SDE (A, Z) is called <u>non-degenerate</u> (elliptic) :⟺

$$A(x) : E \to T_xM \text{ surj. } \forall x \in M$$

<u>Def</u> Let (A, Z) be an SDE on M and

$x_0 : \Omega \to M$ an $\mathcal{F}_0$-measurable r.v.

An adapted continuous process

$$X|[0,\zeta[\;\equiv\; (X_t)_{t<\zeta},$$

taking values in M, and defined up to a stopping time $\zeta > 0$, is called <u>solution to the SDE</u>

$$dX = A(X) \circ dZ$$

with initial condition $X_0 = x_0$ if

i) X is a semimartingale on M with $X_0 = x_0$

ii) $\forall f \in C_c^\infty(M)$

$$d(f(X_t)) = (df)_{X_t} A(X_t) \circ dZ_t$$

<u>Note</u> a) $E \xrightarrow{A(x)} T_x M \xrightarrow{(df)_x} \mathbb{R}$

b) $U \circ dZ = U\,dZ + \underbrace{\tfrac{1}{2}\, dU\, dZ}_{= \frac{1}{2} d \underbrace{[U,Z]}_{\text{quadratic covariation}}}$ <u>Stratonovich differential</u>

<u>Rule</u> $V \circ (U \circ dZ) = (VU) \circ dZ$

X is called <u>maximal solution</u> of the SDE if

$$\{\zeta < \infty\} \subset \left\{ \lim_{t \uparrow \zeta} X_t = \infty \text{ in } \hat{M} = M \cup \{\infty\} \right\}$$

Rem $X_t(\omega) := \infty$ for $\zeta(\omega) \leq t < \infty$

$f(\infty) := 0$ for $f \in \mathcal{C}_c^\infty(M)$

Then

$$dX = A(X) \circ dZ, \quad X_0 = x_0$$

$$\iff \forall f \in \mathcal{C}_c(M), \quad f(X_t) = f(x_0) + \int_0^t (df)_{X_s} A(X_s) \circ dZ_s, \quad t \geq 0$$

$$\iff \forall f \in \mathcal{C}_c(M), \quad f(X_t) = f(x_0) + \sum_{i=10}^{r} \int_0^t (df)_{X_s} A_i(X_s) \circ dZ_s^i, \quad t \geq 0$$

where $A_i := A(\cdot) e_i$, $i = 1, \ldots, r$

Why Stratonovich differentials?

Ito-Stratonovich formula Let X be a continuous $\mathbb{R}^n$-valued semimartingale and $f \in \mathcal{C}^3(\mathbb{R}^n)$. Then

$$d(f \circ X) = \sum_{i=1}^n (D_i f)(X) \circ dX^i = \langle \nabla f(X), \circ dX \rangle_{\mathbb{R}^n}$$

Pr $$d\big((D_i f)(X)\big) \underset{\text{Itô}}{=} \sum_{k=1}^n (D_i D_k f)(X)\, dX^k + \frac{1}{2} \sum_{k,l} (D_i D_k D_l f)(X)\, dX^k dX^l$$

$$\Rightarrow \sum_{i=1}^n (D_i f)(X) \circ dX^i = \sum_{i=1}^n (D_i f)(X)\, dX^i + \frac{1}{2} \sum_{i=1}^n \underbrace{d\big((D_i f)(X)\big)\, dX^i}_{= \sum_{k=1}^n (D_i D_k f)(X)\, dX^k dX^i}$$

$$\underset{\text{Itô}}{=} d(f \circ X)$$

Ex $\quad E = \mathbb{R}^{r+1}$, $\quad z = (t, \underbrace{z^1, \dots, z^r}_{\text{BM on } \mathbb{R}^r})$, $\quad (e_0, e_1, \dots, e_r)$ standard basis of $\mathbb{R}^{r+1}$

$A : M \times E \longrightarrow TM$ homom. of vector bundles $/M$

$\rightsquigarrow \; A_i := A(\cdot)e_i \in \Gamma(TM)$, $i = 0, 1, \dots, r$

Then

$$\boxed{dX = A(X) \circ dz} \;\hat{=}\; \boxed{dX = A_0(X)\,dt + \sum_{i=1}^r A_i(X) \circ dz^i}, \qquad (*)$$

i.e. $\forall f \in \mathcal{C}_c^\infty(M)$,

$$\begin{aligned}
d(f \circ X) &= (df)_X A(X) \circ dz \\
&= \sum_{i=0}^r (df)_X A(X)e_i \circ dz^i \\
&= \sum_{i=0}^r \underbrace{(df)_X A_i(X)}_{=(A_i f)(X)} \circ dz^i \\
&= (A_0 f)(X)\,dt + \sum_{i=1}^r (A_i f)(X) \circ dz^i \\
&= (A_0 f)(X)\,dt + \sum_{i=1}^r (A_i f)(X)\,dz^i + \frac{1}{2}\sum_{i=1}^r d\left[(A_i f)(X), z^i\right]
\end{aligned}$$

Note $\quad E \xrightarrow{A(x)} T_x M \xrightarrow{(df)_x} \mathbb{R}$

But $d\big((A_i f)(X)\big) = \sum_{j=1}^r (A_j A_i f)(X)\,dz^j + d(\text{bounded variation})$,

$$\Rightarrow \quad d\left[(A_i f)(X), z^i\right] = (A_i^2 f)(X)\,dt$$

$d[z^i, z^j] = \delta_{ij}\,dt$, $\quad 1 \le i, j \le r$

Conclusion $\quad$ Let $L = A_0 + \frac{1}{2}\sum_{i=1}^r A_i^2$

Then $\quad d(f \circ X) - (Lf)(X)\,dt \overset{m}{=} 0 \quad \forall f \in \mathcal{C}_c^\infty(M)$

modulo differentials of martingales

In other words: maximal solutions to the SDE $(*)$ are L-diffusions to $L = A_0 + \frac{1}{2}\sum_{i=1}^r A_i^2$

M mf, $L = A_0 + \frac{1}{2}\sum_{i=1}^{r} A_i^2$ PDO on M, $A_0, A_1, \dots, A_r$ v.f. on M

$X_t(x)$, $t \geq 0$, $x \in M$ L-diffusion on M, $X_0(x) = x$, if

$\forall f \in C_c^\infty(M)$, $f(X_t(x)) - f(x) - \int_0^t (Lf)(X_s(x))\,ds$ is a martingale, or
$\forall f \in C^\infty(M)$, —— " —— is a local martingale

In other words,

$$d\left(f(X_t(x))\right) - (Lf)(X_t(x))\,dt \overset{m}{=} 0$$

SDE on M, $z = (z^1, \dots, z^r)$ BM$(\mathbb{R}^r)$

$$(*) \quad dX = A_0(X)\,dt + \sum_{i=1}^{r} A_i(X) \circ dz^i$$

Every maximal solution X to the SDE $(*)$, starting at x,
is an L-diffusion $X_t(x)$ on M

More generally, SDE on M
(A, z) where z = semimartingale on $\mathbb{R}^r$
$A : \mathbb{R}^r \times M \to TM$, $(z, x) \longmapsto A(x)z$ s.th. $A(\cdot)z \in \Gamma(TM)$ $\forall z \in E$, $A(x) : T_xM \to \mathbb{R}^r$ linear $\forall x \in M$

$$\boxed{dX = A(X) \circ dz}$$

Theorem (SDEs : Existence & Uniqueness of solutions)

Let (A, Z) be an SDE on M and $x_0 : \Omega \to M$ $\mathcal{F}_0$-measurable

Then $\exists_1$ maximal solution $X|[0,\zeta[$ ($\zeta > 0$ a.s.) of the SDE

$$(*) \qquad dX = A(X) \circ dZ$$

with initial condition $X_0 = x_0$.

Uniqueness holds in the sense : if $Y|[0,\xi[$ is another solution with $Y_0 = x_0$, then $\xi \leq \zeta$ and $X|[0,\xi[= Y$ a.s.

Idea of proof

(1) Consider the case $M = \mathbb{R}^n$

Then $A \in \mathcal{C}^\infty(\mathbb{R}^n, \mathrm{Mat}(n \times r; \mathbb{R}))$ and $(*)$ has a unique solution X with initial condition $X_0 = x_0$

Note that X is a solution of

$$dX = A(X) \circ dZ$$

in the Itô-Stratonovich sense iff

$$\forall f \in \mathcal{C}^\infty_c(M), \quad d(f \circ X) = (df)_X A(X) \circ dZ$$

Indeed if

$dX = A(X) \circ dZ$ in the Itô-Stratonovich sense

then
$$d(f \circ X) = \langle \nabla f(X), \circ dX \rangle = \langle \nabla f(X), A(X) \circ dZ \rangle = (df)_X A(X) \circ dZ$$

(2) General case

Whitney embedding $M \hookrightarrow \mathbb{R}^N$, with N sufficiently large, as closed submanifold

Observation:

$A : M \times \mathbb{R}^r \longrightarrow TM, \; (x,z) \longmapsto A(x)z = \sum_i A_i(x) z^i$ has a continuation to

$\bar{A} : \mathbb{R}^N \times \mathbb{R}^r \longrightarrow \mathbb{R}^N \times \mathbb{R}^N, \; (x,z) \longmapsto \bar{A}(x) = \sum_i \bar{A}_i(x) z^i$

Replace

$$dX = A(X) \circ dZ \quad \text{on } M \qquad (*)$$

by

$$dX = \bar{A}(X) \circ dZ \quad \text{on } \mathbb{R}^N \qquad (\bar{*})$$

Show that every solution to $(\bar{*})$ in $\mathbb{R}^N$ which starts on $M \subset \mathbb{R}^N$ for $t=0$ stays on M up to its lifetime

What can we do with L-diffusions ?

Heat equation

Ex Given $f \in \mathcal{C}(M)$. Find a solution

$$u = u(t,x), \quad t \in \mathbb{R}_+, \; x \in M,$$

to

$$(HE) \quad \begin{cases} \frac{\partial}{\partial t} u = Lu \\ u|_{t=0} = f \end{cases}$$

Suppose that $\zeta(x) = \infty$ a.s $\forall x \in M$

Let u be a bdd solution to (HE)

Fix $t > 0$ and let $\varphi_s(x) = u(t-s, x)$, $0 \leq s \leq t$

$$d\big(\varphi_s(X_s(x))\big) = (\partial_s \varphi_s)(X_s(x))\,ds + (L\varphi_s)(X_s(x))\,ds + (\text{local mart.})_s$$

In other words

$$u(t-s, X_s(x)) = u(t,x) + \underbrace{\int_0^s \Big(\frac{\partial}{\partial r} + L\Big) u(t-r, X_r(x))\,dr}_{=0} + \underbrace{(\text{loc. mart})_s}_{\Longrightarrow \text{ true martingale, zero at time } 0} \qquad (0 \leq s \leq t)$$

$$\Rightarrow u(t,x) = \mathbb{E}\left[u(t-s, X_s(x))\right] \xrightarrow[\substack{\text{dominated}\\ \text{convergence}}]{s \uparrow t} \mathbb{E}\left[u(0, X_t(x))\right] = \mathbb{E}\left[f(X_t(x))\right]$$

Conclusion Under the hypothesis $\zeta(x) = \infty \ \forall x \in M$,
we have uniqueness of bounded solutions to the heat equation (HE):

$$u(t,x) = \mathbb{E}\left[f(X_t(x))\right]$$

Question What happens if $X(x)$ has a nontrivial lifetime $\zeta(x)$?

Note: There $\exists$ always a minimal solution to (HE) in the sense that $u(t,x) \to 0$ as $x \to \infty$ in $\hat{M} = M \dot{\cup} \{\infty\}$

Let $\sigma_n \uparrow \zeta(x)$ be an increasing sequence of stopping times

The argument above shows

$$u(t,x) = \mathbb{E}\left[u(t - t\wedge\sigma_n, X^{(x)}_{t\wedge\sigma_n})\right] \quad (\text{indep. of } n!)$$

$$= \mathbb{E}\left[\lim_{n\to\infty} u(t - t\wedge\sigma_n, X^{(x)}_{t\wedge\sigma_n})\right]$$

$$= \mathbb{E}\left[1_{\{t<\zeta(x)\}} u(0, X_t(x))\right]$$

$$= \mathbb{E}\left[1_{\{t<\zeta(x)\}} f(X_t(x))\right]$$

Note that

$$t\wedge\sigma_n \uparrow \begin{cases} t & \text{if } t < \zeta(\omega) \\ \zeta(x) & \text{if } t \geq \zeta(\omega) \end{cases}$$

$\zeta := \zeta(x)$

2. M-valued semimartingales: quadratic variation & integration along 1-forms

Situation: M differentiable mf, X continuous M-valued semimartingale

$b \in \Gamma(T^*M \otimes T^*M)$, i.e. $b_x : T_xM \times T_xM \to \mathbb{R}$ bilinear $\forall x \in M$

(Ex. $f, g \in \mathcal{C}^\infty(M)$, $df, dg \in \Gamma(T^*M)$. where $df_x v = v(f)$, $v \in T_xM$
$\Rightarrow b = df \otimes dg \in \Gamma(T^*M \otimes T^*M)$)

$\alpha \in \Gamma(T^*M)$, i.e. $\alpha_x : T_xM \to \mathbb{R}$ linear $\forall x \in M$

Goal: Explain $\int b(dX, dX)$ and $\int_X \alpha$

Localization Lemma

Let X be a continuous M-valued semimartingale, $(U_k)_{k\in\mathbb{N}}$ countable open covering of M.
Then $\exists$ an increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times with $\tau_0 = 0$ & $\sup_n \tau_n = \infty$, s.th. on each interval of the form $[\tau_n, \tau_{n+1}] \cap (\mathbb{R}_+ \times \{\tau_n < \tau_{n+1}\})$, X takes values only in U_k

Pr. To $(U_k)_{k\in\mathbb{N}}$ choose a refinement $(W_k)_{k\in\mathbb{N}}$ s.th.
$\forall k \in \mathbb{N}$, $\overline{W}_k \subset U_{n(k)}$ for some $n(k)$

Define stopping times $(\tau_n^k)_{0\le k\le n,\, n\ge 0}$ as follows:

$$\tau_0^0 := 0$$
$$\tau_{n+1}^0 := \tau_n^n$$
$$\tau_{n+1}^k := \inf\{t \geq \tau_{n+1}^{k-1} : X_t \notin W_k\}, \quad k = 1, 2, \dots, n+1$$

Claim: (τ_n^k) has after an appropriate renumbering the wanted properties to show:

$$\sup_{n \geq 0} \sup_{k \leq n} \tau_n^k = \infty$$

Suppose $\exists\ \omega \in \Omega$ s.th. $t_0 := \sup_{n \geq 0} \sup_{k \leq n} \tau_n^k(\omega) < \infty$

$\Rightarrow$ $X_{t_0}(\omega) \in W_\ell$ for some ℓ

$\Rightarrow$ $X_t(\omega) \in W_\ell \quad \forall t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ for some $\varepsilon > 0$

X has continuous paths

$\Rightarrow$ $\exists\ n_0 \in \mathbb{N}$, $n_0 \geq \ell$: $\tau_{n_0}^0(\omega) > t_0 - \varepsilon$,

definition of t_0

But then $\tau_{n_0}^\ell \geq t_0 + \varepsilon$ Contradiction! ▣

<u>Note</u> Then $X|([\tau_n, \tau_{n+1}]\cap(\mathbb{R}_+ \times \{\tau_n < \tau_{n+1}\})$ semimartingale $\forall n$, i.e. $\forall n$,

$X_t^n = X_{(\tau_n + t)\wedge \tau_{n+1}}$, $t \geq 0$ is a semimartingale

w/r to the shifted filtration $(\mathcal{F}_t^n)_{t \geq 0}$

$$\mathcal{F}_t^n := \mathcal{F}_{\tau_n + t}$$

Lemma For each mf M $\exists\ h^1,\dots,h^\ell \in \mathcal{C}^\infty(M)$ s.th.

i) Each $f \in \mathcal{C}^\infty(M)$ factorizes as $f = \bar{f}\circ(h^1,\dots,h^\ell)$ for some $\bar{f} \in \mathcal{C}^\infty(\mathbb{R}^\ell)$

ii) Each $b \in \Gamma(T^*M \otimes T^*M)$ writes as

$$b = \sum_{i,j=1}^{\ell} b_{ij}\, dh^i \otimes dh^j \quad \text{where} \quad b_{ij} \in \mathcal{C}^\infty(M)$$

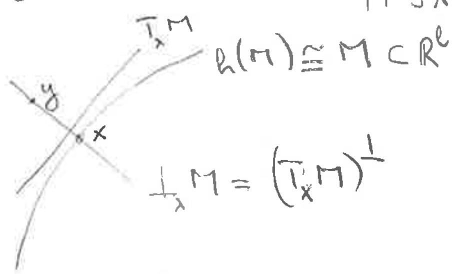
iii) Each $\alpha \in \Gamma(T^*M)$ writes as

$$\alpha = \sum_{i=1}^{\ell} \alpha_i\, dh^i \quad \text{where} \quad \alpha_i \in \mathcal{C}^\infty(M)$$

<u>Pr</u> $M \overset{h}{\hookrightarrow} \mathbb{R}^\ell$ Whitney embedding (as closed sub-mf)

$\exists$ partition $(\varphi_\lambda)_{\lambda\in\Lambda}$ of $\mathbb{1}$ on M and a family $(I_\lambda)_{\lambda\in\Lambda}$ of subsets $I_\lambda \subset \{1,\dots,\ell\}$ such that:

$\forall \lambda\in\Lambda$, $(h^i)_{i\in I_\lambda}$ is a chart on M on some open neighbourhood of $\operatorname{supp}\varphi_\lambda$

i) 

Define $\bar{f}|h(M)$ by $f = \bar{f}\circ h$

$\bar{f}(y) := \bar{f}(x)\,\phi(y)$, $y \in \perp_x M$, $\phi \in \mathcal{C}^\infty(\mathbb{R}^\ell)$

$\phi \equiv 1$ locally about $h(M)$

$\phi \equiv 0$ on a slightly larger neighbourhood

ii) $$\varphi_\lambda b = \sum_{i,j=1}^{\ell} b^\lambda_{ij}\, dh^i \otimes dh^j$$ where $b^\lambda_{ij} \in \mathcal{C}^\infty(M)$:

$\operatorname{supp} b^\lambda_{ij} \subset \operatorname{supp}\varphi_\lambda$

$b^\lambda_{ij} = 0$ for $\{i,j\} \not\subset I_\lambda$

$$\Rightarrow b = \sum_{i,j=1}^{\ell} b_{ij}\, dh^i \otimes dh^j \quad \text{where} \quad b_{ij} := \sum_\lambda b^\lambda_{ij}$$

iii) analogously to ii)

Given a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0}\right)$

$\mathcal{S}$ - vector space of real cont semimartingales

$\mathcal{S} = \mathcal{M} \oplus \mathcal{A}$ where $\mathcal{M}$ = space of cont local martingales

$\mathcal{A} = \{A : A$ cont adapted, pathwise locally of bounded variation, $A_0 = 0$ a.s. $\}$

Theorem 1 M smooth mf & X cont. M-valued semimartingale

Then $\exists_1$ linear mapping

$$\Gamma(T^*M \otimes T^*M) \longrightarrow \mathcal{A}, \quad b \longmapsto \int b(dX, dX) =: \mathbb{I}_X(b)$$

s.th. $\forall f, g \in \mathcal{C}^\infty(M)$

i) $df \otimes dg \longrightarrow [f\circ X, g \circ X]$

ii) $f \cdot b \longrightarrow \int (f \circ X)\, \underbrace{b(dX, dX)}_{:= d\mathbb{I}_X(b)}$

Pr $b \in \Gamma(T^*M \otimes T^*M) \xrightarrow{\text{Lemma}} b = \sum_{\text{finite}} b_{ij}\, dh^i \otimes dh^j$

$\leadsto \int b(dX, dX) = \sum (b_{ij} \circ X)\, d[h^i \circ X, h^j \circ X] \quad (*)$

Uniqueness ✓

Existence: to show $(*)$ well-defined

Let $b = \sum_{\text{finite}} u_\nu\, df^\nu \otimes dg^\nu = 0$, to show $\sum (u_\nu \circ X)\, d[f^\nu \circ X, g^\nu \circ X] \equiv 0$

without restriction: (h, M) global chart (localization lemma)

$u_\nu = \bar{u}_\nu \circ h$, $f_\nu = \bar{f}_\nu \circ h$, $g_\nu = \bar{g}_\nu \circ h$ where $\bar{u}_\nu, \bar{f}_\nu, \bar{g}_\nu \in \mathcal{C}^\infty(\mathbb{R}^r)$

$\leadsto \bar{X} := h \circ X$

$$\Rightarrow \sum_\nu (u_\nu \circ X)\, d[f^\nu \circ X, g^\nu \circ X] = \sum_\nu (\bar{u}_\nu \circ \bar{X})\, d[\bar{f}^\nu \circ \bar{X}, \bar{g}^\nu \circ \bar{X}]$$

$$= \sum_{i,j} \underbrace{\sum_\nu (\bar{u}_\nu \circ \bar{X})(D_i \bar{f}^\nu \circ \bar{X})(D_j \bar{g}^\nu \circ X)}\, d[\bar{X}^i, \bar{X}^j] = 0$$

$$= \left(\sum_\nu u_\nu\, df^\nu \otimes dg^\nu\right)_X \left(\frac{\partial}{\partial h^i}\Big|_X, \frac{\partial}{\partial h^j}\Big|_X\right) = b_X\left(\frac{\partial}{\partial h^i}\Big|_X, \frac{\partial}{\partial h^j}\Big|_X\right) = 0$$

<u>Def</u> $\int b(dX,dX)$ is called <u>integral of b along X</u>
or <u>b-quadratic variation</u>

We write $\int_0^t b(dX,dX)$ instead of $\left(\int b(dX,dX)\right)_t$

<u>Rem</u> $\int b(dX,dX)$ depends only on the symmetric part of b
(in particular: b antisymmetric $\Rightarrow \int b(dX,dX) = 0$)
$\bar{b}(A,B) := b(B,A)$
$b \mapsto \int \bar{b}(dX,dX)$ also satisfies the defining properties i, & ii,

<u>Note</u>: $b = b_{sym} + b_{antisym}$
$b_{sym}(v,w) = \frac{b(v,w)+b(w,v)}{2}$
$b_{antisym}(v,w) = \frac{b(v,w)-b(w,v)}{2}$ $\quad v,w \in T_xM$

<u>Rem</u> (pullback formula)

Let $M \xrightarrow{\phi} N$ & $b \in \Gamma(T^*N \otimes T^*N)$

$\rightsquigarrow \phi^* b \in \Gamma(T^*M \otimes T^*M)$, $(\phi^* b)_x (A,B) = b_{\phi(x)}(d\phi_x \cdot A, d\phi_x \cdot B)$
$\forall A,B \in T_xM$

Let X be a continuous M-valued semimartingale

Then

$$\int (\phi^* b)(dX,dX) = \int b(d\,\phi\circ X ; d\,\phi\circ X)$$

<u>Pr</u>. $b \mapsto \int (\phi^* b)(dX,dX)$ satisfies the defining properties
of the b-quadratic variation along $\phi\circ X$
Clear, since $\phi^*(df \otimes dg) = d(f\circ\phi) \otimes d(g\circ\phi)$

Theorem 2 M smooth mf & X cont. M-valued semimartingale

Then $\exists_1$ linear mapping

$$\Gamma(T^*M) \longrightarrow \mathcal{S}, \quad \alpha \longmapsto \int_X \alpha \;\hat{=}\; \int \alpha(\circ dX) = I_X(\alpha)$$

s.th. $\forall f \in C^\infty(M)$,

i) $df \longmapsto f(X) - f(X_0)$

ii) $f\alpha \longmapsto \int f(X) \circ d\, I_X(\alpha)$

$\int_X \alpha$ is called Stratonovich integral of α along X

Pr $\alpha \in \Gamma(T^*M) \rightsquigarrow \alpha = \sum_i \alpha_i\, dh^i$ where $\alpha_i \in C^\infty(M)$

uniqueness obvious: $\int_X \alpha = \sum_i \alpha_i(X) \circ d(h^i(X))$ $(*)$

To show: $(*)$ is well-defined

Assume $\alpha = \sum_{\text{finite}} u_\nu\, df^\nu = 0$ to show $\sum_\nu u_\nu(X) \circ d(f^\nu(X)) = 0$

without restriction again (h, M) global chart

$$\sum_\nu (u_\nu(X)) \circ d(f^\nu(X)) = \sum_\nu (\bar{u}_\nu(\bar{X})) \circ d(\bar{f}_\nu(\bar{X}))$$

$$= \sum_i \underbrace{\sum_\nu (\bar{u}_\nu(\bar{X}))\, (D_i \bar{f}^\nu(\bar{X}))}_{= \left(\sum u_\nu df^\nu\right)_X \left(\frac{\partial}{\partial h^i}\big|_X\right) = \alpha_X\left(\frac{\partial}{\partial h^i}\big|_X\right) = 0} \circ d\bar{X}^i = 0$$

Pullback formula

$M \xrightarrow{\phi} N$ C^∞ & $\alpha \in \Gamma(T^*N)$

$\rightsquigarrow \phi^*\alpha \in \Gamma(T^*M)$, $(\phi^*\alpha)_x A = \alpha_{\phi(x)}(d\phi_x \cdot A)$, $A \in T_xM$

Then $\int_X \phi^*\alpha = \int_{\phi \circ X} \alpha$

Pr $\alpha \longmapsto \int_X \phi^*\alpha$ satisfies the defining properties of the Stratonovich of α along $\phi \circ X$

clear, since $\phi^* df = d(f \circ \phi)$

Ex. $X_t = x(t)$ deterministic C^1-curve in M

$$\Rightarrow \int_X \alpha = \int \alpha(\dot{x}(t))\,dt$$ line integral of α along $t \mapsto x(t)$

Defining properties

e.g. $\alpha = df$ $\quad \int df(\dot{x}(t))\,dt = \int (f\circ x)^{\cdot}(t)\,dt = f(x(\cdot)) - f(x(0))$

Rem $\int b(dX, dX)$ and $\int_X \alpha$ are compatible with time-change.

More precisely: X cont semimartingale taking values in M

$(\tau_t)_{t\geq 0}$ continuous finite time-change $\rightsquigarrow \hat{X} : \hat{X}_t = X_{\tau_t}$ w/r to the time-changed filtration $\hat{\mathcal{F}}_t = \mathcal{F}_{\tau_t}$

Then $$\int b(d\hat{X}, d\hat{X}) = \left(\int b(dX, dX)\right)^{\wedge}$$

$$\int_{\hat{X}} \alpha = \left(\int_X \alpha\right)^{\wedge}$$

Pr The right-hand sides have the defining properties ∎

In particular, if τ is an arbitrary stopping time,

$X^\tau : X^\tau_t = X_{\tau\wedge t}$, then

$$\int b(dX^\tau, dX^\tau) = \left(\int b(dX, dX)\right)^\tau$$

$$\int_{X^\tau} \alpha = \left(\int_X \alpha\right)^\tau$$

Slightly more generally we have the following:

Let X be a continuous M-valued semimartingale

<u>Thm 1'</u> Let $\mathbb{B}$ = vector space of continuous adapted $T^*M \otimes T^*M$-valued processes B over X

Then $\exists_1$ linear mapping

$$\mathbb{B} \longrightarrow \mathcal{A}, \qquad B \longmapsto \int B(dX, dX),$$

s.th.

$$b \circ X \longmapsto \int b(dX, dX) \qquad \forall\, b \in \Gamma(T^*M \otimes T^*M)$$

$$K \cdot B \longmapsto \int K \underbrace{b(dX, dX)}_{= d \int B(dX, dX)} \qquad \forall \text{ cont. adapted real process } K$$

<u>Thm 2'</u> Let $\mathbb{D}$ = vector space of continuous adapted T^*M-valued processes J over X

Then $\exists_1$ linear mapping

$$\mathbb{D} \longrightarrow \mathcal{S}, \qquad J \longmapsto \int_X J \equiv \int J(\circ dX),$$

s.th.

$$\alpha \circ X \longmapsto \int_X \alpha \equiv \int \alpha(\circ dX) \qquad \forall\, \alpha \in \Gamma(T^*M)$$

$$K \cdot J \longmapsto \int K \circ \underbrace{J(\circ dX)}_{\equiv d\left(\int J(\circ dX)\right)} \qquad \forall \text{ cont. adapted real process } K$$

<u>Pr.</u> Exercise

Ex (Winding of a semimartingale in the plane)

Let $\alpha \in \Gamma(T^*M \otimes \mathbb{C})$ be a complex differential form on M,

i.e. $\alpha = \alpha_1 + i\alpha_2$, $\alpha_i \in \Gamma(T^*M)$

Define $\int_Z \alpha := \int_Z \alpha_1 + i \int_Z \alpha_2$

Let Z be semimartingale taking values in $\mathbb{C}$ s.th.

$Z_0 \neq 0$ & Z doesn't hit 0 a.s.

Write $Z_t = |Z_t| e^{i\Theta_t}$ with Θ_t a pathwise continuous version of $\arg(Z_t)$

Consider the complex differential form

$$\alpha = \frac{dz}{z} \quad \text{on} \quad \mathbb{C} \setminus \{0\}$$

Then $\int_Z \alpha$ is well-defined (as a semimartingale in $\mathbb{C} \setminus \{0\}$)

Claim

$$\boxed{\Theta = \Theta_0 + \underbrace{\operatorname{Im} \int_Z \alpha}_{\text{winding of the semimartingale } Z \text{ about } 0}}$$

Pr To show $\exp\left(\int_Z \alpha\right) = \frac{Z}{Z_0}$

Indeed let $L = \int_Z \alpha \underset{\alpha = \frac{dz}{z}}{=} \int \frac{1}{Z} \circ dZ$

$$\Longrightarrow de^L = e^L \circ dL = \frac{e^L}{Z} \circ dZ$$

$$d\left(\frac{e^L}{Z}\right) = e^L \left(-\frac{1}{Z^2}\right) \circ dZ + \frac{1}{Z} \frac{e^L}{Z} \circ dZ = 0 \quad \blacksquare$$

In other words,

$$\log Z_t - \log Z_0 := \int_Z \frac{dz}{z}$$

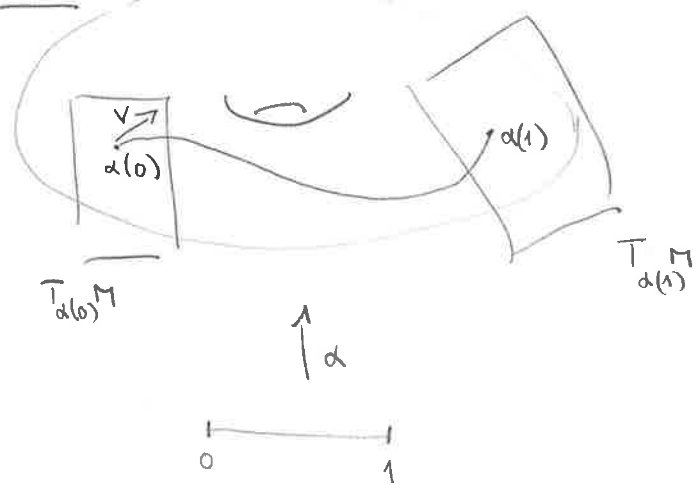
gives a continuous version of a logarithm along the paths of Z

3. Linear connections and martingales on manifolds

Let E be a vector bundle over M, e.g. $E = TM$

Let $\alpha : [0,1] \to M$ $\mathcal{C}^\infty$ curve

$\boxed{E = TM}$



Want to have a canonical procedure to translate (transport) $v \in T_{\alpha(0)}M$ to $T_{\alpha(1)}M$ along α

If we have in addition a Riemannian metric on M, then the translation should preserve angles.

Necessary structure: linear connection in E

There are different (but equivalent) ways to introduce linear connections in E:

- parallel transport $\quad //_{0,t} : E_{\alpha(0)} \to E_{\alpha(t)}$
- covariant derivative on E
- horizontal splitting of TE : $TE = \pi^* E \oplus H$

Recall M, N diffb. mfs & $\varphi \in \mathcal{C}^\infty(N,M)$, $E \xrightarrow{\pi} M$ v.b. $/M$

$$\begin{array}{ccc} \varphi^*E & & E \\ \downarrow & & \downarrow \pi \\ N & \xrightarrow{\varphi} & M \end{array}$$

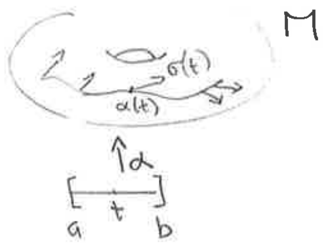
Then $\varphi^* E$ v.b. $/N$ (pullback of the v.b. E via φ).

$(\varphi^* E)_x = E_{\varphi(x)} \quad \forall\ x \in N$

Ex $N = [a,b]$, $\alpha \in \mathcal{C}^\infty([a,b], M)$

$\sigma \in \Gamma(\varphi^* E)$, i.e.

$\sigma(t) \in E_{\alpha(t)}$ vector field along α



<u>Def</u> Let $E \xrightarrow{\pi} M$ be a vector bundle $/M$

A <u>parallel transport</u> in E assigns to each differentiable curve α from p to q in M a linear isomorphism

$$L_\alpha : E_p \longrightarrow E_q \quad \text{s.th.}$$

i) (invariance under <u>reparametrisation</u>)

$\alpha : [a,b] \to M$ $\mathcal{C}^\infty$ & $\varphi : [a',b'] \to [a,b]$ $\mathcal{C}^\infty$ s.th. $\varphi(a') = a$, $\varphi(b') = b$

$\Rightarrow$ $L_{\alpha\circ\varphi} = L_\alpha$

(<u>transitivity</u>)

$a \leq c \leq b \Rightarrow L_{\alpha|[c,b]} \circ L_{\alpha|[a,c]} = L_\alpha$

(behavior under <u>backtransport</u>)

ii) $\alpha^- : [a,b] \to M$, $t \mapsto \alpha(a+b-t)$ $\Rightarrow$ $L_{\alpha^-} = L_\alpha^{-1}$

iii) If α depends differentiably on some parameters then L_α as well (<u>first-order-axiom</u>)

$\forall X \in \Gamma(E)$ & $v \in T_pM$ the <u>covariant derivative</u> of X <u>in direction v</u> <u>w/r</u> to L

$$\nabla_v X = \nabla_D (X\circ\alpha)(0) \in E_p \quad , \quad \alpha : [-\varepsilon,\varepsilon] \to M \ \mathcal{C}^\infty\text{-curve}$$
$$\alpha(0) = p \ \& \ \dot\alpha(0) = v$$

is well-defined & independent of the choice of α

<u>Here</u>: $\alpha : [a,b] \to M$ $\mathcal{C}^\infty$ & $\sigma \in \Gamma(\alpha^* E)$

$\rightsquigarrow$ $\nabla_D \sigma \in \Gamma(\alpha^* E)$, $(\nabla_D\sigma)(t) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L^{-1}_{\alpha|[t,t+\varepsilon]} \sigma(t+\varepsilon) \in E_{\alpha(t)}$

"σ <u>parallel along</u> α" if $\nabla_D \sigma \equiv 0$ along α

Technical lemma

E, F v.b. $/M$ & $K : \Gamma(E) \to \Gamma(F)$ $\mathcal{C}^\infty$-linear

$A, B \in \Gamma(E)$, $p \in M$

Then $A_p = B_p \implies K(A)_p = K(B)_p$

Thus $K \in \Gamma(\mathrm{Hom}(E,F)) \equiv \Gamma(E^* \otimes F)$

Indeed. To show $A_p = 0 \implies K(A)_p = 0$

choose $e_1, \dots, e_m \in \Gamma(E|_U)$ local frame at p

Then $A|_U = \sum_{i=1}^m a^i e_i$, $a^i \in \mathcal{C}^\infty(U)$

$\implies a^1(p) = \dots = a^m(p) = 0$

Let $\psi \in \mathcal{C}^\infty(M)$, $\psi(p) = 1$ & $\operatorname{supp}\psi \subset U$

$\bar{e}_i := \psi e_i \in \Gamma(E)$, $\bar{a}^i := \psi a^i$ $\mathcal{C}^\infty(M)$, well-defined ($:= 0$ outside of U)

$\implies \psi^2 A = \sum_i \bar{a}^i \bar{e}_i$

$\implies K(A)_p = \psi^2(p) K(A)_p = K(\psi^2 A)_p = \sum_i \bar{a}^i(p) K(\bar{e}_i)_p = 0$ ∎

Def E v.b. $/M$

A covariant derivative on E is a $\mathbb{R}$-linear mapping

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

satisfying the product rule

$$\nabla(fX) = df \otimes X + f \nabla X \qquad \forall X \in \Gamma(E), f \in \mathcal{C}^\infty(M)$$

$X \in \Gamma(E)$ "parallel" if $\nabla X = 0$

Rem Since $\Gamma(T^*M \otimes E) \equiv \mathrm{Hom}_{\mathcal{C}^\infty(M)}(\Gamma(TM), \Gamma(E))$ by the Lemma, a covariant derivative ∇ can be seen as an $\mathbb{R}$-linear mapping

$$\Gamma(TM) \times \Gamma(E) \to \Gamma(E), \quad (A, X) \longmapsto \nabla_A X := (\nabla X)(A)$$

Then

$$\left.\begin{aligned} \nabla_{fA} X &= f \nabla_A X \\ \nabla_A (fX) &= (Af) X + f \nabla_A X \end{aligned}\right\} \quad \forall A \in \Gamma(TM),\ X \in \Gamma(E),\ f \in \mathcal{C}^\infty(M)$$

In addition

$(\nabla_A X)_p$ depends only on $A_p \in T_pM$

Hence, for $v \in T_pM$ choosing $A \in \Gamma(TM)$ s.th. $A_p = v$,
then $\nabla_v X := (\nabla_A X)_p \in E_p$ is well-defined (i.e. indep. of the choice of A)

<u>covariant derivative of X in direction v</u>

Rem $\nabla_v X$ depends only on X locally about p

Indeed $p \in U \subset M$, U open & $X|_U = 0$

Let $\psi \in \mathcal{C}^\infty(M)$ s.th. $\mathrm{supp}\,\psi \subset U$, $\psi(p) = 1$

$\Rightarrow \psi X \equiv 0$

$\Rightarrow 0 = (\nabla_A(\psi X))_p = (A\psi)(p) \underbrace{X_p}_{=0} + \underbrace{\psi(p)}_{=1} (\nabla_A X)_p$

$\Rightarrow (\nabla_A X)_p = \nabla_v X = 0$

Lemma $N \xrightarrow{f} M$ $\mathcal{C}^\infty$ & E v.b. $/M$

$$\begin{array}{ccc} f^*E & & E \\ \vdots & & \downarrow \pi \\ N & \xrightarrow{f} & M \end{array}$$

$X \in \Gamma(E) \rightsquigarrow f^*X \in \Gamma(f^*E)$, $(f^*X)_p = X_{f(p)}$, $p \in N$

To each covariant derivative ∇ on E $\exists_1$ covariant derivative on f^*E (again denoted by ∇) s.th.

$$\nabla_w(f^*X) = \nabla_{df_p w} X \in E_{f(p)} \quad \forall\, X \in \Gamma(E),\ w \in T_pN,\ p \in N$$

Pr. Exercise

Def ∇ covariant derivative on a v.b. E over M & $\alpha : I \to M$ $\mathcal{C}^\infty$ curve

(a) $X \in \Gamma(\alpha^*E) \rightsquigarrow \nabla_D X \in \Gamma(\alpha^*E)$ covariant derivative of X along α

$D :=$ canonical v.f. on I

(b) $X \in \Gamma(\alpha^*E)$ parallel along α (w/r to ∇) if $\nabla_D X \equiv 0$

Def ∇ covariant derivative on $E = TM$ & $\gamma : I \to M$ $\mathcal{C}^\infty$

γ geodesic if $\dot\gamma \in \Gamma(\gamma^*TM)$ is parallel along γ (w/r to ∇) (i.e. $\nabla_D \dot\gamma \equiv 0$)

Theorem ∇ covariant derivative on E, $\alpha : I \to M$ $\mathcal{C}^\infty$ curve , $e \in E_{\alpha(t_0)}$, $t_0 \in I$

$\Rightarrow \exists_1$ parallel $X \in \Gamma(\alpha^*E)$ s.th. $X(t_0) = e$

$s,t \in I$: $/\!/_{s,t} : E_{\alpha(s)} \to E_{\alpha(t)}$, $/\!/_{s,t}\, e = X(t)$, linear isomorphisms

$/\!/_{s,t}^{-1} = /\!/_{t,s}$, $/\!/_{t,t} = \mathrm{id}_{E_{\alpha(t)}}$

Rem $/\!/$-transport in E (as defined at the beginning of the Section) & covariant derivative in E define equivalent structures on E: if $X \in \Gamma(E)$, $v \in T_pM$ & $\alpha : I \to M$ C^∞ curve s.th. $\dot\alpha(0) = v$, then

$$\nabla_v X = \frac{d}{dt}\Big|_{t=0} \left(/\!/_{0,t}^{-1} X_{\alpha(t)} \right)$$

We talk about a linear connection in E (if $E = TM$ also "linear connection on M")

3rd equivalent structure: horizontal splitting of TE

Def $E \xrightarrow{\pi} M$ v.b.

A subbundle $H \subset TE$ is called horizontal splitting of TE if

i) $TE = H \oplus \pi^* E$ (i.e. $\forall e \in E$, $T_eE = H_e \oplus E_{\pi(e)}$)

ii) $\forall s \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ H is compatible with the operation $g_s : E \to E$, $e \mapsto se$, (i.e. $(dg_s) H_e = H_{se}$ $\forall e \in E$ & $s \in \mathbb{R}^*$)

Note Let $e \in E$, $\pi(e) = p$

The projection $\pi : E \to M$ is submersive at e,

i.e. $(d\pi)_e : T_eE \to T_pM$ surj.

& $\ker (d\pi)_e = T_e(\pi^{-1}p) = T_e(E_p) \cong E_p \subset T_eE$

This gives an exact sequence of v.b. over E

$$(*) \qquad 0 \to \pi^*E \to TE \underset{h}{\overset{d\pi}{\rightleftarrows}} \pi^*TM \to 0 \qquad \boxed{d\pi \circ h = id}$$

A horizontal splitting of TE gives a splitting of the sequence $(*)$:

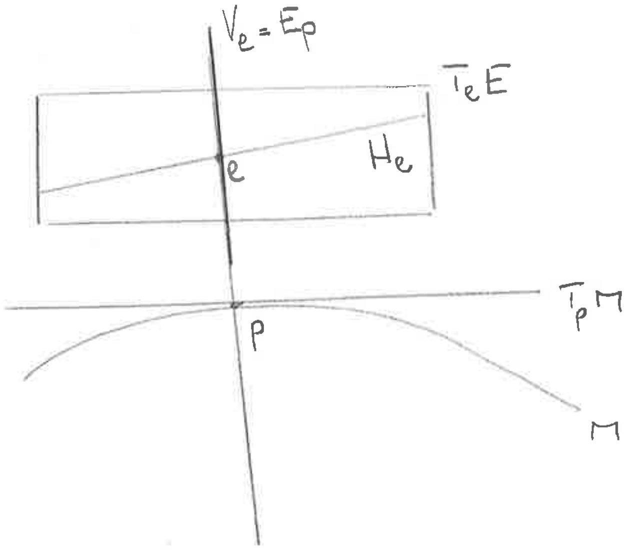
$$h = (d\pi|_H)^{-1}$$

$h : \pi^*TM \xrightarrow{\sim} H \subset TE$ "horizontal lift"

(Note that fibrewise

$$(d\pi)_e : H_e \xrightarrow{\sim} T_pM \quad \& \quad h_e : T_pM \xrightarrow{\sim} H_e)$$

$TE = H \oplus V$ where $V = \pi^* E$



$w \in T_eE$: w "horizontal" if $w \in H_e$

w "vertical" if $w \in V_e \equiv E_p$

in the sense "tangential to the submanifold E_p of E"

Proposition

1. ∇ covariant derivative on E $\longrightarrow$ horizontal splitting of TE

For $e \in E$, $\pi(e) = p$,

$$H_e := \left\{ (dX)_p v \;\middle|\; v \in T_pM,\ X \in \Gamma(E) \text{ with } X(p) = e \;\&\; \nabla_v X = 0 \right\} \subset T_eE$$

2. horizontal splitting of TE $\longrightarrow$ covariant derivative ∇ on E:

For $X \in \Gamma(E)$ the covariant derivative $\nabla X \in \Gamma(T^*M \otimes E)$ is given by

$$TM \xrightarrow{dX} X^*TE \equiv X^*H \oplus X^*V \xrightarrow{pr_V} X^*V = X^*\pi^*E = E$$

In particular, for $\sigma \in \Gamma(\alpha^*E)$, α C^∞-curve in M,

$\nabla_D \sigma \in \Gamma(\alpha^*E)$ is defined via:

$$\begin{array}{lll} \mathbb{R} \longrightarrow \sigma^*TE \equiv \sigma^*H \oplus \sigma^*V \xrightarrow{pr_V} \sigma^*V = \sigma^*\pi^*E = E \\ t \longmapsto \dot\sigma(t) \longmapsto (\nabla_D\sigma)(t) \in E_{\alpha(t)} \end{array}$$

1. & 2. are inverse to each other

Cor $E \xrightarrow{\pi} M$ v.b $/M$

i) $X \in \Gamma(E)$ parallel

$\iff \nabla_v X = 0 \quad \forall v \in TM$

$\iff (dX)v \in H \quad \forall v \in TM$

ii) $\sigma \in \Gamma(\alpha^* E)$ parallel

$\iff \nabla_D \sigma \equiv 0$

$\iff \dot{\sigma}(t) \in H_{\sigma(t)} \quad \forall t \in I$

In particular: given $\alpha : I \to M$ C^∞ curve & $e \in E_{\alpha(t_0)}$ with $t_0 \in I$, then $\exists_1$ lift of α to a horizontal curve $u : I \to E$ with $u(t_0) = e$

(i.e. $\pi \circ u = \alpha$, $\dot{u}(t) \in H_{u(t)}$ $\forall t$, $u(t_0) = e$)

Linear connection in E
- covariant derivative on E
- parallel transport in E
- horizontal splitting of TE

Rem Every covariant derivative on E induces a covariant derivative on E^*.

$$\nabla : \Gamma(TM) \times \Gamma(E^*) \longrightarrow \Gamma(E^*)$$
$$(A,b) \longmapsto \nabla_A b : (\nabla_A b)(B) = A(bB) - b(\nabla_A B), \quad B \in \Gamma(E)$$

In particular,

$E = TM$, $\alpha \in \Gamma(T^*M)$

$$\nabla\alpha \in \Gamma(T^*M \otimes T^*M), \quad (\nabla\alpha)(A,B) := (\nabla_A \alpha)(B), \quad A, B \in \Gamma(TM)$$

Ex $\alpha = df$, $f \in \mathcal{C}^\infty(M)$

$$\rightsquigarrow \nabla df \equiv \mathrm{Hess}(f) \in \Gamma(T^*M \otimes T^*M)$$ Hessian of f

$$\nabla df(A,B) = ABf - (\nabla_A B) f$$ second fundamental form of f

Def Let ∇ be a covariant derivative on TM ("linear connection on M")

X cont. M-valued semimartingale

X ∇-martingale

$$:\Longleftrightarrow \quad d(f(X)) - \tfrac{1}{2} (\nabla df)(dX, dX) \overset{m}{=} 0 \qquad \forall\, f \in \mathcal{C}^\infty(M)$$

<u>Remark</u> $M = \mathbb{R}^n$, ∇ canonical linear connection on $\mathbb{R}^n$.

$\nabla_{D_i} D_j \equiv 0$ where $D_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$

$\Longrightarrow (\nabla df)(D_i, D_j) = D_i D_j f$

<u>Then</u>: ∇-martingales $\hat{=}$ continuous local martingales on $\mathbb{R}^n$

Indeed; by Itô's[1] formula,
for a continuous $\mathbb{R}^n$-valued semimartingale X are equivalent:

X local martingale

$\iff \forall f \in \mathcal{C}^\infty(M)$, $d(f(X)) - \frac{1}{2} \sum_{i,j} (D_i D_j f)(X)\, d[X^i, X^j] \in d\mathcal{M}$

$\iff \forall f \in \mathcal{C}^\infty(M)$, $d(f(X)) - \frac{1}{2} \nabla d f\,(dX, dX) \overset{m}{=} 0$

Definition (torsion)

M mf. & ∇ linear connection on M

$$T : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM),$$
$$(X,Y) \longmapsto \nabla_X Y - \nabla_Y X - [X,Y] \qquad \mathcal{C}^\infty(M)\text{-linear}$$

$T \in \Gamma(T^*M \otimes T^*M \otimes TM)$ torsion tensor

Here: $[X,Y] := XY - YX \in \Gamma(TM)$ Lie product

∇ torsion-free (symmetric) if $T \equiv 0$

Remark $\nabla df \in \Gamma(T^*M \otimes T^*M)$,
$$\nabla df(A,B) = ABf - (\nabla_A B)f \quad , \quad A,B \in \Gamma(TM)$$

∇df symmetric $\forall f \iff \nabla$ is torsion-free

Remark A connection ∇ with torsion can be symmetrized:

$$\nabla \longmapsto \bar{\nabla} : \bar{\nabla}_A B := \frac{1}{2}\left(\nabla_A B + \nabla_B A + [A,B]\right)$$
$$\equiv \nabla_A B - \frac{1}{2} T(A,B) \quad , \quad \bar{\nabla} \text{ torsion-free}$$

Then $(\bar{\nabla} df)(A,B) = \frac{1}{2}\left[\nabla df(A,B) + \nabla df(B,A)\right]$

$\nabla df(dX,dX)$ depends only on the symmetric part of ∇df

$\implies$ ∇-martingales $\hat{=}$ $\bar{\nabla}$-martingales

Ex (∇-martingales as solutions of SDEs)

M mf equipped with a linear connection ∇ in TM (without restrictions torsion free)

X solution to

$$dX = A_0(X) + \sum_{i=1}^r A_i(X) \circ dZ^i \quad , \quad A_0, A_1, \dots, A_r \in \Gamma(TM)$$

(Z $\mathbb{R}^r$-valued continuous semimartingale)

$$\Rightarrow d(f(X)) = (A_0 f)(X)\, dt + \sum_{i=1}^r (A_i f)(X)\, dZ^i + \frac{1}{2} \sum_{i,j=1}^r (A_i A_j f)(X)\, d[Z^i, Z^j]$$

$$(\nabla df)(dX, dX) = \sum_{i,j=1}^r (\nabla df)(A_i, A_j)(X)\, d[Z^i, Z^j]$$

$$\nabla df(A_i, A_j) = A_i A_j f - (\nabla_{A_i} A_j) f$$

$$\Rightarrow d(f(X)) - \frac{1}{2}(\nabla df)(dX, dX)$$
$$= (A_0 f)(X)\, dt + \sum_{i=1}^r (A_i f)(X)\, dZ^i + \frac{1}{2} \sum_{i,j=1}^r (\nabla_{A_i} A_j f)(X)\, d[Z^i, Z^j]$$

Z^{drift} := drift component of Z

Then X ∇-martingale

$$\Leftrightarrow (A_0 f)(X)\, dt + \sum_{i=1}^r (A_i f)(X)\, d(Z^{drift})^i + \frac{1}{2} \sum_{i,j=1}^r (\nabla_{A_i} A_j f)(X)\, d[Z^i, Z^j] = 0$$

In particular: Z $BM(\mathbb{R}^r)$

$$\Rightarrow X \ \nabla\text{-martingale if } A_0 = -\frac{1}{2} \sum_{i=1}^r \nabla_{A_i} A_i$$

4. Riemannian metrics and Brownian motions

A <u>Riemannian manifold</u> (M,g) is a diffb mf M equipped with a <u>Riemannian metric</u> g on TM, i.e.

$g \in \Gamma(T^*M \otimes T^*M)$ s.th.

$\forall x \in M$, $g_x : T_xM \times T_xM \to \mathbb{R}$ symmetr. & positive definite

<u>Notation</u>

$$g_x = \langle \, , \rangle_x \equiv \langle \, , \rangle_{T_xM}$$

$$(M,g) \equiv (M, \langle \, , \rangle)$$

$$|A| = \sqrt{g(A,A)} \, , \quad A \in \Gamma(TM)$$

$\alpha \; [a,b] \to M$ C^∞ curve : $\text{length}(\alpha) := \int_a^b |\dot\alpha(t)|_{\alpha(t)} \, dt$

<u>Def</u> X cont. semimartingale taking values in a Riem mf (M,g). Then

$$[X,X] = \int g(dX,dX) \equiv \int \langle dX, dX \rangle$$

is called <u>Riemannian quadratic variation</u> of X

<u>Def</u> (M,g) Riem mf & ∇ linear connection on M

∇ is called a <u>Riemannian connection</u> if

$$Z\langle A,B \rangle = \langle \nabla_Z A, B \rangle + \langle A, \nabla_Z B \rangle \quad \forall A,B,Z \in \Gamma(TM)$$

<u>Proposition</u> (M,g) Riem mf & ∇ lin. connection on M

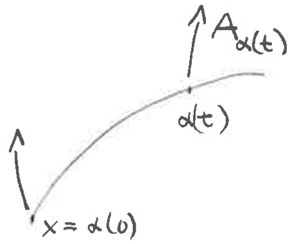
Equivalent:

i) ∇ Riem. connection

ii) Parallel transports $/\!/_{0,t} : T_{\alpha(0)}M \to T_{\alpha(t)}M$ along diffb. curves are isometries

Pr $x \in M$ & α $\mathcal{C}^\infty$-curve, $\alpha(0) = x$, $\dot\alpha(0) = v \in T_xM$

$$\rightsquigarrow \nabla_v A = \frac{d}{dt}\Big|_{t=0} /\!/_{0,t}^{-1} A_{\alpha(t)} , \quad A \in \Gamma(TM)$$



ii) => i) Let $v = z_x$

$$\left(v(f) = \frac{d}{dt}\Big|_{t=0} f(\alpha(t)) , \ f \in \mathcal{C}^\infty(M) \right)$$

$$\underbrace{v\langle A, B\rangle}_{=f} = \frac{d}{dt}\Big|_{t=0} \langle A_{\alpha(t)}, B_{\alpha(t)} \rangle_{T_{\alpha(t)}M}$$

$$= \frac{d}{dt}\Big|_{t=0} \langle /\!/_{0,t}^{-1} A_{\alpha(t)}, /\!/_{0,t}^{-1} B_{\alpha(t)} \rangle_{T_{\alpha(0)}M}$$

$$= \langle \underbrace{\frac{d}{dt}\Big|_{t=0} /\!/_{0,t}^{-1} A_{\alpha(t)}}_{= \nabla_v A}, B_{\alpha(0)} \rangle + \langle A_{\alpha(0)}, \underbrace{\frac{d}{dt}\Big|_{t=0} /\!/_{0,t}^{-1} B_{\alpha(t)}}_{= \nabla_v B} \rangle$$

i) => ii) Ex.

Theorem (of Levi-Civita)

On a Riem. mf (M,g) $\exists_1$ torsion-free Riem. connection ∇ (Levi-Civita connection)

Def (M,g) Riem mf & ∇ LC-connection

$f \in \mathcal{C}^\infty(M) \rightsquigarrow \nabla df \in \Gamma(T^*M \otimes T^*M)$

$\Delta f = \operatorname{trace} \nabla df \in \mathcal{C}^\infty(M)$ Laplace-Beltrami operator on M

$\Delta f(x) = \sum_{i=1}^n \nabla df(e_i, e_i)$, $(e_1, \dots, e_n)$ onb. for T_xM

Def (M,g) Riem mf, X cont. adapted M-valued process with maximal lifetime (on some filtered probability space)

X is called Brownian motion on (M,g), if

$$d(f(X)) - \tfrac{1}{2}\Delta f(X)\,dt \overset{m}{=} 0 \quad \forall f \in \mathcal{C}^\infty(M).$$

$BM(M,g) \mathrel{\hat{=}}$ class of Br. motions on (M,g)

Def (M,g) Riem. mf & ∇ Riem. connection

$f \in \mathcal{C}^\infty(M) \longrightarrow \operatorname{grad} f \in \Gamma(TM)$, $\langle \operatorname{grad} f, A\rangle := Af$, $A \in \Gamma(TM)$

Then $(\nabla df)(A,B) = \langle \nabla_A \operatorname{grad} f, B\rangle$, $A,B \in \Gamma(TM)$

Indeed: $\underbrace{A\langle \operatorname{grad} f, B\rangle}_{= ABf} \underset{\nabla \text{ Riemannian}}{=} \langle \nabla_A \operatorname{grad} f, B\rangle + \underbrace{\langle \operatorname{grad} f, \nabla_A B\rangle}_{=(\nabla_A B) f}$

$\Rightarrow \langle \nabla_A \operatorname{grad} f, B\rangle = ABf - (\nabla_A B) f = (\nabla df)(A,B)$

Lévy-Characterization of $BM(M,g)$

(M,g) Riem mf & ∇ LC-connection on M

X cont. M-valued semimartingale (of maximal lifetime)

Equivalent:

i) X $BM(M,g)$

ii) X ∇-martingale and $[f(X), f(X)] = \int |\operatorname{grad} f|^2(X)\,dt \quad \forall f \in \mathcal{C}^\infty(M)$

iii) X ∇-martingale and $\int b(dX, dX) = \int (\operatorname{trace} b)(X)\,dt \quad \forall b \in \Gamma(T^*M \otimes T^*M)$

In particular, for X $BM(M,g)$,

$$[X,X] := \int g(dX,dX) = n\cdot t \quad \text{where } n = \dim M$$

Pr

1) Claim: For an M-valued semimartingale X are equivalent

a) $[f(X), f(X)] = \int \|(\operatorname{grad} f)\|^2(X)\, dt \quad \forall f \in \mathcal{C}^\infty(M)$

b) $\int b(dX, dX) = \int (\operatorname{trace} b)(X)\, dt \quad \forall b \in \Gamma(T^*M \otimes T^*M)$

Pr $f, h \in \mathcal{C}^\infty(M)$

$$\begin{aligned}\operatorname{trace}(df \otimes dh) &= \sum_i (df \otimes dh)(e_i, e_i) \\ &= \sum_i df(e_i)\, dh(e_i) \\ &= \sum_i \langle \operatorname{grad} f, e_i\rangle \langle \operatorname{grad} h, e_i\rangle \\ &= \langle \operatorname{grad} f, \operatorname{grad} h\rangle\end{aligned}$$

b) $\Rightarrow$ a) $\quad b = df \otimes df$

a) $\Rightarrow$ b) By polarization

$$\underbrace{[f(X), h(X)]}_{= \int (df \otimes dh)(dX, dX)} = \underbrace{\int \langle \operatorname{grad} f, \operatorname{grad} h\rangle (X)\, dt}_{= \int \operatorname{trace}(df \otimes dh)(X)\, dt}$$

$$\overset{\uparrow}{\Rightarrow} \quad \int b(dX, dX) = \int (\operatorname{trace} b)(X)\, dt$$

characterization of the quadratic variation (uniqueness part)

2) iii) $\Rightarrow$ i) Let X be a ∇-martingale s.th. $b(dX, dX) = (\operatorname{trace} b)(X)\, dt$

Take $b = \nabla df$

$$\Rightarrow d(f(X)) \underset{\substack{\uparrow \\ X\ \nabla\text{-mart.}}}{=} \frac{1}{2} \nabla df(dX, dX) = \frac{1}{2} \underbrace{(\operatorname{trace} \nabla df)(X)}_{= \Delta f(X)}\, dt \quad \Rightarrow X \ BM(M, g)$$

i) $\Rightarrow$ ii) X $BM(M,g)$ & $f \in \mathcal{C}^\infty(M)$

$$\frac{1}{2}\nabla d f^2 = f \nabla d f + df \otimes df \Rightarrow \frac{1}{2}\Delta f^2 = f\Delta f + |\mathrm{grad} f|^2$$

$$\Rightarrow d(f^2(X)) \overset{m}{=} \frac{1}{2}(\Delta f^2)(X)\, dt = (f\Delta f)(X)\, dt + |\mathrm{grad} f|^2(X)\, dt$$

On the other hand, by Itô,

$$d(f^2(X)) = 2 f(X)\, d(f(X)) + d[f(X), f(X)]$$

$$\overset{m}{=} f(X)\, \Delta f(X)\, dt + d[f(X), f(X)]$$

Uniqueness of the Doob-Meyer decomposition

$$d[f(X), f(X)] = |\mathrm{grad} f|^2(X)\, dt$$

By 1 $\quad \nabla d f(dX, dX) \overset{m}{=} \underbrace{(\mathrm{tr}\, \nabla d f)(X)}_{\Delta f(X)}\, dt$

$$\Rightarrow d(f(X)) - \frac{1}{2}\nabla d f(dX, dX) \overset{m}{=} 0$$

$\Rightarrow X$ ∇-martingale

Theorem (BM as solutions of SDEs)

(M,g) Riem mf & ∇ LC-connection on M

$$(*)\quad \boxed{dX = A_0(X)\,dt + A(X)\circ dZ}$$

where $A_0 \in \Gamma(TM)$, $A : M\times\mathbb{R}^r \to TM$, $(x,e) \mapsto A(x)e$ bundle map $/M$

Z $BM(\mathbb{R}^r)$

$A(x) : \mathbb{R}^r \to T_xM$ linear $\forall x\in M$

$A(\cdot)e \in \Gamma(TM)$ $\forall e\in E$

In other words,

$$dX = A_0(X)dt + \sum_{i=1}^r A_i(X)\circ dZ^i \qquad \text{where} \quad A_i := A(\cdot)e_i \in \Gamma(TM)$$

Maximal solutions of $(*)$ are $BM(M,g)$ if

i) $A_0 = -\frac{1}{2}\sum_{i=1}^r \nabla_{A_i} A_i$

ii) $A(x)^* : T_xM \to \mathbb{R}^r$ isometr. embedding $\forall x\in M$

(i.e. $A(x)A(x)^* = \mathrm{id}_{T_xM}$ $\forall x\in M$ with $A(x)^*$ the adjoint to $A(x): \mathbb{R}^r \to T_xM$)

Pr X solution to $(*)$ $\Rightarrow$ X ∇-martingale,

i.e $d f(X) - \frac{1}{2}\underbrace{\nabla df(dX,dX)}_{} \overset{m}{=} 0 \quad \forall f\in C^\infty(M)$

$$= \sum_i \nabla df(A_i,A_i)(X)\,dt$$

To show: $\sum_i \nabla df(A_i,A_i) = \Delta f$

Fix $x\in M$ & $(a_1,\dots,a_n)$ onb of T_xM

$$\Rightarrow \Delta f(x) = (\operatorname{trace}\nabla df)(x) = \sum_i (\nabla df)_x(a_i,a_i)$$
$$= \sum_i (\nabla df)_x(A(x)A^*(x)a_i,\ A(x)A^*(x)a_i)$$

Extend $(A(x)^*a_1,\dots,A(x)^*a_n)$ to an o.n.b. $(\tilde e_1,\dots,\tilde e_r)$ of $\mathbb{R}^r$

Note that: $(\operatorname{im}A(x)^*)^\perp = \ker A(x)$

$$\Rightarrow \Delta f(x) = \sum_i (\nabla df)_x(A(x)\tilde e_i,\ A(x)\tilde e_i)$$
$$= \sum_i (\nabla df)_x(A(x)e_i,\ A(x)e_i) \qquad (e_1,\dots,e_r) \text{ standard basis of } \mathbb{R}^r$$
$$= \sum_i (\nabla df)_x(A_i(x),\ A_i(x))$$

Martingales & BM on embedded submfs

M diffb mf, $M \overset{\iota}{\hookrightarrow} \mathbb{R}^l$ embedding

Canonical connection on $\mathbb{R}^l$

$$\nabla_{D_i} D_j \equiv 0 \qquad (\text{LC-connection on } \mathbb{R}^l)$$

∇ induced connection on M

$A \in \Gamma(TM)$, $X \in \Gamma(T\mathbb{R}^l)$: $\nabla_A(\iota^* X) = \nabla_{\bar{A}} X$ where $\bar{A} = d\iota \cdot A \equiv \iota_* A$

$\nabla df(A,B) = (\mathrm{Hess}_{\mathbb{R}^l} \bar{f})(\bar{A}, \bar{B})$ where $f = \bar{f} \circ \iota$

<u>Proposition</u> (∇-martingales on M)

X cont. M-valued semimartingale, $\bar{X} = \iota \circ X$

$\bar{X} = \bar{X}_0 + N + C$ Doob-Meyer decomposition on $\bar{X}$ in $\mathbb{R}^l$

Then:

X ∇-martingale $\iff \underbrace{dC_t \perp T_{X_t} M \ \forall t \text{ a.s.}}_{\text{i.e. } \int \langle H_t, dC_t \rangle = 0 \ \forall \text{ cont. adapted process } H \text{ s.t. } H_t \in T_{X_t}M \text{ a.s}}$ ($T_{X_t}M \subset \mathbb{R}^l$ linear subspace)

<u>Pr.</u> Let $h \in \mathcal{C}^\infty(M)$

$$\underset{\text{Itô}}{\Rightarrow} \ d(h(X)) = d(\bar{h}(\bar{X}))$$
$$= \sum_i D_i \bar{h}(\bar{X})\, d\bar{X}^i + \frac{1}{2} \sum_{i,j} D_i D_j \bar{h}(\bar{X})\, d\bar{X}^i d\bar{X}^j$$
$$= \underbrace{\langle \mathrm{grad}_{\mathbb{R}^l} \bar{h}(\bar{X}), d\bar{X} \rangle}_{\overset{m}{=} \langle \mathrm{grad}_{\mathbb{R}^l} \bar{h}(\bar{X}),\, dC \rangle} + \underbrace{\frac{1}{2} (\mathrm{Hess}_{\mathbb{R}^l} \bar{h})(d\bar{X}, d\bar{X})}_{= \frac{1}{2} (\nabla dh)(dX, dX)}$$

pullback formula

Hence: X ∇-martingale $\iff \langle \mathrm{grad}_{\mathbb{R}^l} \bar{h}(\bar{X}), dC \rangle = 0 \quad \forall h \in \mathcal{C}^\infty(M)$

Applied to the coordinate functions $h^1, \dots, h^l$ of the embedding $M \overset{\iota}{\hookrightarrow} \mathbb{R}^l$ gives the claim.

$M \hookrightarrow \mathbb{R}^l$ embedded subm$f$

M is canonically a Riem mf where $g = \iota^* \mathrm{eucl}$

$g(A,B) = \mathrm{eucl}(\bar{A}, \bar{B}) := \sum_{i=1}^{l} \bar{A}_i \bar{B}_i$, where $\bar{A} = \iota_* A$
$A, B \in \Gamma(TM)$

$g_x(u,v) = \langle d\iota_x u, d\iota_x v \rangle_{\mathbb{R}^l}$, $u, v \in T_x M$

($\Rightarrow d\iota_x : T_x M \to \mathbb{R}^l$ isometry $\forall x \in M$)

$A \in \Gamma(TM) \rightsquigarrow \bar{A} = \iota_* A \in \mathcal{C}^\infty(M, \mathbb{R}^l)$

LC-connection on (M, g)

$$\nabla_v A = P(x) \, (d\bar{A})_x v \quad , \quad A \in \Gamma(TM), \ v \in T_x M$$
$P(x) : \mathbb{R}^l \to T_x M$ orth. projection

<u>Proposition</u> (BM (M,g) as solutions of SDEs)

$(*)$ $\boxed{dX = P(X) \circ dZ}$, Z BM $(\mathbb{R}^l)$

Solutions to $(*)$ are BM(M,g)

<u>Pr</u> $A(x) = P(x) : \mathbb{R}^l \to T_x M$ $(\Rightarrow A(x)^* = d\iota_x)$

$\Rightarrow A(x) A(x)^* = \mathrm{id}_{T_x M}$

To show: $\sum_i \nabla_{A_i} A_i = 0$ where $A_i = A(\cdot) e_i \in \Gamma(TM)$

Let $A^e := A(\cdot) e \in \Gamma(TM)$

To show: $\nabla_v A^e = 0 \quad \forall e \in \underbrace{\mathrm{im}\, A(x)^*}_{= (\ker A(x))^\perp}$, $v \in T_x M$

$Q(x) := A^*(x) A(x) : \mathbb{R}^l \to \mathbb{R}^l$

$Q(x)^2 = Q(x)$, $e \in \mathrm{im}\, A(x)^* \Rightarrow Q(x) e = e$

$$(dQ)_x e \underset{QQ=Q}{=} \underbrace{(dQ)_x Q(x) e}_{= (dQ)_x e} + Q(x) (dQ)_x e$$

$$\Rightarrow \underbrace{Q(x) (dQ)_x e}_{= A(x)^* A(x) d(A^*(\cdot) A(\cdot) e)_x} = 0 \quad \forall e \in \mathrm{im}\, A(x)^*$$

$$= A(x)^* A(x) \underbrace{d(\bar{A}^e)_x}_{= (\nabla A^e)_x} \Rightarrow (\nabla A^e)_v = \nabla_v A^e = 0 \quad \forall v \in T_x M$$

5. Parallel transport & stochastically moving frames

Def. G Lie group

$\pi : P \to M$ principal G-bundle over M, i.e.

i) P, M diffb. mfs & $\pi : P \to M$ $\mathcal{C}^\infty$

ii) $\exists$ atlas $(\varphi_i, U_i)_{i\in I}$ of bundle charts, i.e.

$U_i \subset M$ open, $\bigcup_{i\in I} U_i = M$ & $\varphi_i : \pi^{-1}U_i \xrightarrow{\sim} U_i \times G$ over M $\forall i \in I$,

i.e.

$$\begin{tikzcd} \pi^{-1}U_i \arrow[rr, "\varphi_i"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "///"] & U_i \times G \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

iii) If (φ_i, U_i), (φ_j, U_j) are bundle charts, then

$\exists\ \Phi_{ij} : U_i \cap U_j \to G$ $\mathcal{C}^\infty$ s.th.

$$\begin{tikzcd} & \pi^{-1}(U_i \cap U_j) \arrow[dl, "\varphi_i"'] \arrow[dr, "\varphi_j"] \arrow[d, phantom, "///"] & \\ (U_i\cap U_j)\times G \arrow[rr, dashed] & & (U_i \cap U_j)\times G \end{tikzcd}$$

$$(x,g) \longmapsto (x, \Phi_{ij}(x)g)$$

$G \triangleq$ structure group of the principal bundle

Rem $\pi : P \to M$ principle G-bundle

G operates on P from the right

$$P \times G \to P, \quad (u,a) \mapsto ua.$$

In bundle charts (φ, U) the operation is given by

$$u \in \pi^{-1}U \cong U \times G \quad \rightsquigarrow \quad \begin{array}{l} u \equiv (x,g) \\ ua \equiv (x, ga) \end{array}$$

Ex M diffb. mf, $\dim M = n$

(1) The frame bundle $\pi : L(M) \longrightarrow M$ is a principal $GL(n,\mathbb{R})$-bundle, defined as follows:

$$P = L(M) := \dot{\bigcup_{x\in M}} P_x \quad \text{where } P_x = \pi^{-1}x = \{u : \mathbb{R}^n \longrightarrow T_xM \mid u \text{ lin. isomorphism}\}$$

Identify $u \in P_x$ with $(u_1, \dots, u_n) := (ue_1, \dots, ue_n)$ basis of T_xM

Bundle charts (φ, U) for $L(M)$ are obtained from charts (h,U) for M as follows:

$\left(\frac{\partial}{\partial h^1}, \dots, \frac{\partial}{\partial h^n}\right)$ section of $L(M)$ over U

$u \in L(M)$, $\pi(u) = x \in U$

$$\longrightarrow \quad u_j = \sum_{i=1}^{n} a_{ij}(u) \left.\frac{\partial}{\partial h^i}\right|_x \quad \text{where } a(u) = (a_{ij}(u)) \in GL(n;\mathbb{R})$$

Then: $\varphi : \pi^{-1}U \xrightarrow{\sim} U \times G$, $u \longmapsto (\pi(u), a(u))$ bundle chart

$GL(n;\mathbb{R})$ operates on $L(M)$ from the right

$$ug : \mathbb{R}^n \xrightarrow{g} \mathbb{R}^n \xrightarrow{u} T_{\pi(u)}M, \quad u \in L(M),\ g \in GL(n;\mathbb{R})$$

(2) M Riem. mf, $\dim M = n$

The orthonormal frame bundle $\pi : O(M) \longrightarrow M$ is an principal $O(n)$-bundle, defined as follows:

$$O(M) = \dot{\bigcup_{x\in M}} P_x \quad \text{where } P_x = \pi^{-1}x = \{u : \mathbb{R}^n \longrightarrow T_xM \mid u \text{ isometry}\}$$

Identify $u \in P_x$ with $(u_1, \dots, u_n) = (ue_1, \dots, ue_n)$ o.n.b of T_xM

Construction as above:

$\pi : O(M) \longrightarrow M$ principle $O(n)$-bundle

<u>Def</u> Let $\pi : P \to M$ be a principal G-bundle. A <u>G-connection</u> in P is a diffb. G-invariant splitting h of the following exact sequence of v.b. over P

$$\begin{tikzcd} 0 \arrow[r] & \ker d\pi \arrow[r] & TP \arrow[r] & \pi^*TM \arrow[r] \arrow[l, bend left, dashed, "h"] & 0 \end{tikzcd},$$

i.e. $d\pi \circ h = \mathrm{id}$ & $(R_g)_* h = h$ where $R_g : P \to P$, $u \mapsto ug$ is the operation of $g \in G$ from the right on P

<u>Remarks</u>

(1) Splitting h induces a decomposition

$$TP = V \oplus H := \ker d\pi \oplus h(\pi^* M)$$

fiberwise $\quad T_uP = V_u \oplus H_u$, $\quad u \in P$

The bundle isomorphism

$$h : \pi^*TM \xrightarrow{\sim} H$$

is called <u>horizontal lift</u> of the G-connection,

$$h_u : T_{\pi(u)}M \xrightarrow{\sim} H_u, \quad u \in P$$

$H_u = h_u(T_{\pi(u)}M)$ horizontal space at u

$V_u = \{ v \in T_uP : d\pi\, v = 0 \}$ <u>vertical space</u> at u

(2) G-invariance of the splitting: $(R_g)_* h = h$

i.e. $(R_g)_* h_u = h_{ug}$ $\quad \forall u \in P, g \in G$

$$\Rightarrow \boxed{(R_g)_* H_u = H_{ug}}, \quad u \in P$$

<u>Interpretation</u> $\forall u \in P$, V_u is canonically given, no canonical choice of a complement H_u

G-connection in P $\hat{=}$ $\underbrace{G\text{-invariant}}_{(R_g)_* H_u = H_{ug}}$ choice of a horizontal space H_u $\forall u \in P$

<u>Def</u> $\pi : P \to M$ principal G-bundle

$u \in P \rightsquigarrow I_u : G \to P$ embedding
$g \mapsto ug$

$e :=$ unit element $\in G$

$$\iota_u := (dI_u)_e : \underbrace{T_eG}_{=: \mathfrak{g} \text{ Lie algebra of } G} \to T_uP \,, \quad A \mapsto \hat{A}(u)$$

ι_u defines an identification

$$\kappa_u : \mathfrak{g} \xrightarrow{\sim} V_u$$

$A \in \mathfrak{g} \rightsquigarrow \hat{A} \in \Gamma(TP)$ <u>standard-vertical</u> v.f. on P

<u>Notation</u> $X \in \Gamma(TP) \xrightarrow[TP = V \oplus H]{} X = \underbrace{\mathrm{vert}\, X}_{\in \Gamma(TP)} + \underbrace{\mathrm{hor}\, X}_{\in \Gamma(TP)}$

<u>Def</u> $\pi : P \to M$ G-principal bundle equipped with a G-connection

Then

$$\omega_u(X_u) := \kappa_u^{-1}(\mathrm{vert}\, X)_u \,, \qquad X \in \Gamma(TP)$$

defines a $\mathfrak{g}$-valued 1-form $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$

"<u>connection form</u>" of the G-connection

ω is horizontal in the sense that

$$\omega(X) = 0 \iff X \text{ horiz. VF on } P$$

ω determines the G-connection

$u \in P$; $\omega_u : T_uP \to \mathfrak{g}$ linear & $\ker \omega_u = H_u$

Exact sequence of v.b. / P

$$\begin{tikzcd}
 & P\times\mathfrak{g} \arrow[d, phantom, "\parallel"] & & & \\
0 \arrow[r] & \ker d\pi \arrow[r, "\iota"] & TP \arrow[r, "\pi_*"] \arrow[l, bend left, "\omega"] \arrow[d, phantom, "\parallel"] & \pi^*TM \arrow[r] \arrow[l, bend left, "h"] & 0 \qquad (*) \\
 & & V\oplus H & &
\end{tikzcd}$$

Splitting of $(*)$

- horizontal lift h $\qquad \pi_* \circ h = \mathrm{id}$
- $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ $\qquad \omega \circ \iota = \mathrm{id}$
- $TP = V \oplus H$

G connection in P $\triangleq$ G-invariant splitting of $(*)$

Def P principal G-bundle$/M$

$\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ connection form of a G-connection in P

A P-valued semimartingale U is called horizontal if

$$\int_U \omega \equiv \int \omega(\circ dU) = 0 \quad \text{a.s}$$

Note: $\omega = (\omega^1, \dots, \omega^r)$ w/r to a basis of $\mathfrak{g}$

$$\int_U \omega \equiv \left(\int_U \omega^1, \dots, \int_U \omega^r \right)$$

Def Let X be an M-valued semimartingale

A P-valued semimartingale U is called horizontal lift of X if

i) $\pi(U) = X$ a.s.

ii) U horizontal, i.e. $\int_U \omega = 0$

Ex (Deterministic case)

$t \longmapsto x(t)$ curve in M

$t \longmapsto u(t) \in P$ horizontal lift of $t \longmapsto x(t)$ if

i) $\pi \circ u = x$

ii) $\omega(\dot{u}) = 0$

Theorem $\pi : P \to M$ principal G-bundle $/M$ with a G-connection.

Let x_0 be an M-valued, $\mathcal{F}_0$-measurable r.v. & X a M-valued semimartingale s.th. $X_0 = x_0$.

Then, to each P-valued, $\mathcal{F}_0$-measurable r.v. u_0, $\exists_1$ horizontal lift U to P s.th. $U_0 = u_0$ a.s.

Pr (1) Realize X as solution to an SDE on M of the type

$$dY = A(Y) \circ dZ$$

for a suitable A & Z.

e.g. $M \overset{\iota}{\hookrightarrow} \mathbb{R}^r$ Whitney embedding, $Z := \iota(X)$

$A : M \times \mathbb{R}^r \to TM$, $A(x) :=$ orth projection of $\mathbb{R}^r$ onto T_xM

To show: $dX = A(X) \circ dZ$

Let $f \in \mathcal{C}_c^\infty(M)$ & $\bar{f} \in \mathcal{C}_c^\infty(\mathbb{R}^r)$, $\bar{f} \circ \iota = f$

without restriction

$$\bar{f}(y) = f(x) \quad \text{for } y \in (T_xM)^\perp \text{ \& } \|y\| \text{ sufficiently small}$$

(i.e. $\bar{f}$ is const locally about M on the fibers of $\perp M$)

$z \in \mathbb{R}^r$: $\quad df_x\, A(x)z = (d\bar{f})_{\iota(x)} (d\iota)_x A(x) z$

$z = z^M + z^\perp$

$z^M \in T_xM \quad\quad = (d\bar{f})_{\iota(x)}\, z^M = (d\bar{f})_{\iota(x)}\, z$

$z^\perp \in \perp_x M$

$$\Rightarrow d(f(X)) = d(\bar{f}(Z)) = \sum_{i=1}^r (D_i\bar{f})\big(\underbrace{\iota(X)}_{=Z}\big) \circ dZ^i$$

$$= \sum_{i=1}^r (df)_X\, A(X) e_i \circ dZ^i = (df)_X\, A(X) \circ dZ$$

(2) According to (1)

$$(*) \quad dX = \sum_{i=1}^r A_i(X) \circ dZ^i, \quad X_0 = x_0 \qquad \big(\text{where } A_i := A(\cdot)e_i \in \Gamma(TM)\big)$$

Consider on P the "horizontally lifted SDE"

$$(\bar{*}) \qquad dU = \sum_{i=1}^{r} \bar{A}_i(U) \circ dZ^i \quad , \quad U_0 = u_0$$

where $\bar{A}_i \in \Gamma(TP)$ horiz. lift of $A_i \in \Gamma(TM)$,

ie $\bar{A}_i(u) := h_u(A_i(\pi(u)))$, $u \in P$

Let U be the solution to $(\bar{*})$

i) $\pi(U) = X$, since

$$d(\pi(U)) = \sum_{i=1}^{r} (d\pi)_U \bar{A}_i(U) \circ dZ^i = \sum_{i=1}^{r} A_i(\pi(U)) \circ dZ^i$$

$\Rightarrow$ $\pi(U)$ solves $(*)$

$\Rightarrow$ $\pi(U) = X$ a.s

uniqueness of solutions to $(*)$

ii) To show $\int_U \omega = 0$,

but $$\int_U \omega = \sum_{i=1}^{r} \int \underbrace{\omega_U(\bar{A}_i(U))}_{=0} \circ dZ^i = 0$$

iii) To show uniqueness of U

+ verification that U lives as long as X.

Exercise!

Now: M n-dim mf

$P = L(M)$ frame bundle over M, $G = GL(n;\mathbb{R})$

$P = O(M)$ orth. frame bundle over M, $G = O(n)$

$\Rightarrow$ $\mathfrak{g} = M(n\times n, \mathbb{R})$, resp.

$\mathfrak{g} = \{A \in M(n\times n;\mathbb{R}) : A \text{ skew symmetric}\}$

Fix a G-connection in P. Then

$\omega \in \Gamma(T^*P \otimes \mathfrak{g})$, $\omega_u(X_u) = \kappa_u^{-1}(\mathrm{vert}\, X)_u$, $u \in P$, $X \in \Gamma(TP)$

<u>connection form</u>

$\vartheta \in \Gamma(T^*P \otimes \mathbb{R}^n)$, $\vartheta_u(X_u) = u^{-1}\big((d\pi)_u X_u\big)$, $u \in P$, $X \in \Gamma(TP)$

<u>canonical 1-form of the principal bundle</u>

<u>Standard vertical vfs</u> on P

$\hat{A} \in \Gamma(TP)$, $\hat{A}(u) = \kappa_u(A)$, $A \in \mathfrak{g}$

<u>Standard horizontal vfs on P</u>

$L_i \in \Gamma(TP)$, $L_i(u) = h_u(ue_i)$, $i = 1,\dots,n$

<u>Remark</u> The standard vertical / horizontal vfs are determined by

$\vartheta(\hat{A}) = 0$ & $\omega(\hat{A}) = A$, resp.

$\vartheta(L_i) = e_i$ & $\omega(L_i) = 0$

<u>Def</u> The canonical differential operator

$$\Delta^{hor} := \sum_{i=1}^{n} L_i^2$$

is called <u>horizontal Laplacian</u> on $P = L(M)$, resp. $P = O(M)$

Def Let X be an M-valued semimartingale & U a horizontal lift of X taking values in $P = L(M)$, resp $P = O(M)$

The $\mathbb{R}^n$-valued semimartingale

$$Z := \int_U \vartheta = \int \theta(\circ dU)$$

is called <u>anti-development of X in $\mathbb{R}^n$</u> (with initial basis U_0)

<u>Note</u> With respect to the standard basis of $\mathbb{R}^n$

$$Z = (Z^1, \dots, Z^n), \quad Z^i = \int_U \vartheta^i$$

<u>Proposition</u> Let X be an M-valued semimartingale,
U a horizontal lift of X to $P = L(M)$, resp. $P = O(M)$
Z anti-development of X in $\mathbb{R}^n$

<u>Then</u>

(1) $$\int_U \sigma = \sum_{i=1}^n \int \sigma_U (L_i(U)) \circ dZ^i \qquad \forall\, \sigma \in \Gamma(T^*P)$$

(2) $$\int_X \alpha = \sum_{i=1}^n \int \alpha_X (Ue_i) \circ dZ^i \qquad \forall\, \alpha \in \Gamma(T^*M)$$

In particular,

- for $\sigma = df$, $f \in \mathcal{C}^\infty(P)$

$$d(f(U)) = \sum_{i=1}^n \underbrace{(df)_U \, L_i(U)}_{=(L_i f)(U)} \circ dZ^i = \sum_{i=1}^n (L_i f)(U) \circ dZ^i$$

i.e. $$\boxed{dU = \sum_{i=1}^n L_i(U) \circ dZ^i}$$

- for $\alpha = df$, $f \in \mathcal{C}^\infty(M)$

$$d(f(X)) = \sum_{i=1}^n \underbrace{(df)_X (Ue_i)}_{=(Ue_i)(f)} \circ dZ^i = \sum_{i=1}^n (Ue_i)(f) \circ dZ^i$$

i.e. $$\boxed{dX = U \circ dZ}$$

Pr ① To show the right-hand side of ① has the defining properties of $\int_u \sigma$

i.e. to show

$$d(f(u)) = \sum_{i=1}^{n} (L_i f)(u) \circ dz^i \qquad \forall f \in C^\infty(P),$$

but $f(u) - f(u_0) = \int_u df = \int_u df \circ pr_H + \underbrace{\int_u df \circ pr_V}_{=0}$ since u is horizontal

$$(df \circ pr_V)_u = (df)_u \circ pr_{V_u} = (df)_u \, \kappa_u \, \omega_u = (df \circ T_u)_e \, \omega_u$$

To show:

$$\int_u df \circ pr_H = \int_u \sigma \quad \text{where } \sigma \in \Gamma(T^*P), \; \sigma_u = \sum_i (L_i f)(u)\, \vartheta^i_u ,$$

but for $A \in T_u P$:

$$\begin{aligned}
\sigma_u(A) &= \sum_i (df)_u \, L_i(u) \, \vartheta^i_u(A) \\
&= \sum_i (df)_u \, h_u(ue_i) \left(u^{-1} (d\pi)_u A\right)^i \\
&= (df)_u \, h_u\left(u u^{-1} (d\pi)_u A\right) \\
&= (df)_u \, h_u\left((d\pi)_u A\right) = (df)_u \circ pr_{H_u}(A) = (df \circ pr_H)_u
\end{aligned}$$

② To show

$$d(f(x)) = \sum_{i=1}^{n} (ue_i)(f) \circ dz^i,$$

but

$$\begin{aligned}
d(f \circ \pi \circ u) &\overset{①}{=} \sum_i d(f \circ \pi)_u \, L_i(u) \circ dz^i \\
&= \sum_i (df)_{\pi(u)} (d\pi)_u \, L_i(u) \circ dz^i \\
&= \sum_i (df)_x \, ue_i \circ dz^i
\end{aligned}$$

$(d\pi)_u L_i(u) = ue_i$

<u>Remark</u> Each of the processes X, U, Z determines the two others (modulo starting value):

Let u_0 be a $\mathcal{F}_0$-mb, P-valued r.v & $x_0 = \pi(u_0)$

$Z \longmapsto U$: $dU = \sum L_i(U) \circ dZ^i$, $U_0 = u_0$

$U \longmapsto X$: $X = \pi(U)$

$X \longmapsto Z$: $Z = \int_U \vartheta$ where U is the unique horizontal lift of X with $U_0 = u_0$

<u>Def</u> $P = L(M)$, resp. $P = O(M)$ in the Riemannian case

X M-valued semimartingale $\longmapsto$ U horiz. lift of X to P

$s \leq t$: $//_{s,t} : T_{X_s}M \to T_{X_t}M$, $//_{s,t} = U_t \circ U_s^{-1}$

$$\begin{tikzcd} T_{X_s}M \arrow[rr, dashed, "//_{s,t}"] & & T_{X_t}M \\ & \mathbb{R}^n \arrow[ul, "U_s", "\cong"'] \arrow[ur, "U_t"', "\cong"] & \end{tikzcd}$$

linear isomorphisms, resp isometries

$//_{t,s} = //_{s,t}^{-1}$

<u>(stoch.) parallel transport along</u> X

M diffb mf : $P = L(M)$ frame bundle, $G = GL(n, \mathbb{R})$
M Riem mf : $P = O(M)$ orthonormal frame bundle, $G = O(n)$

linear connection on M $\longleftrightarrow$ $GL(n,\mathbb{R})$-connection in $P = L(M)$
Riem. connection on M $\longleftrightarrow$ $O(n)$-connection in $P = O(M)$

Indeed: i) α curve in TM

$$\alpha \text{ parallel} \iff \dot\alpha(t) \in H^{TM}_{\alpha(t)} \ \forall t$$
$$TTM = H^{TM} \oplus V^{TM}$$

ii) α curve in P, $\alpha(t) = (\alpha_1(t), \dots, \alpha_n(t))$, $\alpha_i(t) = \alpha(t)e_i$

$$\alpha \text{ parallel} \iff \dot\alpha(t) \in H^P_{\alpha(t)} \ \forall t$$
$$TP = H^P \oplus V^P$$
$$\overset{!}{\iff} \dot\alpha(t) = (\dot\alpha_1(t), \dots, \dot\alpha_n(t)) \ \& \ \dot\alpha_i(t) \in H^{TM}_{\alpha_i(t)} \ \forall t$$

$\omega \in \Gamma(T^*P \otimes \mathfrak{g})$, $\omega_u(X_u) = \kappa_u^{-1}(\operatorname{vert} X)_u$, $u \in P$, $X \in \Gamma(TP)$
$\vartheta \in \Gamma(T^*P \otimes \mathbb{R}^n)$, $\vartheta_u(X_u) := u^{-1}((d\pi)_u X_u)$, —"—
$L_i \in \Gamma(TP)$, $L_i(u) = h_u(ue_i)$, $i = 1, \dots, n$

X M-valued semimartingale
U horizontal lift to $P = L(M), O(M)$, i.e. $\pi(U) = X$ & $\int_U \omega = 0$
Z antidevelopment of X, $Z = \int_U \vartheta$

$$\begin{tikzcd} T_{X_0}M \arrow[rr, dashed, "/\!/_t"] & & T_{X_t}M \\ & \mathbb{R}^n \arrow[ul, "U_0"] \arrow[ur, "U_t"'] & \end{tikzcd}$$

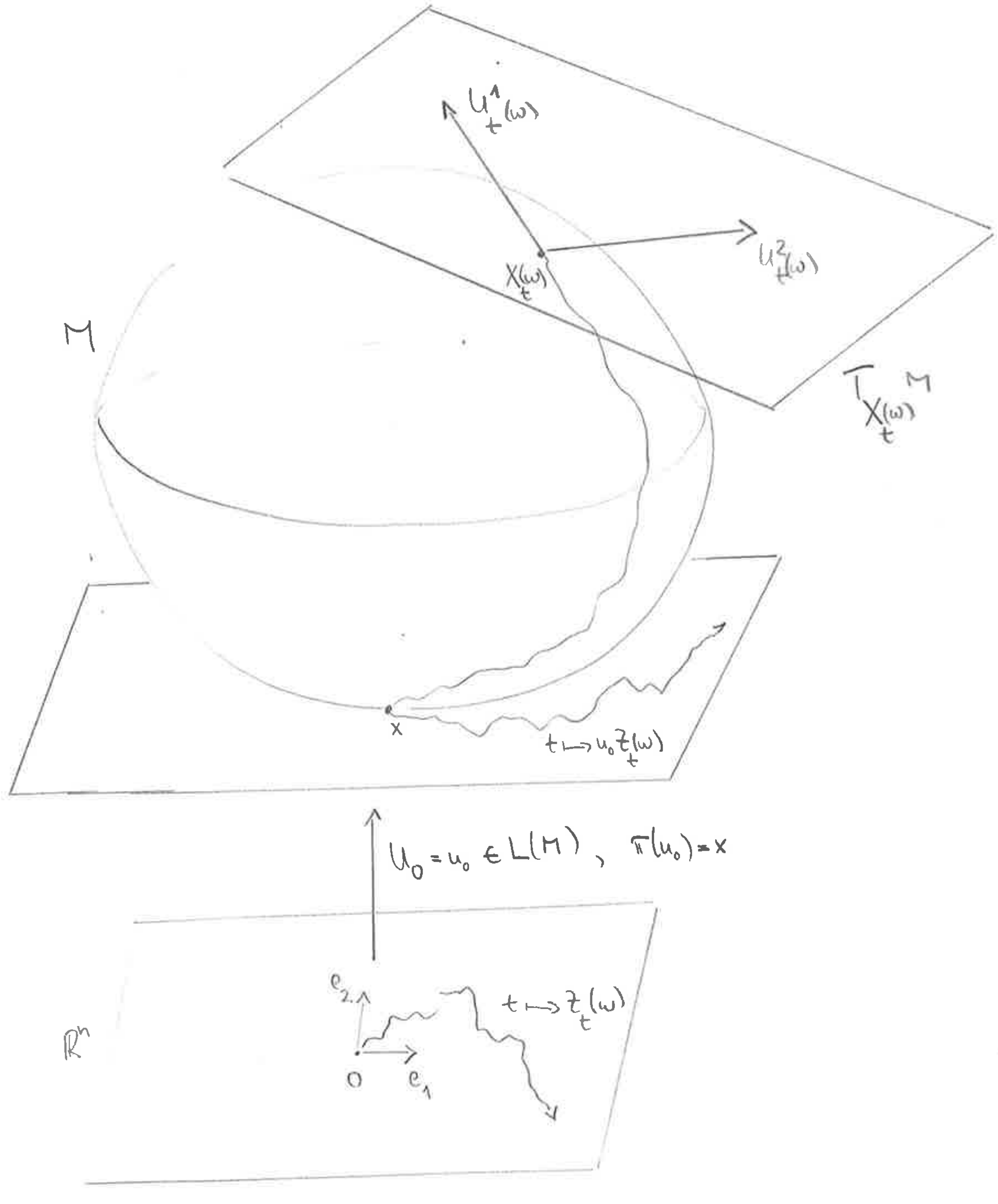
$/\!/_{0,t} = U_t \circ U_0^{-1}$ <u>stochastic $/\!/$-transport</u>

$Z \longmapsto U$: $dU = \sum_i L_i(U) \circ dZ^i$, $U_0 = u_0$
$U \longmapsto X$: $X = \pi(U)$
$X \longmapsto Z$: $Z = \int_U \vartheta$

<u>"X stochastic development of Z"</u>

Geometric picture $\quad P = L(M),\ O(M),$

$P \ni u : \mathbb{R}^n \xrightarrow[u]{\sim} T_{\pi(u)}M$



$$dX = \underbrace{U \circ dZ}_{U u_0^{-1} \circ d(u_0 Z)} \equiv \sum_i U e_i \circ dZ^i \quad \Bigg\} \quad \boxed{dX_t = /\!/_{0,t} \circ d(u_0 Z_t)}$$

$X \mathrel{\hat{=}}$ trace on M printed onto M, under identification of $\mathbb{R}^n \xrightarrow[\sim]{U_t} T_{X_t}M$, when M is "rolled" along $t \mapsto Z_t$

Note $\quad dX_t = /\!/_{0,t} \circ d\big(A(X)\big)_t \quad$ where $A(X) = U_0 \int_U \vartheta \quad$ (indep. of the choice of U_0)

Deterministic case $\quad z$ smooth curve $t \mapsto z(t)$

Cartan development of

$$t \mapsto z(t) \in \mathbb{R}^n, \quad z(0) = 0$$

Find $x : t \mapsto x(t) \in M$ & $u : t \mapsto u(t) \in P$ s.th.

i) $\dot{x} = u\dot{z}$

ii) u parallel along x, i.e. $\nabla_D u \equiv (\nabla_D u_1, \dots, \nabla_D u_n) = 0$

Note i) $\iff z(t) = \int_0^t u(s)^{-1} dx(s) = \int_0^t u(s)^{-1} \dot{x}(s)\, ds$

$\iff z = \int_u \vartheta \equiv \int \vartheta(\dot{u})\, ds$

ii) $\iff u(\cdot)$ horizontal curve, i.e. $\underbrace{\dot{u}(t) \in H_{u(t)} \ \forall t}_{\text{i.e. } \dot{u} = h_u(\dots)}$

but $\dot{x} = (\pi \circ u)^{\cdot} = \underbrace{\pi_*}_{= id|_{T_{\pi(u)}M}} h_u(\dots) \implies \dot{u} = h_u(\dot{x}) \overset{i)}{=} h_u(u\dot{z})$

Hence i) & ii) $\iff \dot{u} = h_u(u\dot{z})$

But $\dot{u} = h_u(u\dot{z}) = \sum_{i=1}^n h_u(ue_i)\,\dot{z}^i = \sum_{i=1}^n L_i(u)\,\dot{z}^i$

Hence
$$\begin{cases} du = \sum_{i=1}^n L_i(u)\, dz^i, & u(0) = u_0 \\ x(\cdot) = (\pi \circ u)(\cdot) \end{cases}$$

Stoch. development of $z(\cdot) \triangleq$ classical Cartan development of $z(\cdot)$

Theorem (Geometric Itô-formula)

X M-valued semimartingale $\leadsto$ U horizontal lift to $P = L(M), O(M)$

$Z = \int_U \vartheta$, resp. $\mathcal{A}(X) = U_0 \int_U \vartheta$

Then $\forall\, f \in \mathcal{C}^\infty(M)$,

$$d(f(X)) = \sum_{i=1}^n (df)_X (Ue_i)\, dZ^i + \frac{1}{2}\sum_{i,j=1}^n (\nabla df)_X (Ue_i, Ue_j)\, dZ^i dZ^j$$

More intrinsically,

$$d(f(X)) = \underbrace{df(U\,dZ)}_{= df(/\!/_{0,t}\, d\mathcal{A}(X))} + \frac{1}{2}\nabla df(dX, dX)$$

Pr

1) **Lemma** Let $\alpha \in \Gamma(T^*M)$; $P = L(M), O(M)$

$\leadsto F_\alpha : P \to \mathbb{R}^n$, $F_\alpha(u)^i = \alpha_{\pi(u)}(ue_i)$

Then $\forall\, A \in \Gamma(TM)$,

$$(\nabla_A \alpha)_{\pi(u)}(ue_i) = \bar{A}_u F^i_\alpha$$

where $\bar{A} \in \Gamma(TP)$ horiz. lift of A (i.e. $\bar{A}_u = h_u(A_{\pi(u)})$, $u \in P$)

Pr Note that

$v \in T_xM$: $\displaystyle \nabla_v \alpha = \lim_{\varepsilon \downarrow 0} \frac{/\!/_{0,\varepsilon}^{-1} \alpha_{\gamma(\varepsilon)} - \alpha_{\gamma(0)}}{\varepsilon}$

where γ is a curve on M s.th. $\gamma(0) = x$ & $\dot\gamma(0) = v$

Here: $\left(/\!/_{0,\varepsilon}^{-1} \alpha_{\gamma(\varepsilon)}\right)(Y) = \alpha_{\gamma(\varepsilon)}(/\!/_{0,\varepsilon} Y)$, $Y \in T_{\gamma(0)}M$

$t \mapsto u(t)$ horizontal lift of $t \mapsto \gamma(t)$ s.th. $u(0) = u$ & $/\!/_{0,\varepsilon} = u(\varepsilon)u(0)^{-1} : T_{\gamma(0)}M \to T_{\gamma(\varepsilon)}M$

Hence

$$(\nabla_A \alpha)_{\pi(u)}(ue_i) = \lim_{\varepsilon \downarrow 0} \frac{/\!/_{0,\varepsilon}^{-1} \alpha_{\gamma(\varepsilon)}(ue_i) - \alpha_{\gamma(0)}(ue_i)}{\varepsilon}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{\alpha_{\gamma(\varepsilon)}(/\!/_{0,\varepsilon} ue_i) - \alpha_{\gamma(0)}(ue_i)}{\varepsilon}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{\alpha_{(\pi\circ u)(\varepsilon)}(u(\varepsilon)e_i) - \alpha_{(\pi\circ u)(0)}(u(0)e_i)}{\varepsilon}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{F^i_\alpha(u(\varepsilon)) - F^i_\alpha(u(0))}{\varepsilon}$$

$= \bar{A}_u F^i_\alpha$, since $\dot u(t) = h_{u(t)}(\dot\gamma(t))$

$\Longrightarrow \dot u(0) = h_u(A_x) = \bar{A}_u$

(2) $d(f(X)) = d\big((f\circ\pi)(u)\big) = \sum_i L_i (f\circ\pi)(u) \circ dz^i$

$$= \sum_i L_i (f\circ\pi)(u)\, dz^i + \frac{1}{2} \sum_{i,j} L_i L_j (f\circ\pi)(u)\, dz^i dz^j$$

But $L_i (f\circ\pi)(u) = d(f\circ\pi)_u\, L_i(u)$

$$= (df)_{\pi(u)} (d\pi)_u\, h_u(ue_i) = (df)_{\pi(u)}(ue_i) = F^i_{df}(u)$$

where $F_{df} : P \to \mathbb{R}^n$, $F_{df}(u) = (df)_{\pi(u)}(ue_i)$

$\overline{ue_i} := h_u(ue_i)$

$\Rightarrow$ $L_i L_j (f\circ\pi)(u) = (L_i F^j_{df})(u) =$

$$= (\overline{ue_i})\, F^j \underset{(1)}{=} (\nabla_{ue_i} df)_{\pi(u)}(ue_j) = (\nabla df)(ue_i, ue_j)$$

In particular,

$$L_i^2 (f\circ\pi)(u) = \nabla df\,(ue_i, ue_i)$$

Remark (M,g) Riem mf with LC-connection

$$\Delta^{hor} := \sum L_i^2 \quad \text{on } O(M)$$

Then $\boxed{\Delta^{hor}(f\circ\pi) = (\Delta f)\circ\pi} \quad \forall f \in C^\infty(M)$

Pr. $\sum_i L_i^2 (f\circ\pi)(u) = \sum_i (\nabla df)(ue_i, ue_i)$

$$= (\operatorname{trace} \nabla df)\, \pi(u)$$

$$= (\Delta f)\circ\pi\,(u) \quad , \quad u \in O(M)$$

Theorem (1) M diffb mf with a connection (without restriction torsion free)

X M-valued semimartingale $\longrightarrow$ U horiz. lift to $L(M)$

$Z = \int_U \vartheta$

Then:

X ∇-martingale $\iff$ Z local martingale on $\mathbb{R}^n$

$\iff$ $\mathcal{A}(X)$ local martingale on $T_{X_0}M$

(2) (M,g) Riem mf with LC-connection & X, U, Z as above

Then:

X $BM(M,g)$ $\iff$ Z $BM(\mathbb{R}^n)$ (more precisely, a $BM(\mathbb{R}^n)$ stopped at ζ = lifetime of X)

Pr (1) X ∇-martingale $\iff$ $d(f(X)) - \frac{1}{2}\nabla df(dX, dX) \overset{m}{=} 0 \quad \forall f \in \mathcal{C}^\infty(M)$

$\underset{\text{geom. Itô formula}}{\iff} \sum_i (df)_X (Ue_i)\, dZ^i \overset{m}{=} 0 \quad \forall f \in \mathcal{C}^\infty(M)$

$\iff$ Z local martingale

(2) X $BM(M,g) \iff d(f(X)) - \frac{1}{2}\Delta f(X)\, dt \overset{m}{=} 0 \quad \forall f \in \mathcal{C}^\infty(M)$

Suppose Z $BM(\mathbb{R}^n)$: $d(f(X)) = \underbrace{df(U\,dZ)}_{d(\text{loc. mart.})} + \underbrace{\frac{1}{2}\sum_{i,j} (\nabla df)_X (Ue_i, Ue_j)\, dZ^i dZ^j}_{(\text{trace } \nabla df)(X)\,dt \;=\; (\Delta f)(X)\,dt}$

Conversely. suppose X $BM(M,g)$

$\Rightarrow X$ ∇-martingale $\overset{(1)}{\Longrightarrow}$ Z local martingale

To show: $dz^i dz^j = \delta_{ij} dt$ where $z^i = \int_U \vartheta^i = \int \vartheta^i(\circ dU)$

& $\vartheta^i_u = \langle u^{-1} (d\pi)_u (\cdot), e_i \rangle$

$= \langle (d\pi)_u (\cdot), ue_i \rangle = \pi^* \langle \cdot, ue_i \rangle$

But $dz^i dz^j = \vartheta^i(\circ dU)\, \vartheta^j(\circ dU)$

$= (\vartheta^i \otimes \vartheta^j)(dU, dU)$

$= \pi^* (\langle \cdot, Ue_i \rangle \otimes \langle \cdot, Ue_j \rangle)(dU, dU)$

$\underset{\substack{\uparrow \\ \text{pullback formula}}}{=} \langle \cdot, Ue_i \rangle \otimes \langle \cdot, Ue_j \rangle (dX, dX)$

$= \operatorname{trace}(\langle \cdot, Ue_i \rangle \otimes \langle \cdot, Ue_j \rangle)(X)\, dt$

$= \delta_{ij}\, dt$ ∎