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The Bonnet Plancherel formula for monomial representations for classes of completely solvable Lie groups

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Abstract

We compute the Bonnet Plancherel formula associated to a monomial representation of a nilpotent Lie group. We give also the corresponding formula for finite multiplicity monomial representation for a class of completely solvable Lie groups.

0. Introduction

Let G be a connected Lie group having a smooth dual. Given a unitary representation π of G acting in a Hilbert space $\mathcal{H}_\pi$, we denote by $\mathcal{H}_\pi^\infty$ the Fréchet space of smooth vectors for π, and $\mathcal{H}_\pi^{-\infty}$ the space of continuous anti-linear functionals on $\mathcal{H}_\pi^\infty$. Let α be any positive distribution on G of finite order. Bonnet's Plancherel formula ([Bon.]) tells us that for $\varphi \in \mathcal{D}(G)$

$$\alpha(\varphi) = \int_{\hat{G}} \operatorname{tr}(\pi(\varphi)U_\pi)d\nu(\pi) \tag{1}$$

where for ν almost everywhere, $U_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^{-\infty},\ \pi \in \hat{G}$, is a certain uniquely determined nuclear operator (see [Bon.] Theorem 4-1).

We recall that Penney's and Bonnet's Plancherel formulas have been described for nilpotent groups and exponential groups in ([Pen], [Fu.1,3,4], [F.Y.], [Gr,1,2], [Li.2,3], [B.L.2]). Furthermore, Fujiwara has given an explicit expression by duality of Bonnet's operators in the case of monomial representations of nilpotent Lie groups.

In the first part of this paper we take a closed connected subgroup $H = \exp(\mathfrak{h})$ of a nilpotent connected simply connected Lie group $G = \exp(\mathfrak{g})$, a unitary character

$\chi = \chi_f$ of H (where $f \in \mathfrak{g}^*$ is such that $\langle f, [\mathfrak{h}, \mathfrak{h}] \rangle = 0$) and we consider the positive distribution

$$\langle S_{H,f}, \varphi \rangle = \int_H \varphi(h)\chi_f(h)dh,\ \varphi \in \mathcal{D}(G).$$

To describe the measure ν given in (1), we use the result of [B.L. 1] where it has been shown that there exists a certain affine subspace $\mathcal{V}$ of $(f + \mathfrak{h}^\perp)$ such that $Ind_H^G \chi_f \simeq \int_{\mathcal{V}}^{\oplus} \pi_\phi d\phi$ ($d\phi$ denotes the Lebesgue measure on $\mathcal{V}$). There exists a Borel cross-section Σ of G-orbits in $G \cdot \mathcal{V}$ and it turns out that the measure ν of Bonnet's formula is supported on Σ. We show in (6) that for $\sigma \in \Sigma$ the operator U_σ is an integral of rank one operators:

$$U_\sigma = \int_{\Gamma_\sigma} Q_{s,\sigma} d\lambda_\sigma(\dot{s})$$

where Γ_σ is defined in paragraph (1.1.b), the operators $Q_{s,\sigma}$ and the measure $d\lambda_\sigma$ in 1.3.

In the exponential case, the determination of Bonnet's operators U_π is difficult. One of the reasons is that there exists no easy way to determine explicitly the C^∞ vectors of a representation.

Several authors have studied in the past the disintegration of induced representations for exponential solvable Lie groups. In ([D.R.]) Duflo and Rais computed the Plancherel formula for $L^2(G)$ of an exponential solvable Lie group. Bonnet's operators have been explicitly described for a normal monomial representation induced from a normal subgroup of an exponential solvable Lie group in [G.H.L.S.].

In the second part of this paper we take the semi-direct product $G = NH$; where $N = \exp(\mathfrak{n})$ is nilpotent and normal in G, and $H = \exp(\mathfrak{h})$ is abelian and acts semi-simply on N with real eigenvalues. Let $\chi = \chi_f$ be a unitary character of H (where $f \in \mathfrak{g}^*$). We consider the representation $\tau_f = Ind_H^G \chi_f$ and we assume that τ_f has finite multiplicity. The first precise formulas in this case have been given by Currey in ([Cu.2]). To describe the measure ν given in (1) we use the main results of this reference, where it has been shown that the set of generic H-orbits in the disintegration of τ_f admits a natural smooth algebraic cross-section Σ. We derive a cross-section Γ of $G-$ orbits in $G.(f + \Sigma)$, and the measure ν of Bonnet's formula will be explicitly described as a measure on Γ. We take $\sigma \in (f + \Sigma)$ and for $l \in G \cdot \sigma \cap (f + \Sigma)$ we define an operator β'_l on the space of the smooth vectors $\mathcal{H}_\sigma^\infty$ of π_σ. We show in (13) that the operators $U_{\pi_\sigma} = U_\sigma$ in Bonnet's formula are determined as a finite sum of rank one operators: $P_{\beta'_l, \beta'_l}$.

1. The Bonnet Plancherel formula for nilpotent Lie group

1.1 Notations and definitions

1.1.a Quotient measures

Let G be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and let $K = \exp(\mathfrak{k})$ be a closed subgroup of $\mathfrak{g}$. We choose a Jordan-Hölder basis $\mathcal{Z} = \{Z_1, \cdots, Z_n\}$ of $\mathfrak{g}$. Let $B = \{X_1, \cdots, X_r\}$ be a Malcev-basis relative to $\mathfrak{k}$, i.e. $\mathfrak{g} = \sum_{1\leq i\leq r}^{\oplus} \mathbb{R}X_i \oplus \mathfrak{k}$ and for any $j = 1, \cdots, r$, the subspace $\mathfrak{g}_j = span\{X_j, \cdots, X_r, \mathfrak{k}\}$ is a subalgebra. The mapping $E_B : \mathbb{R}^r \to G/K : E_B(t_1, \cdots, t_r) = E'_B(t_1, \cdots, t_r)K$, where $E'_B(t_1, \cdots, t_r) = \exp(t_1X_1)\cdots\exp(t_rX_r)$, is then a diffeomorphism. We obtain a $G-$invariant measure $d\dot{g}$ on the quotient space G/K by setting

$$\int_{G/K} \xi(g)d\dot{g} = \int_{\mathbb{R}^r} \xi(E_B(T))dT,\ \xi \in C_c(G/K),$$

where $C_c(G/K)$ denotes the space of the continuous fonctions with compact support on G/K.

It is not difficult to see the following:

1.1.a.1 Proposition *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n. Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$, B_1 and B_2 be two Malcev-basis of $\mathfrak{g}$ relative to $\mathfrak{k}$. Then $E_{B_2}^{-1} \circ E_{B_1}$ is a polynomial mapping from $\mathbb{R}^r$ to $\mathbb{R}^r$ (where r is the codimension of $\mathfrak{k}$ in $\mathfrak{g}$) whose total degree is bounded by a constant M which depends only on the dimension of $\mathfrak{g}$.*

1.1.b Induced representation

Let G be a nilpotent connected simply connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$; $f \in \mathfrak{g}^*$ such that $\langle f, [\mathfrak{h}, \mathfrak{h}]\rangle = 0$ and let χ_f be the unitary character of $H = \exp(\mathfrak{h})$ associated to f. Let $\tau = Ind_H^G \chi_f$. It has been shown in [B.L.1] that there exists a certain affine subspace $\mathcal{V}$ of $\Gamma_f = f + \mathfrak{h}^\perp \subset \mathfrak{g}^*$, such that

$$\tau = \mathrm{Ind}_H^G \chi_f \simeq \int_{\mathcal{V}}^{\oplus} \pi_\phi d\mu(\phi) \tag{2}$$

where $d\mu$ denotes Lebesgue measure on $\mathcal{V}$ and where π_ϕ is the irreducible representation associated to ϕ ($\phi \in \mathcal{V}$).

One has:

Lemma ([Bour], [Fuj. 3])

$$\mu = \int_{G\cdot\mathcal{V}/G} \nu_\Omega d\nu(\Omega)$$

where ν_Ω *is a certain measure on* $\Omega \cap \mathcal{V}$.

Let Σ be a borel cross-section of the G-orbits in $G\cdot\mathcal{V}$. We can consider the measure ν as a measure on Σ and write $\mu = \int_\Sigma \nu_{G\sigma} d\nu(\sigma)$.

Hence for a continuous function F with compact support on $\mathcal{V}$ we get

$$\int_{\mathcal{V}} F(\phi)d\mu(\phi) = \int_\Sigma \int_{G\cdot\sigma\cap\mathcal{V}} F(l)d\nu_{G\cdot\sigma}(l)d\nu(\sigma).$$

We identify $G\cdot\sigma\cap\mathcal{V}$ with the space $G_\sigma/G(\sigma)$, where $G_\sigma = \{g \in G;\ \ g\cdot\sigma \in \mathcal{V}\}$ and $G(\sigma) = \{g \in G,\ \ g\cdot\sigma = \sigma\}$ and we consider the measure λ_σ on $\Gamma_\sigma = G_\sigma/G(\sigma)$ which corresponds to the measure $\nu_{G\cdot\sigma}$, we write:

$$\int_{\mathcal{V}} F(\phi)d\mu(\phi) = \int_\Sigma \int_{\Gamma_\sigma} F(s\cdot\sigma)d\lambda_\sigma(s)d\nu(\sigma). \tag{3}$$

Let now $\mathcal{S}(G/H, f)$ be the space of all C^∞-function ξ on G, such that $\xi(gh) = \chi_f(h^{-1})\xi(g)$ for all $g \in G, h \in H$ and such that the function $T \mapsto \xi(E_B(T))$ is a Schwartz-function on $\mathbb{R}^r$. Let $\mathcal{S}(G)$ denote the Schwartz-space of G, i.e. the space of all complex valued functions φ on G, such that $\varphi \circ \exp$ is an ordinary Schwartz-function on the vector space $\mathfrak{g}$.

Denote for $\varphi \in \mathcal{S}(G)$ $\ P_{H,f}(\varphi)(g) = \int_H \varphi(gh)\chi_f(h)dh$ and let $S_{H,f}$ be the tempered distribution on G defined by the projection $P_{H,f}(\varphi)$ of φ on $\mathcal{S}(G/H, f)$, i.e:

$$\langle S_{H,f}, \varphi\rangle = \int_H \varphi(h)\chi_f(h)dh = P_{H,f}\varphi(e).$$

Let for $\phi \in \mathcal{V}$ $B(\phi)$ denote the Vergne polarization at ϕ for the basis $\mathcal{Z}$. It has been shown in [B.L.1] that there exists for $\phi \in \mathcal{V}$ an invariant measure $d\dot{b}$ on $B(\phi)/B(\phi)\cap H$ such that for the mapping

$$T_\phi : \mathcal{S}(G/H, f) \to \mathcal{S}(G/B(\phi), \phi) \qquad (\phi \in \mathcal{V})$$

given by

$$T_\phi(\xi)(g) = \int_{B(\phi)/B(\phi)\cap H} \xi(bg)\chi_\phi(b)d\dot{b}, \quad \xi \in \mathcal{S}(G/H, f), g \in G,$$

and for $\xi \in \mathcal{S}(G/H, f)$ we have:

$$\int_{\mathcal{V}} \langle T_\phi(\xi), T_\phi(\xi)\rangle_{\mathcal{H}_\phi} d\phi = \|\xi\|^2_{\mathcal{H}_\tau}. \tag{4}$$

We recall also that from [B.L.2] $S_{H,f}$ is disintegrated as an integral $\int_{\mathcal{V}} S_\phi d\mu(\phi)$, where S_ϕ denotes the tempered distribution on $\mathcal{S}(G)$ defined by :

$$\langle S_\phi, \varphi\rangle = \int_{H/H\cap B(\phi)} T_\phi(P_{H,f}(\varphi))(h)\chi_f(h)d\dot{h}. \tag{5}$$

1.2 Main results

This section is based on the paragraph 7.5 in [L.M.].

Let $\mathfrak{g}$ be a nilpotent Lie algebra. Let $\mathcal{B}$ be an algebraic subset of finite dimensional real vector space W, the pair $(\mathfrak{g}, \mathcal{B})$ is a rationnally variable nilpotent Lie algebra (or r.v.n.) if the following holds true:

For every $b \in \mathcal{B}$, a Lie bracket $[,]_b$ on $\mathfrak{g}$ is given such that $(\mathfrak{g}, [,]_b)$ forms a nilpotent Lie algebra. Moreover there exists a fixed basis $\mathcal{Z} = \{Z_1, \cdots, Z_n\}$ of $\mathfrak{g}$, so that the structure constants $(a_{ij}^k(b))$, given by $[Z_i, Z_j]_b = \sum_{k=1}^n a_{ij}^k(b)Z_k$, are rational functions in b, satisfying $a_{ij}^k(b) = 0$ for $i < j, k \leq j$ (so that $\mathcal{Z}$ is a Jordan-Hölder-basis for $(\mathfrak{g}, [,]_b)$ (see [L.M]).

A mapping on $\mathcal{B}$ is called polynomial if it is the restriction of a polynomial mapping on W to $\mathcal{B}$ and it is called rational if it is the restriction of a rational mapping on W to $\mathcal{B}$, such that the denominators of the corresponding rational functions do not vanish on $\mathcal{B}$.

For every $b \in \mathcal{B}$ we choose m elements $V_1(b), \cdots, V_m(b)$, in $\mathfrak{g}^*$ depending rationally on b. Let $V(b) = span(V_1(b), \cdots, V_m(b))$ and $\phi^b : \mathbb{R}^m \to V(b)$ defined by $\phi^b(X) = \sum_{i=1}^m x_i V_i(b)$, where $X = (x_1, \cdots, x_m) \in \mathbb{R}^m$.

Let us denote for $(X, b) \in \mathbb{R}^m \times \mathcal{B}$ and for a polarization $\mathfrak{b}$ at $\phi^b(X)$ in $\mathfrak{g}_b$ the induced representation $\pi_{\phi^b(X),\mathfrak{b}}$ by $\pi_{(X,b),\mathfrak{b}}$. Given any Malcev basis B of $\mathfrak{g}_b$ relative to $\mathfrak{b}$, we can realize the representation in a canonical way on $L^2(\mathbb{R}^r)$ and for every element u in the enveloping $U(\mathfrak{g}_b)$ of $\mathfrak{g}_b$, the operator $d\pi_{\phi^b(X),b}(u)$ becomes a partial differential operator with polynomial coefficients on $\mathbb{R}^r$.

In the following theorem we generalize the theorem 7.7 of [L.M.] by replacing the generic points in $\mathfrak{g}^*$ by the generic points of the forms $\phi^b(X), \quad X \in \mathbb{R}^m$:

1.2.1 Theorem *There exists a Zariski-open subset O in $\mathbb{R}^m \times \mathcal{B}$ such that:*

i) For every $(X, b) \in O$ there exists a polarization $\mathfrak{b}(X, b) = \mathfrak{b}(\phi^b(X))$ at $\phi^b(X)$ and a Malcev basis $B(X, \phi^b(X))$ of $\mathfrak{g}$ relative to $\mathfrak{b}(\phi^b(X))$ depending rationally on (X, b).

ii) For every partial differential operator D on $\mathbb{R}^d$ with polynomial coefficients there exists a rational mapping

$$A : O \to \mathcal{U}(\mathfrak{g}_b), \quad A(X,b) = \sum_{|I| \le n_d} a_I(X,b) Z^I$$

such that $\pi_{(X,b),\mathfrak{b}}(A(X,b)) = D$.

Proof. We use the notations and the proof of [L.M.].

Let $b \in \mathcal{B}$ and $X \in \mathbb{R}^m$; we can construct the indices $j_i(X,b) = j_i(\phi^b(X)) = j_i(\phi^b(X), b); k_i(X,b) = k_i(\phi^b(X)) = k_i(\phi^b(X), b)$ as well as $j_1(X,b)$ and $k_1(X,b)$ corresponding to $(\mathfrak{g}, [,]_b)$ as in [L.M]. We put

$$j_1 := max\{j_1(X,b) : X \in \mathbb{R}^m; b \in \mathcal{B}\},$$

$$k_1 := max\{k_1(X,b) : X \in \mathbb{R}^m; b \in \mathcal{B}\},$$

and put $\mathcal{B}^1 := \{(X,b) \in \mathbb{R}^m \times \mathcal{B} : j_1(\phi^b(X), b) = j_1$ and $k_1(\phi^b(X), b) = k_1\}$. Then

$$\mathcal{B}^1 = \{(X,b) : \phi^b(X)([Z_{j_1}, Z_{k_1}]_b) \neq 0\}$$

is a Zariski-open in $\mathbb{R}^m \times \mathcal{B}$. Next, for $(X,b) \in \mathcal{B}^1$, we put $(\mathfrak{p}_1(\phi^b(X), b), [,]_b) := \{Y \in \mathfrak{g} : \phi^b(X)([Z_{j_1}, Y]_b) = 0\}$, and

$$Z_i^1(X,b) := Z_i - \frac{\phi^b(X)([Z_{j_1}, Z_i]_b)}{\phi^b(X)([Z_{j_1}, Z_{k_1}]_b)} Z_{k_1}, \ i \neq k_1.$$

Then $Z_i^1(X,b), i \neq k_1$, form a Jordan-Hölder-basis of $(\mathfrak{p}_1(\phi^b(X), b), [,]_b)$.

We identify $(\mathfrak{p}_1(\phi^b(X)), b)$ with $\mathfrak{p}_1 := \mathbb{R}^q$, where $q = \dim(\mathfrak{p}_1(\phi^b(X)), b)$, we obtain a new r.v.n.$(\mathfrak{p}_1, \mathcal{B}^1)$. Now for $b^1 = (X,b) \in \mathcal{B}^1$, we get m linear forms: $(V_i^1(b^1))_{i=1}^m$ in $\mathbb{R}^q$ given by: $V_i^1(b^1) = V_i^1(X,b) =$

$(\langle V_i(b), Z_1^1(X,b)\rangle, \cdots, \langle V_i(b), Z_{k_1-1}^1(X,b)\rangle, \langle V_i(b), Z_{k_1+1}^1(X,b)\rangle, \cdots, \langle V_i(b), Z_n^1(X,b)\rangle)$.

We put $V^1(b^1) = \text{span}\ (V_1^1(b^1), \cdots, V_m^1(b^1))$, and $\phi^{b^1} : \mathbb{R}^m \to V^1(b^1) : \quad \phi^{b^1}(Y) = \sum_{i=1}^m y_i V_i^1(b^1)$.

Applying the same procedure now to $(\mathfrak{p}_1, \mathcal{B}^1)$ instead of $(\mathfrak{g}, \mathcal{B}^1)$, and iterating this process, which stops after a finite number d of steps, we construct indices $j_i(X,b)$ and $k_i(X,b)$ for $i = 1, \cdots, d$, and finally stop at some r.v.n $(\mathfrak{p}_d, \mathcal{B}^d)$ where $\mathcal{B}^d \subset \mathbb{R}^m \times \mathcal{B}^{d-1}$ is Zariski-open. We put $O = \mathcal{B}^d$.

Moreover, it has been shown in [L.M.] that for $(X, b^{d-1}) \in O$ the subalgebra $\mathfrak{p}_d(\phi^{b^{d-1}}(X), b^{d-1}) = \mathfrak{b}(\phi^{b^{d-1}}(X))$ is the Vergne polarization for $\phi^{b^d}(X)$ associated to the basis $\mathcal{Z}$ and there exist rational mappings $Y_i : \mathbb{R}^m \to \mathfrak{g}, \quad 1 \le i \le d$, such that $\{Y_1(X), \cdots, Y_d(X)\}$ forms a Malcev basis of $\mathfrak{g}$ relative to $\mathfrak{b}(\phi^{b^{d-1}}(X))$.

One continues as in the proof of theorem 7.7 in [L.M.]. ■

1.2.2 Proposition *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Let $\mathfrak{h}$ and $\mathfrak{b}$ be two subalgebras of $\mathfrak{g}$. There exists a Malcev-basis $\mathcal{U}$ of $\mathfrak{g}$ relative to $\mathfrak{b}$, which contains a Malcev-basis of $\mathfrak{h}$ relative to $\mathfrak{h} \cap \mathfrak{b}$.*

Proof. We proceed by induction on $\dim(\mathfrak{g})$.

Let $\mathfrak{g}_0$ be an ideal of $\mathfrak{g}$ with codimension one containing $\mathfrak{b}$.

i) If $\mathfrak{h} \subset \mathfrak{g}_0$, the induction hypothesis gives us a Malcev basis $\mathcal{U}_0$ of $\mathfrak{g}_0$ relative to $\mathfrak{b}$ which contains a Malcev-basis of $\mathfrak{h}$ relative to $\mathfrak{h} \cap \mathfrak{b}$. Hence we put $\mathcal{U} = \{\mathcal{U}_0, X\}$, where $X \in \mathfrak{g} \setminus \mathfrak{g}_0$.

ii) If $\mathfrak{h} \not\subset \mathfrak{g}_0$, we can choose $X \in \mathfrak{h}$ such that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathbb{R}X$. The induction hypothesis gives us a Malcev-basis $\mathcal{U}_0$ of $\mathfrak{g}_0$ relative to $\mathfrak{b}$, which contains a Malcev-basis of $\mathfrak{h} \cap \mathfrak{g}_0$ relative to $\mathfrak{h} \cap \mathfrak{b}$. Hence we put $\mathcal{U} = \{\mathcal{U}_0, X\}$. ■

1.3 The Bonnet Plancherel Formula

The aim of this section is to describe explicitly the Bonnet Plancherel Formula associated to the disintegration (2). Let $G, H, f, \mathcal{V}, \Sigma$ be as in (1.1.b).

For $\sigma \in \Sigma$, $g \in G_\sigma$ we define the operator: $q_{g,\sigma} : \mathcal{H}_\sigma^\infty \to \mathbb{C}$ by

$$\langle q_{g,\sigma}, \xi \rangle = \int_{H/B(g\cdot\sigma)\cap H} \overline{\xi(hg)\chi_f(h)} d\dot{h}.$$

It has already been shown in [Fuj.1] that the integral on the right is well defined (here it suffices to use that for $g \in G_\sigma$ $\chi_{g\cdot\sigma}(h) = \chi_f(h)$ for all $h \in H$), the operator $q_{g,\sigma}$ is continuous and for all $h \in H$, $\pi_\sigma(h) q_{g,\sigma} = \chi_f(h) q_{g,\sigma}$.

Let φ be in $\mathcal{S}(G)$. For $\sigma \in \mathcal{V}$, the operator $\pi_\sigma(\varphi)$ is a kernel-operator, whose kernel $K_{\pi_\sigma(\varphi)}$ is given by

$$K_{\pi_\sigma(\varphi)}(x, y) = \int_{B(\sigma)} \varphi(xby^{-1}) \chi_\sigma(b) db, \; x, y \in G.$$

Furthermore, for any Malcev-basis $\mathcal{Y} = \{Y_1, \cdots, Y_d\}$ of $\mathfrak{g}$ relative to $\mathfrak{b}(\sigma)$, the function

$$(s, t) \mapsto K_{\pi_\sigma(\varphi)}(\prod_{i=1}^{d} \exp(s_i Y_i), \prod_{i=1}^{d} \exp(t_i Y_i))$$

is a Schwartz-function on $\mathbb{R}^d \times \mathbb{R}^d$ (see [C.G.]).

Let us recall some results from [L.M.].

Let $Z_1, \cdots, Z_n \in \mathfrak{g}$ be a basis of $\mathfrak{g}$, and put

$$L := \sum_{j=1}^{n} Z_j^2 \in U(\mathfrak{g}),$$

where $U(\mathfrak{g})$ is the envelopping algebra of $\mathfrak{g}$. Let $N \in \mathbb{N}$. Since $(1-L)^N$ is hypoelliptic for every $N \in \mathbb{N}^*$, there exists a local fundamental solution $E_N \in \mathcal{D}'(U)$ of $(1-L)^N$ on a neighbourhood U of $e \in G$, i.e.

$$(1-L)^N E_N = \delta_e \text{ in } U.$$

Since $(1-L)^N$ is hypoelliptic, we have that E_N is C^∞ on $G \setminus \{e\}$ and for $d \in \mathbb{N}$, if N is big enough E_N is in $C^d(G)$. Hence E_N is in $L^1(G) \cap L^2(G)$ and is even of class C^d in $L^1(G)$.

We recall that the $N'th$ Sobolev L^1-norm on G is defined by

$$\|f\|_{N,1} = \sum_{|\alpha| \leq N} \|Z^\alpha * f\|_1 + \sum_{|\alpha| \leq N} \|f * Z^\alpha\|_1,$$

where $Z^\alpha = Z_1^{\alpha_1} * \cdots * Z_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$.

1.3.1 Proposition *There exists $N \in \mathbb{N}$ such that for almost all $\sigma \in \Sigma$ and all $g \in G_\sigma$ the distribution $q_{g,\sigma}$ is an element of $\mathcal{H}_\sigma^{-N}$ and*

$$\int_{G/B(\sigma)} \int_{H/H \cap B(g\sigma)} |K_{\pi_\sigma(\varphi)}(u, hg)| d\dot{h} d\dot{u} \leq C_{\sigma,g} \|\varphi\|_{N,1} \;\; \varphi \in \mathcal{S}(G),$$

for some constant $C_{\sigma,g}$.

Proof. Let $\sigma \in \Sigma$ and $g \in G$, such that $g\sigma \in \mathcal{V}$. Let $B_1 =< Y_1, \cdots, Y_d >$ be a Malcev-basis of $\mathfrak{g}$ relative to $\mathfrak{b}(g\sigma)$ such that $B_1' =< Y_{i_1}, \cdots, Y_{i_r} >$ is a Malcev-basis of $\mathfrak{h}$ relative to $\mathfrak{h} \cap \mathfrak{b}(g\sigma)$ (according to 1.2.2). Then we have

$$\begin{aligned} \langle q_{g,\sigma}, \xi \rangle &= \int_{\mathbb{R}^r} \overline{\xi(\exp(t_1 Y_{i_1}) \cdots \exp(t_r Y_{i_r}) g) e^{-i<f, \sum_{j=1}^r t_j Y_{i_j}>}} dt_1 \cdots dt_r \\ &= \int_{\mathbb{R}^r} \overline{(\xi_g \circ E_{B_1'})(t_1, \cdots, t_r)(\chi_f \circ E_{B_1'})(t_1, \cdots, t_r)} dt_1 \cdots dt_r \end{aligned}$$

where $\xi_g(g') = \xi(g'g)$.

Let P_σ be a function on $G/B(\sigma)$ such that $S \mapsto P_\sigma(E'_{B_1}(S))$ is a polynomial on $\mathbb{R}^d$ of degree $\leq 2r$ such that

$$c_{g\sigma} = \int_{\mathbb{R}^r} \frac{1}{|P_\sigma(E'_{B_1}(T)g)|} dT < \infty.$$

Then we get

$$|\langle q_{g,\sigma}, \xi\rangle| \leq c_{g\sigma}\|(P_\sigma \cdot \xi)_g\|_\infty.$$

Let now $B_2 = B_2(g\sigma)$ be the Malcev-basis of $\mathfrak{g}$ relative to $\mathfrak{b}(g\sigma)$ obtained by theorem (1.2.1) applied to the affine subspace $\mathcal{V}$ and the one point set $\mathcal{B}$.

Let $\tilde{Q}_\sigma(S) = P_\sigma(E'_{B_2}(S)g) = P_\sigma(E'_{B_1}\circ({E'_{B_1}}^{-1} \circ E'_{B_2})(S)g),\ S \in \mathbb{R}^d$, whose coefficients depend on g, σ and whose degree is bounded by an integer M_1 independent of g, σ (by 1.1.a.1).

Moreover we can see that for some constant $c'_{g\sigma}$ big enough the polynomial $c'_{g\sigma}F = c'_{g\sigma}(1 + \|T\|^2)^{M_1}, T \in \mathbb{R}^d$, dominates the function $\tilde{Q}_\sigma$ on $\mathbb{R}^d$. Hence

$$\|\tilde{Q}_\sigma \xi_g\|_\infty \leq c'_{g\sigma}\|F\xi_g \circ E_{B_2}\|_\infty \leq c'_{g\sigma}\|D(F\xi_g \circ E_{B_2})\|_2$$

for some fixed partial differential operator with constant coefficients on $\mathbb{R}^d$. Now by theorem (1.2.1) we have that for almost all $\sigma \in \mathcal{V}$ there exists $a(\sigma) \in \mathcal{U}(\mathfrak{g})$ such that : $d\pi_\sigma(a(\sigma)) = D\circ$ multiplication by F. Moreover the degree of $d\pi_\sigma(a(\sigma))$ is bounded by a constant N independent of σ. Thus for almost all $\sigma \in \Sigma$ and all $g \in G$

$$|\langle q_{g,\sigma}, \xi\rangle| \leq c''_\sigma \|\xi\|_N$$

for some big enough constant c''_σ.

For the second statement, we remark that, since $K_{\pi_\sigma(\varphi)}$ is a Schwartz-function on $G \times G$ modulo $B(\sigma) \times B(\sigma)$ by Howe's result (see [C.G.]), the function

$$G \ni u \mapsto \eta_v(u) = K_{\pi_\sigma(\varphi)}(u, v),\ v \in G,$$

is in $\mathcal{S}(G/B(\sigma), \chi_f)$ and so by the arguments for the first statement

$$\int_{G/B(\sigma)} \int_{H/H\cap B(g\sigma)} |K_{\pi_\sigma(\varphi)}(hg, v)| d\dot{h} d\dot{v} = \int_{H/H\cap B(g\sigma)} \left(\int_{G/B(\sigma)} |K_{\pi_\sigma(\varphi)}(hg, v)| d\dot{v} \right) d\dot{h}$$

$$= \int_{H/H\cap B(g\sigma)} \frac{1}{|P_\sigma(hg)|} \left(\int_{G/B(\sigma)} |P_\sigma(hg)\eta_v(hg)| d\dot{v} \right) d\dot{h}$$

$$\leq \int_{H/H\cap B(g\sigma)} \frac{1}{|P_\sigma(hg)|} \left(\int_{G/B(\sigma)} |\pi_\sigma(a'(g\sigma))\eta_v(hg)| d\dot{v} \right) d\dot{h}$$

for some element $a'(g\sigma)$ in the enveloping algebra of $\mathfrak{g}$, whose degree is bounded by a constant N which does not depend on $g\sigma$ according to (1.2.1). Since

$$\int_{G/B(\sigma)} |\pi_\sigma(a'(g\sigma))\eta_v(hg)| d\dot{v} = \int_{G/B(\sigma)} | \int_{B(\sigma)} a'(g\sigma) * \varphi(hgbv^{-1})\chi_\sigma(b) db | d\dot{v}$$

$$\leq \int_G |a'(g\sigma) * \varphi(hgv)| dv = \int_G |a'(g\sigma) * \varphi(v)| dv \leq c_{g\sigma}\|\varphi\|_{N,1},$$

(for some constant $c_{g\sigma}$ depending on $a'(g\sigma)$) for all $h \in H$, it follows that

$$\int_{G/B(\sigma)} \int_{H/H\cap B(g\sigma)} |K_{\pi_\sigma(\varphi)}(v, hg)| d\dot{h} d\dot{v} = \int_{G/B(\sigma)} \int_{H/H\cap B(g\sigma)} |K_{\pi_\sigma(\varphi^*)}(hg, v)| d\dot{h} d\dot{v}$$

$$\leq c_{g\sigma} \|\varphi^*\|_{N,1} \int_{H/H\cap B(g\sigma)} \frac{1}{|P_\sigma(hg)|} d\dot{h} \leq C_{g\sigma} \|\varphi\|_{N,1}$$

(for some new constant $C_{g\sigma}$). ■

This gives us the one dimensional operators:

$$Q_{g,\sigma} = P_{q_{g,\sigma}, q_{g,\sigma}} : \mathcal{H}_\sigma^N \to \mathcal{H}_\sigma^{-N},\ Q_{g,\sigma}(\xi) = \langle \xi, q_{g,\sigma} \rangle q_{g,\sigma}; \qquad \xi \in \mathcal{H}_\sigma^N.$$

In particular for $\varphi \in \mathcal{S}(G)$, $\quad \pi_\sigma(\varphi) \circ Q_{g,\sigma} = P_{\pi_\sigma(\varphi) q_{g,\sigma}, q_{g,\sigma}}$ (see [G.H.L.S.]).

For $\sigma \in \Sigma$, we define the operator $U_\sigma : \mathcal{H}_\sigma^N \to \mathcal{H}_\sigma^{-N}$ as the integral of these operators:

$$U_\sigma = \int_{\Gamma_\sigma} Q_{g,\sigma} d\lambda_\sigma(\dot{g}). \tag{6}$$

We have the following:

1.3.2 Proposition *For almost all $\sigma \in \Sigma$ we have: $U_\sigma : \mathcal{H}_\sigma^N \to \mathcal{H}_\sigma^{-N}$ is trace class.*

Proof. Let $\sigma \in \Sigma, s \in G_\sigma$. We recall that the rank one operator $Q_{s,\sigma}$ has a trace which is given by:

$$\mathrm{tr}(Q_{s,\sigma}) = \mathrm{tr}(A_\sigma^{-N} \circ Q_{s,\sigma} \circ A_\sigma^{-N}) = \langle \pi_\sigma(E_N) q_{s,\sigma}, \pi_\sigma(E_N) q_{s,\sigma} \rangle,$$

where $A_\sigma^{-N} = \pi_\sigma(E_N)$ (see [G.H.L.S.]).

On the other hand for $\psi \in \mathcal{H}_\sigma^\infty$ we have:

$$\langle \pi_\sigma(E_N) q_{s,\sigma}, \psi \rangle = \langle q_{s,\sigma}, \pi_\sigma(E_N^*) \psi \rangle = \int_{H/B(s\cdot\sigma)\cap H} \overline{\pi_\sigma(E_N^*)\psi(hs)\chi_f(h)} d\dot{h}$$

$$= \int_{H/B(s\cdot\sigma)\cap H} \int_{G/B(\sigma)} \overline{K_{\pi_\sigma(E_N^*)}(hs, u)\psi(u) du \chi_f(h) d\dot{h}}$$

$$= \int_{H/B(s\cdot\sigma)\cap H} \int_{G/B(\sigma)} K_{\pi_\sigma(E_N)}(u, hs) \overline{\psi(u) du \chi_f(h)} dh.$$

As N is increasing, the function E_N becomes smoother and smoother and the kernel function

$$(u,h) \mapsto K_{\pi_\sigma(E_N)}(u,hs)$$

is decreasing more and more rapidly at infinity, and so for N big enough, this function is in $L^1(G/B(\sigma),\sigma) \otimes L^1(H/B(s\cdot\sigma)\cap H, f)$ for almost all $\sigma \in \mathcal{V}$ (see 1.3.1). Hence, using Fubini, we can deduce that

$$\begin{aligned} \langle \pi_\sigma(E_N) q_{s,\sigma}, \psi \rangle &= \int_{G/B(\sigma)} \int_{H/B(s\cdot\sigma)\cap H} K_{\pi_\sigma(E_N)}(u,hs) \overline{\chi_f(h) dh \psi(u)} du \\ &= \langle \eta_{s,\sigma}, \psi \rangle \qquad (*) \end{aligned}$$

where $\eta_{s,\sigma}(u) = \int_{H/B(s\cdot\sigma)\cap H} K_{\pi_\sigma(E_N)}(u,hs)\chi_f(h^{-1}) d\dot{h}$ is in $L^2(G/B(s\cdot\sigma), s\cdot\sigma)$.

Hence

$$\begin{aligned} \mathrm{tr}(Q_{s,\sigma}) &= \langle \eta_{s,\sigma}, \eta_{s,\sigma} \rangle \\ &= \int_{G/B(\sigma)} \eta_{s,\sigma}(g) \overline{\eta_{s,\sigma}(g)} d\dot{g} \\ &= \int_{G/B(\sigma)} \int_{H/B(s\cdot\sigma)\cap H} K_{\pi_\sigma(E_N)}(g,h's)\chi_f(h'^{-1}) d\dot{h}' \\ &\quad \overline{\int_{H/B(s\cdot\sigma)\cap H} K_{\pi_\sigma(E_N)}(g,hs)\chi_f(h^{-1}) d\dot{h}} d\dot{g} \\ &= \int_{G/B(\sigma)} \int_{H/B(s\cdot\sigma)\cap H} \int_{B(\sigma)} E_N(gbs^{-1}h'^{-1})\chi_\sigma(b) db \chi_f(h'^{-1}) d\dot{h}' \\ &\quad \overline{\int_{H/B(s\cdot\sigma)\cap H} \int_{B(\sigma)} E_N(gbs^{-1}h^{-1})\chi_\sigma(b) db \chi_f(h^{-1}) d\dot{h}} d\dot{g} \\ &= \int_{G/B(s\cdot\sigma)} \int_{H/B(s\cdot\sigma)\cap H} \int_{B(s\cdot\sigma)} E_N(gbh'^{-1})\chi_{s\cdot\sigma}(b) db \chi_f(h'^{-1}) d\dot{h}' \\ &\quad \overline{\int_{H/B(s\cdot\sigma)\cap H} \int_{B(s\cdot\sigma)} E_N(gbh^{-1})\chi_{s\cdot\sigma}(b) db \chi_f(h^{-1}) d\dot{h}} d\dot{g}. \end{aligned}$$

Now for $q \in C_c(G)$, it has been shown in [B.L.2] that

$$\int_{H/B(s\cdot\sigma)\cap H} \int_{B(s\cdot\sigma)} q(bh^{-1})\chi_{s\cdot\sigma}(b) db \chi_f(h^{-1}) d\dot{h} =$$

$$\int_{B(s\cdot\sigma)/B(s\cdot\sigma)\cap H} \int_H q(bh^{-1})\chi_{s\cdot\sigma}(b)\chi_f(h^{-1}) dh d\dot{b} \qquad (**)$$

We obtain:

$$\mathrm{tr}(Q_{s,\sigma}) = \langle T_{s\cdot\sigma}(P_{H,f}(E_N)), T_{s\cdot\sigma}(P_{H,f}(E_N)) \rangle_{\mathcal{H}_{s\cdot\sigma}} = \|T_{s\cdot\sigma}(P_{H,f}(E_N))\|^2_{\mathcal{H}_{s\cdot\sigma}}$$

On the other hand one has by (3)

$\int_\Sigma \int_{\Gamma_\sigma} \|T_{s\cdot\sigma}(P_{H,f}(E_N))\|^2_{\mathcal{H}_{s\cdot\sigma}} d\lambda_\sigma(\dot{s}) d\nu(\sigma) = \int_{\mathcal{V}} \langle T_\phi(P_{H,f}(E_N)), T_\phi(P_{H,f}(E_N)) \rangle_{\mathcal{H}_\phi} d\phi =$ $\|(P_{H,f}(E_N))\|^2_{\mathcal{H}_\tau}$ by (4).

Hence for almost all $\sigma \in \Sigma$

$$\|U_\sigma\|_1 = \int_{\Gamma_\sigma} \mathrm{tr}(Q_{g,\sigma}) d\lambda_\sigma(\dot{g}) < \infty$$

and the integral

$$U_\sigma = \int_{\Gamma_\sigma} Q_{g,\sigma} d\lambda_\sigma(\dot{g})$$

converges in the space of the trace-class operators. ■

1.3.4. Theorem *There exists $N \in \mathbb{N}$, such that for every $\varphi \in \mathcal{S}(G)$ and for almost all $\sigma \in \Sigma$, we have that the operator $\pi_\sigma(\varphi) \circ U_\sigma : \mathcal{H}_\sigma^N \to \mathcal{H}_\sigma^N$ is trace class and*

$$< S_{H,f}, \varphi >= \int_\Sigma tr(\pi_\sigma(\varphi) \circ U_\sigma) d\nu(\sigma).$$

Proof. Let $\sigma \in \Sigma, s \in G_\sigma$ and $\varphi \in \mathcal{S}(G)$. An argument similar to (*) permits us to write $\pi_\sigma(\varphi) q_{s,\sigma}(u) = \varphi_{s,\sigma}(u) = \int_{H/B(s\cdot\sigma)\cap H} K_{\pi_\sigma(\varphi)}(u, hs)\overline{\chi_f(h)} dh$, for all $u \in G$.

Then

$$\begin{aligned} \langle \pi_\sigma(\varphi) q_{s,\sigma}, q_{s,\sigma} \rangle &= \int_{H/B(s\cdot\sigma)\cap H} \varphi_{s,\sigma}(hs)\chi_f(h) d\dot{h} \\ &= \int_{H/B(s\cdot\sigma)\cap H} \int_{H/B(s\cdot\sigma)\cap H} K_{\pi_\sigma(\varphi)}(hs, h's)\chi_f(h'^{-1}) d\dot{h}'\chi_f(h) d\dot{h} \\ &= \int_{H/B(s\cdot\sigma)\cap H} \int_{H/B(s\cdot\sigma)\cap H} K_{\pi_{s\cdot\sigma}(\varphi)}(h, h')\chi_f(hh'^{-1}) d\dot{h}' d\dot{h}. \end{aligned}$$

We recall that $\pi_\sigma(\varphi) \circ U_\sigma = \pi_\sigma(\varphi) \circ \int_{\Gamma_\sigma} P_{q_{s,\sigma}, q_{s,\sigma}} d\lambda_\sigma(\dot{s}) = \int_{\Gamma_\sigma} P_{\pi_\sigma(\varphi) q_{s,\sigma}, q_{s,\sigma}} d\lambda_\sigma(\dot{s})$.

Hence we deduce that

$$\mathrm{tr}(\pi_\sigma(\varphi) \circ U_\sigma) = \int_{\Gamma_\sigma} \int_{H/B(s\cdot\sigma)\cap H} \int_{H/B(s\cdot\sigma)\cap H} K_{\pi_{s\cdot\sigma}(\varphi)}(h, h')\chi_f(hh'^{-1}) d\dot{h}' d\dot{h} d\lambda_\sigma(\dot{s}). \tag{$***$}$$

Now we recall that, from [B.L.2] one has

$$\begin{aligned} \langle S_{H,f}, \varphi \rangle &= \int_{\mathcal{V}} \langle S_\phi, \varphi \rangle d\mu(\phi) \\ &= \int_\Sigma \int_{\Gamma_\sigma} \int_{H/H\cap B(s\cdot\sigma)} T_{s\cdot\sigma}(P_{H,f}(\varphi))(h)\chi_f(h) d\dot{h} d\lambda_\sigma(\dot{s}) d\nu(\sigma) \end{aligned}$$

(by (3) and (5)).

On the other hand

$$\begin{aligned}
&\int_{\Gamma_\sigma}\int_{H/H\cap B(s\cdot\sigma)} T_{s\cdot\sigma}(P_{H,f}(\varphi))(h)\chi_f(h)d\dot{h}d\lambda_\sigma(\dot{s})\\
&= \int_{\Gamma_\sigma}\int_{H/H\cap B(s\cdot\sigma)}\int_{B(s\cdot\sigma)/B(s\cdot\sigma)\cap H} P_{H,f}(\varphi)(hb)\chi_{s\cdot\sigma}(b)db\chi_f(h)d\dot{h}d\lambda_\sigma(\dot{s})\\
&= \int_{\Gamma_\sigma}\int_{H/H\cap B(s\cdot\sigma)}\int_{B(s\cdot\sigma)/B(s\cdot\sigma)\cap H}\int_H \varphi(hbh')\chi_f(h')dh'\chi_{s\cdot\sigma}(b)db\chi_f(h)d\dot{h}d\lambda_\sigma(\dot{s})\\
&= \int_{\Gamma_\sigma}\int_{H/H\cap B(s\cdot\sigma)}\int_{B(s\cdot\sigma)/B(s\cdot\sigma)\cap H}\int_H \varphi(hbh'^{-1})\chi_f(h'^{-1})dh'\chi_{s\cdot\sigma}(b)db\chi_f(h)d\dot{h}d\lambda_\sigma(\dot{s}).
\end{aligned}$$

Then by $(**), (***)$

$$\begin{aligned}
&\int_{\Gamma_\sigma}\int_{H/H\cap B(s\cdot\sigma)} T_{s\cdot\sigma}(P_{H,f}(\varphi))(h)\chi_f(h)d\dot{h}d\lambda_\sigma(\dot{s})\\
&= \int_{\Gamma_\sigma}\int_{H/H\cap B(s\cdot\sigma)}\int_{H/B(s\cdot\sigma)\cap H}\int_{B(s\cdot\sigma)} \varphi(hbh'^{-1})\chi_{s\cdot\sigma}(b)db\chi_f(hh'^{-1})d\dot{h}'d\dot{h}d\lambda_\sigma(\dot{s})\\
&= \int_{\Gamma_\sigma}\int_{H/H\cap B(s\cdot\sigma)}\int_{H/B(s\cdot\sigma)\cap H} K_{\pi_{s\cdot\sigma}(\varphi)}(h,h')\chi_f(hh'^{-1})d\dot{h}'d\dot{h}d\lambda_\sigma(\dot{s})\\
&= \operatorname{tr}(\pi_\sigma(\varphi)\circ U_\sigma).
\end{aligned}$$

Whence

$$\langle S_{H,f},\varphi\rangle = \int_\Sigma \operatorname{tr}(\pi_\sigma(\varphi)\circ U_\sigma)d\nu(\sigma).$$

■

2. The Bonnet Plancherel formula for a class of completely solvable Lie group

In this part we take, as mentioned in the introduction, the semi-direct product $G = NH$; where $N = \exp(\mathfrak{n})$ is nilpotent and normal in G, and $H = \exp(\mathfrak{h})$ is abelian and acts semi-simply on N with real eigenvalues. Let $\chi = \chi_f$ be a unitary character of H (where $f \in \mathfrak{g}^*$). We consider the representation $\tau_f = Ind_H^G\chi_f$ and we assume that τ_f has finite multiplicity.

Let us recall some results given in the paper [Cu.2].

2.1 Generalities and main results

2.1.1 C^∞ vectors

Let G be an exponential solvable Lie group and K a closed subgroup of G. Fix a choice of right Haar measures dg, dk on G and K. We write Δ_G, Δ_K for the modular functions of G, K (respectively). If χ is a unitary character of K, the induced representation $\pi_\chi = Ind_K^G \chi$ acts in the space $C_c^\infty(G,K,\chi) = \{f \in C^\infty(G) : f(kg) = \chi(k)f(g) \ \forall k \in K, g \in G; f$ compactly supported mod $K\}$, by the formula

$$\pi_\chi(g)f(x) = f(xg)q(g)^{1/2}.$$

Here $q = q_{K,G} : G \to \mathbb{R}_+^*$ is a smooth function on G satisfying $q(e) = 1, \quad q(kg) = \Delta_{K,G}(k)q(g)$.

The space $K \setminus G$ carries a relatively invariant measure $d\gamma$ with modulus q^{-1} which satisfies:

$$\int_{K\setminus G} f(\gamma g)d\gamma = \int_{K\setminus G} f(\gamma)q(g^{-1})d\gamma$$

where $f \in C_c(K \setminus G)$.

The Hilbert space $\mathcal{H}_{\pi_\chi} = L^2(G,K,\chi)$ is the completion of $C_c^\infty(G,K,\chi)$ under the norm $\|f\|_2 = (\int_{K\setminus G} |f(\gamma)|^2 d\gamma)^{1/2}$.

Now let π be a unitary representation of G on a Hilbert space $\mathcal{H}_\pi$, we denote by $\mathcal{H}_\pi^\infty$ the Fréchet space of smooth vectors of π. Its anti-dual space is denoted by $\mathcal{H}_\pi^{-\infty}$. It is well known that $\pi(D(G))\mathcal{H}_\pi^{-\infty} \subset \mathcal{H}_\pi^\infty$ where $D(G) = C_c^\infty(G)$.

2.1.2 Algebraic structure

Let $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ where $\mathfrak{n}$ is nilpotent, $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{n}$ and where $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ such that $ad(\mathfrak{h})$ consists of semi-simple endomorphisms with real eigenvalues.

In [Cu.2] it has been shown that if τ_f is of finite multiplicity then the Lie algebra $\mathfrak{g}$ has a basis $\mathcal{B} = \{C_1, \cdots, C_a, V_1, \cdots, V_\nu, X_1, \cdots, X_u, Y_1, \cdots, Y_u, A_1, \cdots, A_u, B_1, \cdots, B_\nu\}$ such that

$\mathfrak{n} = vect < C_1, \cdots, C_a, V_1, \cdots, V_\nu, X_1, \cdots, X_u, Y_1, \cdots, Y_u >$

and $\mathfrak{h} = vect < A_1, \cdots, A_u, B_1, \cdots, B_\nu >$. Furthermore we have:

i) $[X_h, Y_{h'}] = 0$ if and only if $h \neq h'$ and $[X_h, Y_h]$ is central in $\mathfrak{n}$ for $1 \leq h \leq u$.

ii) For every $h, h' \quad [X_h, X_{h'}] = [Y_h, Y_{h'}] = 0$.

iii) $cent(\mathfrak{g}) = vect < C_1, \cdots, C_a >$, and $cent(\mathfrak{n}) = vect < C_1, \cdots, C_a, V_1, \cdots, V_\nu >$.

iv) $[A_h, X_h] = -X_h; \ [A_h, Y_h] = Y_h; \quad [A_h, X_{h'}] = [A_h, Y_{h'}] = 0 \quad$ for $h \neq h'$.

v) $[B_k, X_h] = \alpha_{k,h}X_h, \quad \alpha_{k,h} \in \mathbb{R}; \ [B_k, Y_h] = 0; \ [A_h, V_k] = 0; \ [B_k, V_k] = V_k$ and $[B_k, V_{k'}] = 0$ for $k \neq k'$

(see Theorem 1.8 in [Cu.2]), we have simplified here the notations of Currey).

2.1.3 Plancherel formula

Let τ be the monomial representation: $\tau = \tau_f = Ind_H^G \chi_f$. To decompose τ means to describe the spectrum of τ, the multiplicities and the equivalence class of the Plancherel measure in terms of the coadjoint orbit picture.

In the case of a completely solvable Lie group, it has been shown in [Li.1] that the spectral decomposition formula is given by $\tau = \int_{(f+\mathfrak{h}^\perp)/H}^{\oplus} \pi_\theta d\nu(\theta)$ where ν is a pushforward of a finite measure on $(f + \mathfrak{h}^\perp)$ which is equivalent to Lebesgue measure.

In the case with which we are concerned where $G = NH$ and τ_f has finite multiplicity, it has been shown in [Cu.2] that the set of generic $H-$orbits in the decomposition of τ_f admits a natural algebraic cross-section Σ and the measure ν is given as an explicit measure on Σ.

Furthermore we can choose $f|\mathfrak{n} = 0$.

The cross-section in $f + \mathfrak{h}^\perp$ is $f + \Sigma$ and is given as follows:

Fixing a choice of signs $\theta = (\epsilon, \delta) = (\epsilon_1, \cdots, \epsilon_u, \delta_1, \cdots, \delta_\nu) \in \{1, -1\}^d; d = u + \nu$, one has $\Sigma = \bigcup_{\theta\in\{1,-1\}^d} \Sigma_\theta$ where $\Sigma_\theta = \{l \in \Omega \cap \mathfrak{h}^\perp;\ \ l(Y_k) = \epsilon_k,\quad 1 \le k \le u$ and $l(V_i) = \delta_i,\quad 1 \le i \le \nu\}$. Here $\Omega = \Omega_0 \cap \Omega_1$, where Ω_0 is the set of $G-$orbits having maximal dimension in $\mathfrak{g}^*$ and Ω_1 consists with $H-$orbits of maximal dimension. The irreducible representations which correspond to $G-$orbits $G{\cdot}l,\ l \in \Omega \cap (f+\mathfrak{h}^\perp)$, are sufficient to decompose τ_f.

There exists a dense open subset D_θ of $\mathbb{R}^a \times \mathbb{R}^u$ such that

$$\Sigma_\theta = \{\sum_{h=1}^{a} \xi_h C_h^* + \sum_{i=1}^{\nu} \delta_i V_i^* + \sum_{k=1}^{u} \epsilon_k Y_k^* + \sum_{k=1}^{u} \mu_k X_k^*;\quad (\xi, \mu) \in D_\theta\} \tag{7}$$

(see [Cu.2], we have made a small change of notations).

Let F be a function on $f + \mathfrak{h}^\perp$. One has

$$\int_{f+\Sigma} F(l)dl = \sum_{\theta\in\{1,-1\}^d} \int_{\mathbb{R}^a\times\mathbb{R}^u} F(f+\sum_{h=1}^{a} \xi_h C_h^*+\sum_{i=1}^{\nu} \delta_i V_i^*+\sum_{k=1}^{u} \epsilon_k Y_k^*+\sum_{k=1}^{u} \mu_k X_k^*)\ d\xi d\mu.$$

Now for $l \in \Sigma$, an $H-$covariant generalized vector for π_l is defined formally by; for $\psi \in \mathcal{H}_l^\infty$

$$\beta_l(\psi) = \int_H \overline{\psi(h)} q_{B,G}^{1/2} q_{H,G}^{-1/2} \chi_f(h)dh, \tag{8}$$

(see 2.1 in [Cu.2]).

2.1.3.1. Theorem [Cu.2] *The integral (8) is absolutely convergent for every* $\psi \in \mathcal{H}_l^\infty$ *and* β_l *is continuous on* $\mathcal{H}_l^N$ *for a certain integer* N (see [Cu.2] proof of theorem 2.2).

The distribution-theoretic Plancherel formula which is equivalent to the disintegration of τ_f is

$$\langle \tau_f(\omega)\alpha_\tau, \alpha_\tau \rangle = \int_{f+\Sigma} \langle \pi_l(\omega)\beta_l, \beta_l \rangle |R(l)| dl$$

where $R(l) = ((2\pi)^n l([X_1, Y_1]) l([X_2, Y_2]) \cdots l([X_u, Y_u]))^{-1}$ with $n = \dim(\mathfrak{n})$ and α_τ is the generalized cyclic vector for τ: $\alpha_\tau(\xi) = \overline{\xi(e)}$ for $\xi \in \mathcal{H}_\tau^\infty$ (cf. [Cu.2] Theorem 3.2).

Of course the reference [Cu.2] contains more information than is conveyed here.

2.2 The Bonnet Plancherel formula

The aim of this section is to describe explicitly the Bonnet Plancherel Formula associated to the disintegration of τ_f. Let G, H, f (and so on) be as above. We recall that the distribution S_{H,χ_f}, defined on $D(G)$ by: $\langle S_{H,\chi_f}, \varphi \rangle = \int_H \varphi(h)\chi_f(h) \Delta_{G,H}^{1/2}(h) dh$, is positive.

By the theorem of P. Bonnet [Bon.], there exist positive nuclear operators $U_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^{-\infty}$, such that

$$\langle S_{H,\chi_f}, \varphi \rangle = \int_{\hat{G}} \operatorname{tr}(\pi(\varphi) U_\pi) d\mu(\pi), \quad \varphi \in D(G).$$

We shall show that the operators U_π are finite sum of rank one operators. The first step is a determination of a cross-section for G-orbits in $G.(f + \Sigma)$.

Let $l = f + l_0 \in f + \Sigma$. By (2.1.3) there exists $\theta = (\epsilon, \delta) = (\epsilon_1, \cdots, \epsilon_u, \delta_1 \cdots, \delta_\nu) \in \{-1, 1\}^d$ such that $l_0 \in \Sigma_\theta$: $l_0 = \sum_{h=1}^{a} \xi_h C_h^* + \sum_{i=1}^{\nu} \delta_i V_i^* + \sum_{k=1}^{u} \epsilon_k Y_k^* + \sum_{k=1}^{u} \mu_k X_k^*$; the $G-$orbit of l consists of elements l' of the form:

$$l' = \sum_{h=1}^{a} \xi_h C_h^* + \sum_{i=1}^{\nu} \delta_i w_i V_i^* + \sum_{k=1}^{u} y_k Y_k^* + \sum_{k=1}^{u} x_k X_k^* + \sum_{k=1}^{u} P_k(w, x_k, y_k) A_k^* + \sum_{i=1}^{\nu} b_i B_i^*$$

where $w_i \in]0, +\infty[\quad 1 \le i \le \nu,\ x_k, y_k, b_i \in \mathbb{R}$ and P_k are polynomials in x_k, y_k and rationals in $w_i, \quad 1 \le k \le u$.

It has been shown in [Cu.2] that

$$O_l = G\cdot l \cap (f+\Sigma) = f + \bigcup_{\epsilon' \in \{-1,1\}^u} \{\sum_{h=1}^{a} \xi_h C_h^* + \sum_{i=1}^{\nu} \delta_i V_i^* + \sum_{k=1}^{u} \epsilon'_k Y_k^* + \sum_{k=1}^{u} \epsilon_k \epsilon'_k \mu_k X_k^*\}. \tag{9}$$

We give a cross-section for $G-$ orbits in $G.(f+\Sigma)$ as the set

$$\begin{aligned}\Gamma &= \Big\{f+\sum_{h=1}^{a}\xi_h C_h^* + \sum_{i=1}^{\nu}\delta_i V_i^* + \sum_{k=1}^{u} Y_k^* + \sum_{k=1}^{u}\mu_k X_k^*, \quad (\xi_h,\mu_k)\in\mathbb{R}^a\times\mathbb{R}^u, \\ &\quad \text{and } \delta=(\delta_1,\cdots,\delta_\nu)\in\{-1,1\}^\nu\Big\} \\ &= \bigcup_{\delta\in\{-1,1\}^\nu}\Gamma_\delta.\end{aligned}$$

We see that our cross-section Γ for $G-$ orbits in $G.(f+\Sigma)$ is contained in $f+\Sigma$.

Furthermore we decompose the Lebesgue measure on $f+\Sigma$ into integral of measures on $O_l, l\in\Gamma$: Given a function F on $(f+\Sigma)$ we write:

$$\int_{f+\Sigma}F(l)dl = \int_\Gamma\int_{G\cdot\sigma\cap(f+\Sigma)}F(\phi)d\mu_\sigma(\phi)d\nu(\sigma) \tag{10}$$

$$= \sum_{\delta\in\{1,-1\}^\nu}\int_{\mathbb{R}^a\times\mathbb{R}^u}\sum_{\epsilon'\in\{1,-1\}^u}F(f+\sum_{h=1}^{a}\xi_h C_h^*+\sum_{i=1}^{\nu}\delta_i V_i^*+\sum_{k=1}^{u}\epsilon'_k Y_k^*+\sum_{k=1}^{u}\epsilon'_k\mu_k X_k^*)\ d\xi d\mu.$$

On the other hand recall that for all $\omega\in D(G)$ we have by [Cu.2]:

$$\langle\tau_f(\omega)\alpha_\tau,\alpha_\tau\rangle = \int_{f+\Sigma}\langle\pi_l(\omega)\beta_l,\beta_l\rangle|R(l)|dl,$$

where $R(l)=((2\pi)^n\prod_{k=1}^{u}l([X_k,Y_k]))^{-1}$.

Remarks

i) From the construction of vectors X_k, Y_k one can verify that $l([X_k,Y_k])\neq 0$ for all $l\in\Omega$.

ii) Since for all $1\le k\le u,\quad [X_k,Y_k]\in cent(\mathfrak{n})$ then for every $\sigma\in\Gamma$ by (9) we have $R(\sigma)=R(l)\quad\forall l\in G\cdot\sigma\cap(f+\Sigma)$. Thus we can write $R(l)=R(f,\delta,\xi)$ as a function uniquely depending on $f,\delta=(\delta_1,\cdots,\delta_\nu)$ and $\xi=(\xi_1,\cdots,\xi_a)$.

Let us write $\pi_{(\xi,\delta,\epsilon,\mu)}$ for the irreducible representation associated to the element $l=l(\xi,\delta,\epsilon,\mu)=f+\sum_{h=1}^{a}\xi_h C_h^*+\sum_{i=1}^{\nu}\delta_i V_i^*+\sum_{k=1}^{u}\epsilon_k Y_k^*+\sum_{k=1}^{u}\mu_k X_k^*$ in $\mathfrak{g}^*$. We deduce that:

$$\langle\tau_f(\omega)\alpha_\tau,\alpha_\tau\rangle = \sum_{\delta\in\{1,-1\}^\nu}\int_{\mathbb{R}^a\times\mathbb{R}^u}\sum_{\epsilon\in\{1,-1\}^u}\langle\pi_{(\xi,\delta,\epsilon,\mu)}(\omega)\beta_{(\xi,\delta,\epsilon,\mu)},\beta_{(\xi,\delta,\epsilon,\mu)}\rangle|R(f,\delta,\xi)|d\xi d\mu. \tag{11}$$

Let now $\sigma=f+\sum_{h=1}^{a}\xi_h C_h^*+\sum_{i=1}^{\nu}\delta_i V_i^*+\sum_{k=1}^{u}Y_k^*+\sum_{k=1}^{u}\mu_k X_k^*\ \in\Gamma\subset(f+\Sigma)$.

For every $l \in G \cdot \sigma \cap (f + \Sigma)$ there exists by (9) an $\epsilon \in \{-1, 1\}^u$ such that:

$$l = f + \sum_{h=1}^{a} \xi_h C_h^* + \sum_{i=1}^{\nu} \delta_i V_i^* + \sum_{k=1}^{u} \epsilon_k Y_k^* + \sum_{k=1}^{u} \epsilon_k \mu_k X_k^*.$$

Put for $1 \le k \le u :\quad a_k(\sigma) = \langle \sigma, [X_k, Y_k] \rangle$. Since $[X_k, Y_k] \in cent(\mathfrak{n})$, we have that $a_k(\sigma) = a_k(l)$. Then by the obvious remark (i) one has $a_k(\sigma) \neq 0$.

Let $g_l = \prod_{k=1}^{u} \exp(y_k Y_k) \prod_{k=1}^{u} \exp(x_k X_k) \prod_{h=1}^{\nu} \exp(v_h V_h) \in N$, where $x_k = \frac{1-\epsilon_k}{a_k(l)}$,

$y_k = \frac{\epsilon_k - 1}{a_k(l)} \mu_k$, and $v_h = -\delta_h^{-1} \sum_{k=1}^{u} \frac{1-\epsilon_k}{a_k(l)} \alpha_{h,k} \mu_k$.

2.2.1 Lemma. *We have that:*

$$l = g_l \cdot \sigma$$

Proof. We recall that

$\mathfrak{g} = vect < C_1, \cdots, C_a, V_1, \cdots, V_\nu, X_1, \cdots, X_u, Y_1, \cdots, Y_u, A_1, \cdots, A_u, B_1, \cdots, B_\nu > .$

According to the expressions of σ, l and since the vectors C_h and V_i are central in $\mathfrak{n}$ we have $g_l \cdot \sigma(C_h) = l(C_h),\quad 1 \le \forall h \le a$, and $g_l \cdot \sigma(V_i) = l(V_i),\quad 1 \le \forall i \le \nu$.

Fix $s \in \{1, \cdots, \nu\}$, we have by (2.1.2.v) and the fact that $f_{|\mathfrak{n}} = 0$

$$\begin{aligned} g_l \cdot \sigma(B_s) &= \sigma(Ad(\prod_{h=1}^{\nu} \exp(-v_h V_h) \prod_{k=1}^{u} \exp(-x_k X_k))(B_s)) \\ &= \sigma(Ad(\prod_{h=1}^{\nu} \exp(-v_h V_h))(B_s + \sum_{k=1}^{u} x_k \alpha_{s,k} X_k)) \\ &= \sigma(B_s + v_s V_s + \sum_{k=1}^{u} x_k \alpha_{s,k} X_k) \\ &= \sigma(B_s) + \delta_s v_s + \sum_{k=1}^{u} x_k \alpha_{s,k} \mu_k \\ &= \sigma(B_s) - \sum_{k=1}^{u} x_k \alpha_{s,k} \mu_k + \sum_{k=1}^{u} x_k \alpha_{s,k} \mu_k \\ &= \sigma(B_s) = l(B_s) = f(B_s). \end{aligned}$$

For $1 \leq i \leq u$, we have by (2.1.2.v), (2.1.2.iv), (2.1.2.ii) and by the fact that $f_{|\mathfrak{n}} = 0$:

$$\begin{aligned} g_l \cdot \sigma(A_i) &= \sigma(Ad(\prod_{k=1}^{u} \exp(-x_k X_k))(A_i + y_i Y_i)) \\ &= \sigma(A_i + y_i Y_i - x_i X_i - x_i y_i [X_i, Y_i]) \\ &= \sigma(A_i) + y_i - x_i \mu_i - x_i y_i a_i(\sigma) \\ &= \sigma(A_i) + \frac{\epsilon_i - 1}{a_i(\sigma)}\mu_i + \frac{\epsilon_i - 1}{a_i(\sigma)}\mu_i + \frac{(\epsilon_i - 1)^2}{a_i(\sigma)}\mu_i \\ &= \sigma(A_i) + \frac{\mu_i}{a_i(\sigma)}(2\epsilon_i - 2 + 1 + (\epsilon_i)^2 - 2\epsilon_i) \\ &= \sigma(A_i) = l(A_i), \end{aligned}$$

$$\begin{aligned} g_l \cdot \sigma(X_i) &= \sigma(Ad(\prod_{k=1}^{u} \exp(-x_k X_k))(X_i - y_i [Y_i, X_i])) \\ &= \sigma(X_i + y_i [X_i, Y_i]) \\ &= \sigma(X_i) + (\epsilon_i - 1)\mu_i \\ &= \epsilon_i \mu_i \\ &= l(X_i) \end{aligned}$$

and

$$\begin{aligned} g_l \cdot \sigma(Y_i) &= \sigma(Y_i - x_i [X_i, Y_i]) \\ &= \sigma(Y_i) - (-\epsilon_i + 1) \\ &= \epsilon_i \\ &= l(Y_i). \end{aligned}$$

Thus $g_l \cdot \sigma = l$. ■

We turn now to Bonnet's operators. First we define for every $l \in G \cdot \sigma \cap (f + \Sigma)$ an operator $\beta'_l : \mathcal{H}^\infty_\sigma \to \mathbb{C}$ by

$$\beta'_l(\psi) = \int_H \overline{\psi(g_l^{-1} h)}\ q_{B,G}^{1/2}\ q_{H,G}^{-1/2}\ \chi_f(h) dh \tag{12}$$

and a function ψ_{g_l} by $\psi_{g_l}(g') = \psi(g_l^{-1} g'), g' \in G$. We can see that ψ_{g_l} is an element of $\mathcal{H}^\infty_l$. Indeed, the covariance condition is satisfied.

Let $B(l)$ be the Vergne polarization associated to l and to our Jordan-Hölder basis of $\mathfrak{g}$. For $g' \in G, b \in B(l)$ we have $\psi_{g_l}(bg') = \psi(g_l^{-1} bg') = \psi(g_l^{-1} b g_l g_l^{-1} g')$. Since

$l = g_l \cdot \sigma$, we have that then $B(l) = g_l B(\sigma) g_l^{-1}$ and $b' = g_l^{-1} b g_l \in B(\sigma)$. Hence

$$\begin{aligned} \psi_{g_l}(bg') &= \psi(b'g_l^{-1}g') \\ &= \chi_\sigma(b')\psi(g_l^{-1}g') \quad (\psi \in \mathcal{H}_\sigma^\infty) \\ &= \chi_\sigma(b')\psi_{g_l}(g') \\ &= \chi_l(b)\psi_{g_l}(g'). \end{aligned}$$

Evidently ψ_{g_l} is C^∞ function. We obtain $\beta'_l(\psi) = \beta_l(\psi_{g_l})$ where β_l is as in (8). Then using (2.1.3.1) we have that (12) converges for all $\psi \in \mathcal{H}_\sigma^\infty$ and $\beta'_l \in \mathcal{H}_\sigma^{-\infty}$.

Let $\sigma \in \Gamma, l \in G\cdot\sigma \cap (f+\Sigma)$ and $\epsilon = (\epsilon_1, \cdots, \epsilon_u) \in \{-1,1\}^u$ such that $\epsilon_k = l(Y_k)$. Since l depends only on ϵ we put $\beta'_l = \beta'_\epsilon$ and we define the operator $U_\sigma : \mathcal{H}_\sigma^\infty \to \mathcal{H}_\sigma^{-\infty}$ by:

$$U_\sigma = \sum_{\epsilon \in \{-1,1\}^u} P_{\beta'_\epsilon, \beta'_\epsilon}. \tag{13}$$

Here $P_{\beta'_\epsilon,\beta'_\epsilon} : \mathcal{H}_\sigma^\infty \to \mathcal{H}_\sigma^{-\infty}$ is a rank one operator defined by $P_{\beta'_\epsilon,\beta'_\epsilon}(\psi) = \langle \psi, \beta'_\epsilon \rangle \beta'_\epsilon$.

We have the following:

2.2.2 Theorem *Let $G = exp(\mathfrak{g})$ be the semi direct product; where $N = exp(\mathfrak{n})$ is nilpotent and normal in G, and $H = exp(\mathfrak{h})$ is abelian and acts semi-simply on N with real eigenvalues. Let f be a linear functional of $\mathfrak{g}$ such that $f([\mathfrak{h},\mathfrak{h}]) = \{0\}$ and χ_f the corresponding unitary character of H. Let $\tau_f = Ind_H^G \chi_f$ and assume that τ_f has finite multiplicity. Let $\Sigma \subset \mathfrak{g}^*$ be the cross-section for the $H-$orbit in $\Omega \cap \mathfrak{h}^\perp$ given in [Cu.2]. Then there exists: a cross-section Γ for the $G-$orbit in $G \cdot (f + \Sigma)$, a measure ν on Γ, such that for every $\omega \in D(G)$ we have:*

$$\langle \tau_f(\omega)\alpha_\tau, \alpha_\tau \rangle = \int_\Gamma tr(\pi_\sigma(\omega) \circ U_\sigma) d\nu(\sigma)$$

where $U_\sigma, \sigma \in \Gamma$, is defined in (13).

Proof. Let $\omega \in D(G)$. We have $\pi_\sigma(\omega) \circ U_\sigma = \sum_{\epsilon \in \{-1,1\}^u} P_{\pi_\sigma(\omega)\circ\beta'_\epsilon, \beta'_\epsilon}$. Hence

$$\mathrm{tr}(\pi_\sigma(\omega) \circ U_\sigma) = \sum_{\epsilon \in \{-1,1\}^u} \langle \pi_\sigma(\omega)\beta'_\epsilon, \beta'_\epsilon \rangle.$$

On the other hand, for all $\psi \in \mathcal{H}_\sigma^\infty$, we have:

$\langle \pi_\sigma(\omega)\beta'_\epsilon, \psi \rangle = \langle \beta'_\epsilon, \pi_\sigma(\omega^*)\psi \rangle = \beta_l((\pi_\sigma(\omega^*)\psi)_{g_l})$; where $l = g_l \cdot \sigma$.

Since for all $x \in G$

$$\begin{aligned}(\pi_\sigma(\omega^*)\psi)_{g_l}(x) &= \pi_\sigma(\omega^*)\psi(g_l^{-1}x)\\ &= \int_G \omega^*(y)(\pi_\sigma(y)\psi)(g_l^{-1}x)dy\\ &= \int_G \omega^*(y)\psi(g_l^{-1}xy)q(y)^{1/2}dy\\ &= \int_G \omega^*(y)\psi_{g_l}(xy)q(y)^{1/2}dy\\ &= \pi_l(\omega^*)\psi_{g_l}(x),\end{aligned}$$

it follows that $(\pi_\sigma(\omega^*)\psi)_{g_l} = \pi_l(\omega^*)\psi_{g_l}$.

Thus

$$\begin{aligned}\langle(\pi_\sigma(\omega)\beta'_\epsilon)_{g_l}, \psi_{g_l}\rangle_{\mathcal{H}_l} = \langle\pi_\sigma(\omega)\beta'_\epsilon, \psi\rangle_{\mathcal{H}_\sigma} &= \langle\beta'_\epsilon, \pi_\sigma(\omega^*)\psi\rangle_{\mathcal{H}_\sigma}\\ &= \langle\beta_\epsilon, \pi_l(\omega^*)\psi_{g_l}\rangle_{\mathcal{H}_l}\\ &= \langle\pi_l(\omega)\beta_l, \psi_{g_l}\rangle_{\mathcal{H}_l}.\end{aligned}$$

Hence $\pi_l(\omega)\beta_l = (\pi_\sigma(\omega)\beta'_\epsilon)_{g_l}$ and $\langle\pi_\sigma(\omega)\beta'_\epsilon, \beta'_\epsilon\rangle = \langle(\pi_\sigma(\omega)\beta'_\epsilon)_{g_l}, \beta_l\rangle = \langle\pi_\epsilon(\omega)\beta_\epsilon, \beta_\epsilon\rangle$.

We deduce that

$$\mathrm{tr}(\pi_\sigma(\omega) \circ U_\sigma) = \sum_{\epsilon\in\{-1,1\}^u} \langle\pi_\epsilon(\omega)\beta_\epsilon, \beta_\epsilon\rangle.$$

The formulas (10) and (11) permit us to conclude, the measure ν is given on each Γ_δ by: $|R(f, \delta, \xi)|d\xi d\mu$. ■

2.3 Exemple ([Cu.2])

Let $\mathfrak{g} = vect < B, A, X, Y, Z >$ with non vanishing brackets

$$[A, X] = -X, \quad [A, Y] = Y, \quad [X, Y] = Z, \quad [B, X] = X, \quad [B, Z] = Z.$$

Here $\mathfrak{h} = vect < A, B >$ and $\mathfrak{n} = vect < X, Y, Z >$.

For $l \in \mathfrak{g}^*$ we write $l = (\lambda, \gamma, \mu, \alpha, \theta)$ where $\lambda = l(Z); \gamma = l(Y); \mu = l(X); \alpha = l(A); \theta = l(B)$. $\Omega_0 = \{l \in \mathfrak{g}^*, \lambda \neq 0\}$ and $\Omega_1 = \{l \in \mathfrak{g}^*, \gamma \neq 0\}$ and the set Ω of generic linear functionals is $\Omega = \Omega_0 \cap \Omega_1$.

The cross-section for $H-$orbits in $\mathfrak{h}^\perp \cap \Omega$ is given in [Cu.2] as:

$$\Sigma = \{(\delta, \epsilon, \mu, 0, 0);\ \mu \in \mathbb{R}; \quad (\epsilon, \delta) \in \{-1, 1\}^2\} = \cup\Sigma_\theta.$$

Now the cross-section for $G-$ orbits in $G \cdot \Sigma$ is: $\Gamma = \cup_{\delta\in\{-1,1\}}\Gamma_\delta$ where

$$\Gamma = \{(\delta, 1, \mu, 0, 0); \quad \mu \in \mathbb{R}, \quad \delta \in \{-1, 1\}\}.$$

Let $\sigma \in \Gamma$; there exits $\delta \in \{-1, 1\}$ such that $\sigma = (\delta, 1, \mu, 0, 0)$. The theorem (2.2.2) says that the Bonnet Plancherel measure is given on each Γ_δ by $(2\pi)^{-3}d\mu$.

For $l \in G \cdot \sigma \ \exists \ \epsilon = l(Y)$ such that $l = (\delta, \epsilon, \epsilon\mu, 0, 0)$. Put g_l such that $l = g_l \cdot \sigma$, here we have: $V = Z$; and since $[B, X] = X$ then for $\epsilon = -1$

$$g_l = \exp(\frac{-2\mu}{\delta} Y)\exp(\frac{2}{\delta} X)\exp(\frac{-2\mu}{\delta^2} Z.)$$

The operator β_l is given in [Cu.2]:

$$\beta_l(\psi) = \int_{\mathbb{R}^2} \overline{\psi(\exp(sB)\exp(tA))} e^s e^{\frac{(t-s)}{2}} dsdt.$$

Thus the formula for the operator β'_l is:

$$\beta'_l(\psi) = \beta'_\epsilon(\psi) = \int_{\mathbb{R}^2} \overline{\psi(g_l^{-1}\exp(sB)\exp(tA))} e^s e^{\frac{(t-s)}{2}} dsdt = \beta_l(\psi_{g_l}).$$

Then Bonnet's operator U_σ is given by

$$U_\sigma = \sum_{\epsilon_1 \in \{-1,1\}} P_{\beta'_{\epsilon_1}, \beta'_{\epsilon_1}} \quad \text{where} \quad P_{\beta'_\epsilon, \beta'_\epsilon}(\psi) = \langle \psi, \beta'_\epsilon \rangle \beta'_\epsilon.$$

Furthermore for $\epsilon = 1$, $\beta'_1 = \beta_\sigma$, then

$$U_\sigma = P_{\beta'_{-1}, \beta'_{-1}} + P_{\beta_1, \beta_1}.$$

Now for $\omega \in D(G)$ we have: $\pi_\sigma(\omega) \circ U_\sigma = P_{\pi_{(\delta,1,\mu)}(\omega)\beta'_{-1}, \beta'_{-1}} + P_{\pi_{(\delta,1,\mu)}(\omega)\beta_1, \beta_1}$. Then:

$$\text{tr}(\pi_\sigma \circ U_\sigma) = \langle \pi_{(\delta,1,\mu)}(\omega)\beta'_{-1}, \beta'_{-1} \rangle + \langle \pi_{(\delta,1,\mu)}(\omega) \circ \beta_1, \beta_1 \rangle$$

By theorem (2.2.2) we have the Bonnet Plancherel formula:

$$\langle \tau_f(\omega)\alpha_\tau, \alpha_\tau \rangle = (2\pi)^{-3} \sum_{\delta \in \{-1,1\}} \int_{\mathbb{R}} \text{tr}(\pi_{(\delta,1,\mu)}(\omega) U_{(\delta,1,\mu)}) d\mu.$$

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