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Abelian extensions of infinite-dimensional Lie groups

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Abstract

In the present paper we study abelian extensions of a connected Lie group G modeled on a locally convex space by a smooth G-module A. We parameterize the extension classes by a suitable cohomology group defined by locally smooth cochains and construct an exact sequence that permits us to calculate the group cohomology from the corresponding continuous Lie algebra cohomology and topological data. The obstructions for the integrability of a Lie algebra 2-cocycle to a Lie group 2-cocycle are described in terms of a period and a flux homomorphism. We also characterize the extensions with global smooth sections, resp., those given by globally smooth cocycles. We apply the general theory to abelian extensions of diffeomorphism groups, where the Lie algebra cocycles are given by closed 2-forms on the manifold M. In this case we show that period and flux homomorphism can be described directly in terms of M, and for central extensions of groups of volume preserving diffeomorphisms corresponding to Lichnerowicz cocycles, this entails that the flux homomorphism vanishes on an explicitly described covering group. We also discuss the group cohomology of the diffeomorphism group of the circle and its universal covering with values in modules of λ-densities.

Introduction

In this paper we undertake a detailed analysis of abelian extensions of Lie groups which might be infinite-dimensional, a main point being to derive criteria for abelian extensions of Lie algebras to integrate to extensions of corresponding connected groups. This is of particular interest in the infinite-dimensional theory because not every infinite-dimensional Lie algebra can be 'integrated' to a global Lie group.

The concept of a Lie group used here is that a *Lie group* G is a manifold modeled on a locally convex space, endowed with a group structure for which the group operations are smooth (cf. [Mi83]; see also [Gl01] for non-complete model spaces). An *abelian extension* is an exact sequence of Lie groups $A \hookrightarrow \widehat{G} \twoheadrightarrow G$ which defines a locally trivial smooth principal bundle with the abelian structure group A over the Lie group G. Then A inherits the structure of a *smooth G-module* in the sense that the conjugation action of $\widehat{G}$ on A factors through a smooth map $G \times A \to A$. The extension is called *central* if this action is trivial.

The natural context to deal with abelian extensions of Lie groups is provided by a suitable Lie group cohomology with values in smooth modules: If G is a Lie group and A a smooth G-module, then the space $C^n_s(G,A)$ of (locally smooth) n-cochains consists of maps $G^n \to A$ which are smooth in an identity neighborhood and vanish on all tuples of the form $(g_1, \ldots, \mathbf{1}, \ldots, g_n)$. We thus obtain with the standard group differential d_G a cochain complex $(C^\bullet_s(G,A), d_G)$ with cohomology groups $H^n_s(G,A)$. If G and A are discrete, these groups coincide with the standard cohomology groups of G with values in A ([EiML47]), but if G is a finite-dimensional Lie group, they differ in general from the traditionally considered cohomology groups defined by globally smooth cocycles as in [Gui80] and [HocMo62]. In the following we assume that the identity component A_0 of A is of the form $\mathfrak{a}/\Gamma_A$, where Γ_A is a discrete subgroup of the Mackey complete locally convex space $\mathfrak{a}$. Mackey completeness means that Riemann integrals of smooth curves $[0,1] \to \mathfrak{a}$ exist ([KM97]), which is needed to ensure the existence of $\mathfrak{a}$-valued period integrals. We write $q_A \colon \mathfrak{a} \to A_0 \cong \mathfrak{a}/\Gamma_A$ for the quotient map which is a universal covering of A_0.

It is a key feature of Lie theory that one can calculate complicated objects attached to a Lie group G in terms of linear objects attached to the Lie algebra and additional topological data. This is exactly what we do in the present paper with the cohomology groups $H^1_s(G,A)$ and $H^2_s(G,A)$. Passing to the derived representation of the Lie algebra $\mathfrak{g}$ of G on the Lie algebra $\mathfrak{a}$ of A, we obtain a module of the Lie algebra $\mathfrak{g}$ which is *topological* in the sense that the module structure is a continuous bilinear map $\mathfrak{g} \times \mathfrak{a} \to \mathfrak{a}$. Then the continuous alternating maps $\mathfrak{g}^n \to \mathfrak{a}$ form the (continuous) Lie algebra cochain complex $(C^\bullet_c(\mathfrak{g},\mathfrak{a}), d_{\mathfrak{g}})$, and its cohomology spaces are denoted $H^n_c(\mathfrak{g},\mathfrak{a})$. It is shown in Appendix B that for $n \geq 2$ there is a natural *derivation map*

$$D_n \colon H^n_s(G,A) \to H^n_c(\mathfrak{g},\mathfrak{a}), \quad [f] \mapsto [D_n f] \tag{0.1}$$

from locally smooth Lie group cohomology to continuous Lie algebra cohomology, considered first by van Est in [Est53] (see also [Gui80, III.7.7] and [EK64]). This map is based on the isomorphism

$$H^n_c(\mathfrak{g},\mathfrak{a}) \to H^n_{\mathrm{dR,eq}}(G,\mathfrak{a}), \quad [\omega] \mapsto [\omega^{\mathrm{eq}}]$$

between Lie algebra cohomology and the de Rham cohomology of the complex $(\Omega^\bullet_{\mathrm{dR,eq}}(G,\mathfrak{a}), d)$ of equivariant $\mathfrak{a}$-valued differential forms on G, introduced by Chevalley and Eilenberg for finite-dimensional groups ([CE48]). Here ω^{eq} denotes the unique equivariant $\mathfrak{a}$-valued form on G with $\omega^{\mathrm{eq}}_{\mathbf{1}} = \omega$. For $n = 1$ we only have a map $D_1 \colon Z^1_s(G,A) \to Z^1_c(\mathfrak{g},\mathfrak{a})$, and if A is connected, then this map factors to a map $D_1 \colon H^1_s(G,A) \to H^1_c(\mathfrak{g},\mathfrak{a})$ on the cohomology level. Since the Lie algebra cohomology spaces $H^n_c(\mathfrak{g},\mathfrak{a})$ are by far better accessible than those of G, it is important to understand the amount of information lost by the map D_n, i.e., one is interested in kernel and cokernel of D_n. A determination of the cokernel consists in describing integrability conditions on cohomology classes $[\omega] \in H^n_c(\mathfrak{g},\mathfrak{a})$ which are necessary for the existence of some $f \in Z^n_s(G,A)$ with $D_n f = \omega$.

The present paper consists of three main parts. In Sections 1-8 we describe the classification of abelian extensions of a Lie group G by a smooth G-module A in terms of the cohomology group $H^2_s(G,A)$ and explain how this group can be calculated in terms of $H^2_c(\mathfrak{g},\mathfrak{a})$ and data related to the first two homotopy groups $\pi_1(G)$ and $\pi_2(G)$. Sections 9-11 are devoted to applications to several types of groups of diffeomorphisms of compact manifolds. The remainder of the paper consists of six appendices in which we prove auxiliary results that are used either for the analysis of $H^2_s(G,A)$ or for the applications to diffeomorphism groups in Section 9.

We now describe the main results of the paper in some more detail. After briefly reviewing the relation between abelian extensions of topological Lie algebras and the continuous cohomology space $H^2_c(\mathfrak{g},\mathfrak{a})$, we show in Section 2 that for a connected Lie group G and each 2-cocycle $f \in Z^2_s(G,A)$ the multiplication

$$(a,g)(a',g') := (a + g.a' + f(g,g'), gg')$$

on the product set $A \times G$ defines a Lie group structure, denoted $A \times_f G$. Here a subtle point is that in general the manifold structure on $A \times_f G$ is not the product manifold structure. Only if f is a smooth function on $G \times G$, we can simply take the product structure and obtain a smooth multiplication. If A is a discrete group, then $A \times_f G$ is a covering group of G. Standard arguments show that equivalent cocycles lead to equivalent extensions, and we derive that $H^2_s(G,A)$ parameterizes the equivalence classes of Lie group extensions $A \hookrightarrow \widehat{G} \twoheadrightarrow G$ for which the action of G on A induced by the conjugation action of $\widehat{G}$ on A coincides with the original G-module structure. This was our original motivation to study the (locally smooth) cohomology groups $H^2_s(G,A)$. If G is not connected, then an appropriate subgroup $H^2_{ss}(G,A) \subseteq H^2_s(G,A)$ classifies the extensions of G by A.

In Section 3 we briefly discuss the relation between smooth 1-cocycles on a connected Lie group and the corresponding continuous Lie algebra 1-cocycles. This is instructive for the understanding of the flux homomorphism occurring below as an obstruction to the existence of global group extensions. For a Lie algebra cocycle $\alpha \in Z^1_c(\mathfrak{g},\mathfrak{a})$ we define the *period homomorphism*

$$P_1([\alpha]) = q_A \circ \mathrm{per}_\alpha \colon \pi_1(G) \to A^G, \quad [\gamma] \mapsto q_A\Big(\int_\gamma \alpha\Big) = \int_\gamma \alpha + \Gamma_A.$$

The first main point in Section 3 is the exactness of the sequence

$$\mathbf{0} \to H^1_s(G,A) \xrightarrow{D_1} H^1_c(\mathfrak{g},\mathfrak{a}) \xrightarrow{P_1} \mathrm{Hom}(\pi_1(G), A^G) \tag{0.2}$$

which is valid for a connected Lie group G if A is connected.

If the connected group G acts on a non-connected smooth module A, then it acts trivially on the discrete abelian group $\pi_0(A) \cong A/A_0$ of connected components, but to determine the action on A, one needs more information than the G-action on A_0. As a consequence of the exact sequence (0.2), we show that this information is contained in the characteristic homomorphism

$$\overline{\theta}_A : \pi_0(A) \to H^1_c(\mathfrak{g}, \mathfrak{a}), \quad [a] \mapsto D_1[d_G a].$$

In Sections 4-7 we determine kernel and cokernel of the map D_2 from (0.1). First we show in Section 4 that each cocycle $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$ determines a period homomorphism

$$\mathrm{per}_\omega : \pi_2(G) \to \mathfrak{a}^G, \quad [\sigma] \mapsto \int_\sigma \omega^{\mathrm{eq}},$$

where $\sigma : \mathbb{S}^2 \to G$ is a (piecewise) smooth representative of the homotopy class, whose existence has been shown in [Ne02]. In Section 5 we then show that if G is simply connected and $q_A \circ \mathrm{per}_\omega$ vanishes, then $[\omega] \in \mathrm{im}\, D_2$. For that we use a slight adaptation of the method used in [Ne02] for central extensions and originally inspired by the construction of group cocycles in [Est54] by using the symplectic area of geodesic triangles (see also [DuGu78] for a similar method).

In Section 6 we eventually turn to the refinements needed for non-simply connected groups which leads to the *flux homomorphism*

$$F_\omega : \pi_1(G) \to H^1_c(\mathfrak{g}, \mathfrak{a}).$$

If $q_G : \widetilde{G} \to G$ is the universal covering of G, and we identify $\pi_1(G)$ with $\ker q_G$, then F_ω is the restriction to $\pi_1(G)$ of the flux cocycle,

$$F_\omega : \widetilde{G} \to C^1_c(\mathfrak{g}, \mathfrak{a})/d_{\mathfrak{g}}\mathfrak{a},$$

which is a group cocycle whose "derivative" is the Lie algebra cocycle $f_\omega(x) = [i_x\omega]$. In general we cannot expect the space $C^1_c(\mathfrak{g}, \mathfrak{a})/d_{\mathfrak{g}}\mathfrak{a}$ to carry any reasonable Hausdorff topology. Therefore we cannot directly apply the results from Section 3 and thus have to work our way around this problem. For central extensions the flux homomorphism is much less complicated because it simplifies to a homomorphism $\pi_1(G) \to \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{a})$ ([Ne02]). The main result of Section 6 is the Integrability Criterion (Theorem 6.7) which says that, for a connected group A, $[\omega] \in \mathrm{im}(D_2)$ if and only if

$$P_2([\omega]) := (q_A \circ \mathrm{per}_\omega, F_\omega) = 0. \tag{0.3}$$

We also prove a generalization of this criterion for non-connected groups A which is more complicated because the condition $F_\omega = 0$ has to be modified suitably. If G is smoothly paracompact, then the closed $\mathfrak{a}$-valued 2-form ω^{eq} defines a singular cohomology class in $H^2_{\mathrm{sing}}(G, A_0) \cong \mathrm{Hom}(H_2(G), A_0)$, and from a description of generators of $H_2(G)$ in terms of $\pi_2(G)$ and $\pi_1(G)$, we show that in general the vanishing of this cohomology class is weaker than $P_2([\omega]) = 0$. In Section 7 we combine the results on the cokernel of D_2 with a description of its kernel and get the following exact sequence (Theorem 7.2):

$$\begin{aligned} &\mathbf{0} \to H^1_s(G,A) \xrightarrow{I} H^1_s(\widetilde{G},A) \xrightarrow{R} H^1\big(\pi_1(G),A\big)^G \cong \operatorname{Hom}\big(\pi_1(G),A^G\big) \xrightarrow{\delta} \\ &\xrightarrow{\delta} H^2_s(G,A) \xrightarrow{D} H^2_c(\mathfrak{g},\mathfrak{a}) \xrightarrow{P_2} \operatorname{Hom}\big(\pi_2(G),A\big) \times \operatorname{Hom}\big(\pi_1(G),H^1_c(\mathfrak{g},\mathfrak{a})\big). \end{aligned} \tag{0.4}$$

Here I and R are the natural inflation and restriction maps and δ assigns to a group homomorphism $\gamma\colon \pi_1(G) \to A^G$ the quotient of the semi-direct product $A \rtimes \widetilde{G}$ by the graph $\{(\gamma(d),d)\colon d \in \pi_1(G)\}$ of γ, which is a discrete central subgroup.

In many situations one would like to know when it is possible to integrate Lie algebra cocycles to global smooth group cocycles $f\colon G \times G \to \mathfrak{a}$. In Section 8 we show that, under the assumption that G is smoothly paracompact, this it is possible if and only if $\omega^{\rm eq}$ is an exact 2-form and F_ω vanishes (Proposition 8.4 and Remark 8.5).

Combining the exact sequence from above with the exact Inflation-Restriction Sequence derived in Appendix D, we obtain for connected groups G and $A \cong \mathfrak{a}/\Gamma_A$ the following commutative diagram with an exact second row (Prop. D.8 and the subsequent discussion) and exact columns (Proposition 3.4 for H^1_s and Theorem 7.2 for H^2_s):

$$\begin{array}{ccccccccc}
\mathbf{0} & & \mathbf{0} & & & & \operatorname{Hom}(\pi_1(G),A^G) & & \mathbf{0} \\
\downarrow & & \downarrow & & & & \downarrow{\scriptstyle \delta} & & \downarrow \\
H^1_s(G,A) & \xrightarrow{I} & H^1_s(\widetilde{G},A) & \xrightarrow{R} & \operatorname{Hom}(\pi_1(G),A^G) & \xrightarrow{\delta} & H^2_s(G,A) & \xrightarrow{I} & H^2_s(\widetilde{G},A) \\
\downarrow{\scriptstyle D_1} & & \downarrow{\scriptstyle D_1} & & \downarrow{\scriptstyle \rm id} & & \downarrow{\scriptstyle D_2} & & \downarrow{\scriptstyle D_2} \\
H^1_c(\mathfrak{g},\mathfrak{a}) & \xrightarrow{\rm id} & H^1_c(\mathfrak{g},\mathfrak{a}) & \xrightarrow{P_1} & \operatorname{Hom}(\pi_1(G),A^G) & & H^2_c(\mathfrak{g},\mathfrak{a}) & \xrightarrow{\rm id} & H^2_c(\mathfrak{g},\mathfrak{a}) \\
\downarrow{\scriptstyle P_1} & & \downarrow & & & & \downarrow{\scriptstyle P_2} & & \downarrow{\scriptstyle P_2} \\
\operatorname{Hom}(\pi_1(G),A^G) & & \mathbf{0} & & & & \begin{smallmatrix}\operatorname{Hom}(\pi_2(G),A^G)\oplus \\ \operatorname{Hom}(\pi_1(G),H^1_c(\mathfrak{g},\mathfrak{a}))\end{smallmatrix} & & \operatorname{Hom}(\pi_2(G),A^G)
\end{array}$$

In the remaining Sections 9-11 we apply the general theory to various kinds of diffeomorphism groups. In Section 9 we turn to the special situation arising for the group $G := \operatorname{Diff}(M)_0^{\rm op}$ of diffeomorphism of a compact manifold, its Lie algebra $\mathfrak{g} := \mathcal{V}(M)$ (the smooth vector fields on M), the smooth G-module $A = \mathfrak{a} = C^\infty(M,V)$ (V a Fréchet space), and the special class of 2-cocycles of the form $\omega_{\mathfrak{g}}(X,Y) := \omega_M(X,Y)$, where ω_M is a closed V-valued 2-form on M. In this case we explain how information on the period map and the flux cocycle can be calculated in geometrical terms. The two main results are that that the period map

$$\operatorname{per}_{\omega_{\mathfrak{g}}} \colon \pi_2(\operatorname{Diff}(M)) \to C^\infty(M,V)^{\mathcal{V}(M)} = V$$

factors through the evaluation map $\mathrm{ev}^D_{m_0}\colon \mathrm{Diff}(M) \to M, \varphi \mapsto \varphi(m_0)$ to the map

$$\mathrm{per}_{\omega_M}\colon \pi_2(M, m_0) \to V, \quad [\sigma] \mapsto \int_\sigma \omega_M.$$

Likewise the flux homomorphism can be interpreted as a map

$$F_\omega\colon \pi_1(\mathrm{Diff}(M)) \to H^1_{\mathrm{dR}}(M, V) \cong \mathrm{Hom}(\pi_1(M), V),$$

that vanishes if and only if all integrals of the 2-form ω_M over smooth cycles of the form $H\colon \mathbb{T}^2 \to M, (s,t) \mapsto \alpha(s).\beta(t)$ with a loop α in $\mathrm{Diff}(M)$ and β in M vanish.

In Section 10 we consider the important group $G = \mathrm{Diff}(\mathbb{S}^1)_0$ of orientation preserving diffeomorphisms of the circle and the module $\mathfrak{a}$ of λ-densities for $\lambda \in \mathbb{R}$. The corresponding group cocycles have been discussed by Ovsienko and Roger in [OR98]. Here we extend their results to Lie algebra cocycles not integrable on G which integrate to group cocycles of the universal covering group $\widetilde{G}$, for which we provide explicit formulas. As a byproduct of this construction, we obtain a non-trivial abelian extension of the group $\mathrm{SL}_2(\mathbb{R})$ by an infinite-dimensional Fréchet space. Since all finite-dimensional Lie group extensions of $\mathrm{SL}_2(\mathbb{R})$ by vector spaces split on the Lie algebra level, this example nicely illustrates the difference between the finite and infinite-dimensional theory.

If μ is a volume form on the compact manifold M and $D(M,\mu) := \mathrm{Diff}(M,\mu)_0^{\mathrm{op}}$ the identity component of the group $\mathrm{Diff}(M,\mu)$ of volume preserving diffeomorphisms of M with Lie algebra $\mathcal{V}(M,\mu) := \{X \in \mathcal{V}(M)\colon \mathcal{L}_X\mu = 0\}$, then interesting scalar-valued Lie algebra cocycles (Lichnerowicz cocycles) arise from closed 2-forms $\omega \in Z^2_{\mathrm{dR}}(M, \mathbb{R})$ by

$$\mathcal{V}(M,\mu) \times \mathcal{V}(M,\mu) \to \mathbb{R}, \quad (X,Y) \mapsto \int_M \omega(X,Y)\mu.$$

The existence of corresponding central extensions is addressed in Section 11, where we use the information on the $C^\infty(M,\mathbb{R})$-valued cocycle on the full Lie algebra $\mathcal{V}(M)$ derived in Section 9 to show in particular that if the manifold M is a compact connected Lie group, then each Lichnerowicz cocycle can be integrated to a group cocycle on a certain covering group $\widetilde{D}(M,\mu)$ which is an extension of $D(M,\mu)$ by the discrete group $\pi_1(M)$. The main point is to show that the flux cocycle vanishes on the fundamental group of the covering group $\widetilde{D}(M,\mu)$.

We conclude this paper with several appendices dealing with the relation between differential forms and Alexander–Spanier cohomology (Appendix A), which in turn is used in Appendix B to show that the maps D_n from locally smooth Lie group cochains to Lie algebra cochains intertwine the differentials d_G and $d_\mathfrak{g}$ (cf. [Est53], [EK64]). In Appendix C we describe a general procedure to construct global Lie groups from local data, which is used in Section 2 and in [Ne04a] to obtain Lie group structures on group extensions. For calculations of cohomology groups, the corresponding exact Inflation-Restriction Sequence for Lie group cohomology is provided in Appendix D and the long exact sequence in Lie group cohomology induced from an exact sequence of smooth modules in Appendix E. The latter sequence is obtained from general homological algebra, whereas the former contains certain subtleties related to smoothness conditions that are specific in the Lie theoretic context. Finally we show in Appendix F that multiplication of Lie group and Lie algebra cocycles is compatible with the differentiation maps D_n. This has interesting applications in various contexts, in particular in Section 10 and [Ne04b].

If G is simply connected, the criterion for the integrability of a Lie algebra cocycle ω to a group cocycle is simply that all periods of the equivariant $\mathfrak{a}$-valued 2-form $\omega^{\rm eq}$ are contained in $\Gamma_A = \ker q_A \subseteq \mathfrak{a}$. Similar conditions arise in the theory of abelian principal bundles on smoothly paracompact presymplectic manifolds (M,ω), i.e., ω is a closed 2-form on M. Here the integrality of the cohomology class $[\omega] \in H^2_{\rm sing}(M,\mathbb{R})$ is equivalent to the existence of a $\mathbb{T}$-principal bundle $\mathbb{T} \hookrightarrow \widehat{M} \twoheadrightarrow M$ whose first Chern class is $[\omega]$ (cf. [Bry93]). If G is smoothly paracompact, $\omega \in Z^2_c(\mathfrak{g},\mathfrak{a})$ and $A = A_0$ as above, then the vanishing of the corresponding cohomology class in $H^2_{\rm sing}(G,A)$, which corresponds to the "integrality" of the periods, also implies the existence of a corresponding A-bundle over G, but this does not imply in general that ω corresponds to an extension of G by A. It might be necessary to consider a non-connected group A' with $\pi_0(A') \cong \pi_1(G)$ (see Section 6 for more details).

It is instructive to illustrate the difference between abelian and central extensions of Lie groups in the context of abelian principal bundles. Let $q\colon P \to M$ be a smooth principal bundle with the abelian structure group Z over the compact connected manifold M. Then the group $\mathrm{Diff}(P)^Z$ of all diffeomorphisms of P commuting with Z (the automorphism group of the bundle) is an abelian extension of an open subgroup of $\mathrm{Diff}(M)$ by the gauge group $\mathrm{Gau}(P) \cong C^\infty(M,Z)$ of the bundle. Here the conjugation action of $\mathrm{Diff}(M)$ on $\mathrm{Gau}(P)$ is given by composing functions with diffeomorphisms. In this context central extensions arise as follows. Let $\theta \in \Omega^1(P,\mathfrak{z})$ be a principal connection 1-form and $\omega \in \Omega^2(M,\mathfrak{z})$ its curvature, i.e., $q^*\omega = -d\theta$. Then the subgroup $\mathrm{Diff}(P)^Z_\theta$ of $\mathrm{Diff}(P)^Z$ preserving θ is a central extension of an open subgroup of $\mathrm{Sp}(M,\omega) := \{\varphi \in \mathrm{Diff}(M)\colon \varphi^*\omega = \omega\}$ by Z. Therefore the passage from $\mathrm{Diff}(M)$ to the much smaller subgroup $\mathrm{Sp}(M,\omega)$ corresponds to the passage from an abelian extension by $C^\infty(M,Z)$ to a central extension by Z. Philosophically this means that diffeomorphism groups have natural abelian extensions, whereas symplectomorphism groups have natural central extensions.

This point of view is also crucial in the representation theory of infinite-dimensional Lie groups, where one is forced to consider Lie groups G acting on a manifold M on which we have a circle bundle $q\colon P \to M$ with a connection θ, but its curvature ω is not G-invariant. Then it might happen that each element of G can be lifted to a bundle automorphism on P, but this automorphism will not preserve the connection 1-form. This leads to the abelian extension of G by the gauge group $C^\infty(M, Z)$, instead of the central extension by Z, to which we may reduce if G preserves the curvature form. Note that each abelian extension $A \hookrightarrow \widehat{G} \twoheadrightarrow G$ corresponding to a Lie algebra cocycle $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$ is of this form, because the left translation action of G on itself does not preserve the equivariant 2-form $\omega^{\rm eq}$, which plays the role of the curvature of the A-bundle $\widehat{G} \to G$. We refer to [Mi89] for a detailed discussion of the case where M is a restricted Graßmannian of a polarized Hilbert space and the groups are restricted operator groups of Schatten class $p > 2$, resp., mapping groups $C^\infty(M, K)$, where K is finite-dimensional and M is a compact manifold of dimension ≥ 2 (see also [PS86] for a discussion of related points). It is for the same reason that abelian extensions of vector field Lie algebras occur naturally in mathematics physics (cf. [La99] and also [AI95] for more general applications of Lie group cohomology in physics), and the need for a corresponding global group corresponding to these abelian extensions arises naturally. One way to get these global groups, which is complementary to our direct approach, is to use crossed homomorphisms of Lie groups to pull back central extensions to abelian ones. Y. Billig applied this method quite successfully in [Bi03], where he introduces for orientable manifolds natural analogs of the Virasoro group which are abelian extensions of $\mathrm{Diff}(M)$.

Another motivation for a general study of abelian extensions comes from the fact that for the group $\mathrm{Diff}(M)$, where M is a compact orientable manifold, one has natural modules given by tensor densities and spaces of tensors on M. The corresponding abelian extensions can be used to interprete certain partial differential equations as geodesic equations on a Lie group, which leads to important information on the behavior of their solutions ([Vi02], [AK98]).

If the smooth G-module A is trivial and the space $H^2_c(\mathfrak{g}, \mathfrak{a})$ is trivial, or if at least $D = 0$, then the exact sequence (0.4) leads to

$$H^2_s(G, A) \cong \mathrm{Hom}\,\big(\pi_1(G), A\big)/\,\mathrm{Hom}\,\big(\widetilde{G}, A\big)\,|_{\pi_1(G)},$$

a formula which has first been obtained for connected compact Lie groups by A. Shapiro ([Sh49]). A crucial simplification in the finite-dimensional case is that extensions of simply connected Lie groups have smooth global sections, so that one can get along by using only globally smooth cochains. Along these lines many specific results have been obtained by G. Hochschild ([Ho51]). For finite-dimensional Lie groups our integrability criterion for Lie algebra 2-cocycles simplifies significantly because $\pi_2(G)$ vanishes ([Car52]). This in turn has been used by É. Cartan to construct central Lie group extensions and thus to derive Lie's Third Theorem that each finite-dimensional Lie algebra belongs to a global Lie group. Our characterization of abelian extensions with global smooth sections in Section 8 follows Cartan's construction.

Cohomology theories for topological groups with values in topological modules have been studied from various points of view by several people. In [Se70]

G. Segal defines cohomology of topological groups as a derived functor. In his context the values lie in an abelian group which is compactly generated and locally contractible as an abelian group in the category of k-spaces. The corresponding cohomology groups $H^2(G, A)$ classify topological extensions with continuous local sections. In a similar fashion D. Wigner defines cohomology for topological groups in terms of Ext-functors and explains how it can be described in terms of group cocycles.

In [Mo64] C. C. Moore defines group cohomology for second countable locally compact groups in terms of cochains which are Borel measurable. This is natural in this context, where group extensions have Borel measurable cross sections ([Ma57]). For finite-dimensional Lie groups measurable cocycles are equivalent to locally smooth cocycles ([Va85, Th. 7.21]). In [Mo76] Moore also discusses universal central extensions of finite-dimensional Lie groups and criteria for the triviality of all central extensions. Universal central extensions of finite-dimensional groups are also described in [CVLL98], which is a nice survey of central $\mathbb{T}$-extensions of Lie groups and their role in quantum physics. For infinite-dimensional groups universal central extensions are constructed in [Ne03b] and for locally convex root graded Lie algebras in [Ne03a].

Hochschild and Mostow approach in [HocMo62] cohomology of finite-dimensional Lie groups by injective resolutions in a topological and a differentiable setting, which leads to continuous and differentiable cohomologies $H^\bullet_c(G, V)$ and $H^\bullet_d(G, V)$ with values in a locally convex space V on which G acts continuously, resp., differentiably. Under mild assumptions on V (concerning the existence of V-valued integrals), they show that for a connected finite-dimensional Lie group G there are isomorphisms

$$H^\bullet_c(G, V) \cong H^\bullet_d(G, V) \cong H^\bullet(\mathfrak{g}, \mathfrak{k}, V), \tag{0.5}$$

where the latter term denotes relative Lie algebra cohomology, and $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K of G ([HocMo62, Th. 6.1]).

The isomorphism (0.5) generalizes a result of van Est ([Est55]) to infinite-dimensional modules V (cf. also [Gui80]). In [GW78] A. Guichardet and D. Wigner give an explicit realization of the isomorphism (0.5) for a semisimple group G by writing down an explicit map from $H^n(G, V)$ to $H^n(\mathfrak{g}, \mathfrak{k}, V)$ which is a restriction of the map D_n in (0.1). Here a main point is that after averaging over the maximal compact group K, one can represent group cocycles by functions f for which $D_n f$ is a relative cocycle in $Z^n(\mathfrak{g}, \mathfrak{k}, V)$ (cf. also [Est55, Thm. 1]). This averaging process would not work for locally smooth cochains because they do not form a translation invariant space of functions. The homogeneous space G/K is a Riemannian symmetric space of semi-negative curvature, so that two points are joint by a unique geodesic. This implies that one can assign to each ordered triple in G/K in a G-equivariant fashion a differentiable 2-simplex, and integrating G-invariant closed forms leads directly from relative Lie algebra 2-cocycles to smooth global group cocycles ([DuGu78]). It is interesting to compare this approach with our integration method in Section 5, where we choose some G-invariant system of "line segments" on G, whereas the symmetric space G/K has the natural G-invariant system consisting of geodesic segments.

In all situations where one wants to apply spectral sequence arguments, one is forced to assume that the cohomology spaces occurring as target spaces of cocycles are finite-dimensional. In [HocMo62] this leads to the assumption that the topological group G under consideration is of *finite homology type*, i.e., for each finite-dimensional topological G-module V the cohomology spaces $H^n_c(G,V)$ are finite-dimensional (cf. also [Est58] for similar assumptions). It is clear that for infinite-dimensional groups such assumptions are only met in very rare circumstances. Another important feature of finite-dimensional connected Lie groups G is that for a maximal compact subgroup K the quotient space G/K is contractible. Therefore one can combine averaging over K with the smooth contractibility of G/K, which eventually leads to the van Est Theorem (0.5) above. In [Est53] van Est studies another spectral sequence relating the cohomology $H^\bullet_{gs}(G,\mathfrak{a})$ of a finite-dimensional connected Lie group G with values in a finite-dimensional smooth G-module $\mathfrak{a}$ and defined by globally smooth cochains to the cohomology of its Lie algebra. The group $H^2_{gs}(G,A) \cong H^2_{gs}(G,\mathfrak{a})$ can be viewed as a subgroup of $H^2_s(G,A)$ corresponding to extensions with smooth global sections, but it might be quite small if the first two homotopy groups of G are non-trivial and $A \neq \mathfrak{a}$ (cf. Remark 8.5).

We emphasize that our results hold for Lie groups which are not necessarily smoothly paracompact, so that one cannot use smooth partitions of unity to construct bundles for prescribed curvature forms and de Rham's Theorem is not available (cf. [KM97, Th. 16.10]). This point is important because many interesting Banach–Lie groups are not smoothly paracompact since their model spaces do not permit smooth bump functions (cf. [KM97]). For smooth loop groups central extensions are discussed in [PS86], but in this case many difficulties are absent due to the fact that loop groups are modeled on nuclear Fréchet spaces which are smoothly regular ([KM97, Th. 16.10]), hence smoothly paracompact because this holds for every smoothly Hausdorff second countable manifold modeled on a smoothly regular space ([KM97, 27.4]).

The present paper is a sequel to [Ne02], dealing with central extensions. Fortunately it was possible to use some of the constructions from [Ne02] quite directly in the present paper, but the material on the flux homomorphism developed in Section 6 is completely new. It is quite trivial for central extensions, where it does not play such an important role. In [Ne04a] the results on abelian extensions are used to classify general extensions: Let N be a Lie group and $Z(N)$ its center. Suppose further that $Z(N)$ is an initial Lie subgroup, i.e., that $Z(N)$ carries a Lie group structure and every smooth map $M \to N$ with values in $Z(N)$ defines a smooth map $M \to Z(N)$. Then the group $H^2_{ss}(G,Z(N))$ parameterizes the equivalence classes of extensions of G by N corresponding to a given smooth outer action of G on N. We refer to [Ne04a] for more details and the definition of a smooth outer action.

In the present paper we give a complete description of kernel and cokernel of the map D_2 for a connected Lie group G and a connected module $A \cong \mathfrak{a}/\Gamma_A$. We plan to return in a subsequent paper to this problem for non-connected groups G, which, in view of the present results, means to obtain accessible criteria for the extendibility of a 2-cocycle on the identity component G_0 of G to the whole group G. For trivial modules A, i.e., central extensions of G_0, this leads to obstructions in $H^3(\pi_0(G), A)$ arising as the characteristic class of a crossed module

$$\mathbf{1} \to A \to \widehat{G} \to G \to \pi_0(G) \to \mathbf{1}$$

(cf. [Ne04a]). The crossed module structure contains in particular an action of the whole group G by automorphisms on the central extension $\widehat{G}$ of G_0. The existence of this action is closely related to the invariance of the class $[\omega] \in H^2_c(\mathfrak{g}, \mathfrak{a})$ under the action of $\pi_0(G)$ (see the discussion of automorphisms of extensions in the appendix of [Ne04a]), and this in turn is related to the question whether $H^2_{ss}(G, A)$ is strictly smaller than $H^2_s(G, A)$.

We are grateful to S. Haller for providing a crucial topological argument concerning the flux homomorphism for the group of volume preserving diffeomorphisms (cf. Section 11). We also thank C. Vizman for many inspiring discussions on the subject, and G. Segal for suggesting a different type of obstructions to the integrability of abelian extensions in [Se02]. Many thanks go also to A. Dzhumadildaev for asking for global central extensions of groups of volume preserving diffeomorphisms which correspond to the cocycles he studied on the Lie algebra level in [Dz92]. This led us to the results in Section 11.

0. Preliminaries and notation

In this paper $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ denotes the field of real or complex numbers. Let X and Y be topological $\mathbb{K}$-vector spaces, $U \subseteq X$ open and $f \colon U \to Y$ a map. Then the *derivative of f at x in the direction of h* is defined as

$$df(x)(h) := \lim_{t \to 0} \frac{1}{t}\big(f(x+th) - f(x)\big)$$

whenever the limit exists. The function f is called *differentiable at x* if $df(x)(h)$ exists for all $h \in X$. It is called *continuously differentiable or C^1* if it is continuous, differentiable at all points of U and

$$df \colon U \times X \to Y, \quad (x, h) \mapsto df(x)(h)$$

is a continuous map. It is called a C^n-map if f is C^1 and df is a C^{n-1}-map, and C^∞ (*smooth*) if it is C^n for all $n \in \mathbb{N}$. This is the notion of differentiability used in [Mil83], and [Gl01], where the latter reference deals with the modifications necessary for incomplete spaces.

Since we have a chain rule for C^∞-maps between locally convex spaces ([Gl01]), we can define smooth manifolds M as in the finite-dimensional case. A Lie group G is a smooth manifold modeled on a locally convex space $\mathfrak g$ for which the group multiplication $m_G \colon G \times G \to G$ and the inversion are smooth maps. We write $\mathbf{1} \in G$ for the identity element, $\lambda_g(x) = gx$ for left multiplication, $\rho_g(x) = xg$ for right multiplication, and $c_g(x) := gxg^{-1}$ for conjugation. The tangent map $Tm_G \colon T(G \times G) \cong TG \times TG \to TG$ defines a Lie group structure on TG, and the zero section $G \hookrightarrow TG$ realizes G as a subgroup of TG. In this sense we obtain the natural left and right action of G on TG by restricting Tm_G. We write $(g,v) \mapsto g.v := Tm_G(g,v)$ for the left action and $(v,g) \mapsto v.g := Tm_G(v,g)$ for the right action. In this sense each $x \in T_{\mathbf{1}}(G)$ corresponds to a unique left invariant vector field $x_l(g) = g.x$ and the space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak g := \mathbf{L}(G) := T_{\mathbf{1}}(G)$ a continuous Lie bracket which is uniquely determined by $[x,y]_l = [x_l, y_l]$. For the right invariant vector fields $x_r(g) = x.g$ we then have $[x_r, y_r] = -[x,y]_r$.

We call a Lie algebra $\mathfrak g$ which is a topological vector space such that the Lie bracket is continuous a *topological Lie algebra*. In this sense the Lie algebra of a Lie group is a locally convex topological Lie algebra. If G is a connected Lie group, then we write $q_G \colon \widetilde{G} \to G$ for its universal covering Lie group and identify $\pi_1(G)$ with the kernel of q_G.

Throughout this paper we write abelian groups A additively with $\mathbf{0}$ as identity element. If G is a Lie group, then a *smooth G-module* is an abelian Lie group A, endowed with a smooth G-action $\rho_A \colon G \times A \to A$ by group automorphisms. We sometimes write (A, ρ_A) to include the notation ρ_A for the action map. If $\mathfrak a$ is the Lie algebra of A, then the smooth action induces a smooth action on $\mathfrak a$, so that $\mathfrak a$ also is a smooth G-module, hence also a module of the Lie algebra $\mathfrak g$ of G by the derived representation. In the following we shall mostly assume that the identity component A_0 of A is of the form $A_0 \cong \mathfrak a/\Gamma_A$, where $\Gamma_A \subseteq \mathfrak a$ is a discrete subgroup of the Mackey complete space $\mathfrak a$. Then the quotient map $q_A \colon \mathfrak a \to A_0$ is the universal covering map of A_0, and $\pi_1(A) \cong \Gamma_A$.

A linear subspace W of a topological vector space V is called *(topologically) split* if it is closed and there is a continuous linear map $\sigma \colon V/W \to V$ for which the map

$$W \times V/W \to V, \quad (w,x) \mapsto w + \sigma(x)$$

is an isomorphism of topological vector spaces. Note that the closedness of W guarantees that the quotient topology turns V/W into a Hausdorff space which is a topological vector space with respect to the induced vector space structure. A continuous linear map $f \colon V \to W$ between topological vector spaces is said to be *(topologically) split* if the subspaces $\ker(f) \subseteq V$ and $\operatorname{im}(f) \subseteq W$ are topologically split.

1. Abelian extensions of topological Lie algebras

For the definition of the cohomology of a topological Lie algebra $\mathfrak{g}$ with values in a topological $\mathfrak{g}$-module $\mathfrak{a}$ we refer to Appendix B.

Definition 1.1. Let $\mathfrak{g}$ and $\mathfrak{n}$ be topological Lie algebras. A topologically split short exact sequence

$$\mathfrak{n} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$$

is called an *extension of* $\mathfrak{g}$ *by* $\mathfrak{n}$. We identify $\mathfrak{n}$ with its image in $\widehat{\mathfrak{g}}$, and write $\widehat{\mathfrak{g}}$ as a direct sum $\widehat{\mathfrak{g}} = \mathfrak{n} \oplus \mathfrak{g}$ of topological vector spaces. Then $\mathfrak{n}$ is a topologically split ideal of $\widehat{\mathfrak{g}}$ and the quotient map $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ corresponds to $(n,x) \mapsto x$. If $\mathfrak{n}$ is abelian, then the extension is called *abelian*.

Two extensions $\mathfrak{n} \hookrightarrow \widehat{\mathfrak{g}}_1 \twoheadrightarrow \mathfrak{g}$ and $\mathfrak{n} \hookrightarrow \widehat{\mathfrak{g}}_2 \twoheadrightarrow \mathfrak{g}$ are called *equivalent* if there exists a morphism $\varphi\colon \widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ of topological Lie algebras such that the diagram

$$\begin{array}{ccccc} \mathfrak{n} & \hookrightarrow & \widehat{\mathfrak{g}}_1 & \twoheadrightarrow & \mathfrak{g} \\ \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{n}}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{g}}} \\ \mathfrak{n} & \hookrightarrow & \widehat{\mathfrak{g}}_2 & \twoheadrightarrow & \mathfrak{g} \end{array}$$

commutes. It is easy to see that this implies that φ is an isomorphism of topological Lie algebras, hence defines an equivalence relation. We write $\mathrm{Ext}(\mathfrak{g},\mathfrak{n})$ for the set of equivalence classes of extensions of $\mathfrak{g}$ by $\mathfrak{n}$ denoted $[\widehat{\mathfrak{g}}]$.

We call an extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ with $\ker q = \mathfrak{n}$ *trivial*, or say that the extension *splits*, if there exists a continuous Lie algebra homomorphism $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ with $q \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. In this case the map

$$\mathfrak{n} \rtimes_S \mathfrak{g} \to \widehat{\mathfrak{g}}, \quad (n,x) \mapsto n + \sigma(x)$$

is an isomorphism, where the semi-direct sum is defined by the homomorphism

$$S\colon \mathfrak{g} \to \mathrm{der}(\mathfrak{n}), \quad S(x)(n) := [\sigma(x), n]. \qquad \blacksquare$$

Definition 1.2. Let $\mathfrak{a}$ be a topological $\mathfrak{g}$-module. To each continuous 2-cocycle $\omega \in Z^2_c(\mathfrak{g},\mathfrak{a})$ we associate a topological Lie algebra $\mathfrak{a} \oplus_\omega \mathfrak{g}$ as the topological product vector space $\mathfrak{a} \times \mathfrak{g}$ endowed with the Lie bracket

$$[(a,x),(a',x')] := (x.a' - x'.a + \omega(x,x'), [x,x']).$$

The quotient map $q\colon \mathfrak{a} \oplus_\omega \mathfrak{g} \to \mathfrak{g}, (a,x) \mapsto x$ is a continuous homomorphism of Lie algebras with kernel $\mathfrak{a}$, hence defines an $\mathfrak{a}$-extension of $\mathfrak{g}$. The map $\sigma\colon \mathfrak{g} \to \mathfrak{a} \oplus_\omega \mathfrak{g}, x \mapsto (0,x)$ is a continuous linear section of q. $\blacksquare$

Proposition 1.3. *Let* $(\mathfrak{a}, \rho_{\mathfrak{a}})$ *be a topological* $\mathfrak{g}$*-module and write* $\mathrm{Ext}_{\rho_{\mathfrak{a}}}(\mathfrak{g}, \mathfrak{a})$ *for the set of all equivalence classes of* $\mathfrak{a}$*-extensions* $\widehat{\mathfrak{g}}$ *of* $\mathfrak{g}$ *for which the adjoint action of* $\widehat{\mathfrak{g}}$ *on* $\mathfrak{a}$ *induces the given* $\mathfrak{g}$*-module structure on* $\mathfrak{a}$*. Then the map*

$$Z^2_c(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Ext}_{\rho_{\mathfrak{a}}}(\mathfrak{g}, \mathfrak{a}), \quad \omega \mapsto [\mathfrak{a} \oplus_\omega \mathfrak{g}]$$

factors through a bijection

$$H^2_c(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Ext}_{\rho_{\mathfrak{a}}}(\mathfrak{g}, \mathfrak{a}), \quad [\omega] \mapsto [\mathfrak{a} \oplus_\omega \mathfrak{g}].$$

Proof. Suppose that $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is an $\mathfrak{a}$-extension of $\mathfrak{g}$ for which the induced $\mathfrak{g}$-module structure on $\mathfrak{a}$ coincides with $\rho_{\mathfrak{a}}$. Let $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ be a continuous linear section, so that $q \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. Then

$$\omega(x, y) := [\sigma(x), \sigma(y)] - \sigma([x, y])$$

has values in the subspace $\mathfrak{a} = \ker q$ of $\widehat{\mathfrak{g}}$, and the map $\mathfrak{a} \times \mathfrak{g} \to \widehat{\mathfrak{g}}, (a, x) \mapsto a + \sigma(x)$ is an isomorphism of topological Lie algebras $\mathfrak{a} \oplus_\omega \mathfrak{g} \to \widehat{\mathfrak{g}}$.

It is easy to verify that $\mathfrak{a} \oplus_\omega \mathfrak{g} \sim \mathfrak{a} \oplus_\eta \mathfrak{g}$ if and only if $\omega - \eta \in B^2_c(\mathfrak{g}, \mathfrak{a})$. Therefore the quotient space $H^2_c(\mathfrak{g}, \mathfrak{a})$ classifies the equivalence classes of $\mathfrak{a}$-extensions of $\mathfrak{g}$ by the assignment $[\omega] \mapsto [\mathfrak{a} \oplus_\omega \mathfrak{g}]$ (cf. [CE48]). ∎

2. Abelian extensions of Lie groups

Let A be a smooth G-module. In this section we explain how to assign to a cocycle $f \in Z^2_s(G, A)$ (satisfying some additional smoothness condition if G is not connected) a Lie group $A \times_f G$ which is an extension of A by G for which the induced action of G on A coincides with the original one. We shall see that this assignment leads to a bijection between a certain subgroup $H^2_{ss}(G, A)$ of $H^2_s(G, A)$ with the set of equivalence classes of extensions of G by the smooth G-module A. If G is connected, then $H^2_{ss}(G, A) = H^2_s(G, A)$. We also show that the assignment $f \mapsto A \times_f G$ is compatible with the derivation map $D\colon Z^2_s(G, A) \to Z^2_c(\mathfrak{g}, \mathfrak{a})$ in the sense that $\mathfrak{a} \oplus_{Df} \mathfrak{g}$ is the Lie algebra of $A \times_f G$ (cf. Appendix B for definitions).

Lemma 2.1. *Let* G *be a group,* A *a* G*-module and* $f\colon G \times G \to A$ *a normalized* 2*-cocycle, i.e.,*

$$f(g, \mathbf{1}) = f(\mathbf{1}, g) = \mathbf{0}, \quad f(g, g') + f(gg', gg') = g.f(g', g'') + f(g, g'g''), \quad g, g', g'' \in G.$$

Then we obtain a group $A \times_f G$ *by endowing the product set* $A \times G$ *with the multiplication*

$$(a, g)(a', g') := (a + g.a' + f(g, g'), gg'). \tag{2.1}$$

The unit element of this group is $(\mathbf{0}, \mathbf{1})$*, inversion is given by*

$$(a, g)^{-1} = (-g^{-1}.(a + f(g, g^{-1})), g^{-1}), \tag{2.2}$$

and conjugation by the formula

$$(2.3)\quad (a,g)(a',g')(a,g)^{-1} = \big(a+g.a' - gg'g^{-1}.a + f(g,g') - f(gg'g^{-1},g), gg'g^{-1}\big).$$

The map $q\colon A\times_f G\to G, (a,g)\mapsto g$ *is a surjective homomorphism whose kernel* $A\times\{\mathbf{1}\}$ *is isomorphic to* A*. The conjugation action of* $A\times_f G$ *on the normal subgroup* A *factors through the original action of* G *on* A*.*

Proof. The condition $f(\mathbf{1},g)=f(g,\mathbf{1})=\mathbf{0}$ implies that $(\mathbf{0},\mathbf{1})$ is an identity element in $A\times_f G$, and the associativity of the multiplication is equivalent to the cocycle condition. The formula for the inversion is easily verified. Conjugation in $A\times_f G$ is given by

$$\begin{aligned}&(a,g)(a',g')(a,g)^{-1}\\ &= \big(a+g.a'+f(g,g'),gg'\big)\big(-g^{-1}.(a+f(g,g^{-1})),g^{-1}\big)\\ &= \big(a+g.a'+f(g,g')-gg'g^{-1}.(a+f(g,g^{-1}))+f(gg',g^{-1}),gg'g^{-1}\big).\end{aligned}$$

To simplify this expression, we use

$$f(g,g^{-1}) = f(g,g^{-1}) + f(\mathbf{1},g) = f(g,\mathbf{1}) + g.f(g^{-1},g) = g.f(g^{-1},g)$$

and

$$f(gg',g^{-1}) + f(gg'g^{-1},g) = f(gg',\mathbf{1}) + gg'.f(g^{-1},g) = gg'.f(g^{-1},g)$$

to obtain

$$\begin{aligned}&(a,g)(a',g')(a,g)^{-1}\\ &= \big(a+g.a'+f(g,g')-gg'g^{-1}.a-gg'g^{-1}.f(g,g^{-1})+f(gg',g^{-1}),gg'g^{-1}\big)\\ &= \big(a+g.a'+f(g,g')-gg'g^{-1}.a-gg'.f(g^{-1}.g)+f(gg',g^{-1}),gg'g^{-1}\big)\\ &= \big(a+g.a'+f(g,g')-gg'g^{-1}.a-f(gg'g^{-1},g),gg'g^{-1}\big).\end{aligned}$$

In particular, we obtain

$$(\mathbf{0},g)(a,\mathbf{1})(\mathbf{0},g)^{-1} = (g.a,\mathbf{1}).$$

This means that the action of G on A given by $q(g).a := gag^{-1}$ for $g\in A\times_f G$ coincides with the given action of G on A. ■

Definition 2.2. An *extension of Lie groups* is a surjective morphism $q\colon \widehat{G}\to G$ of Lie groups with a smooth local section for which $N:=\ker q$ has a natural Lie group structure such that the map $N\times\widehat{G}\to\widehat{G}, (n,g)\mapsto ng$ is smooth. Then the existence of a smooth local section implies that $\widehat{G}$ is a smooth N-principal bundle, so that N is a split Lie subgroup of $\widehat{G}$ in the sense of Definition C.4.

We call two extensions $N \hookrightarrow \widehat{G}_1 \twoheadrightarrow G$ and $N \hookrightarrow \widehat{G}_2 \twoheadrightarrow G$ of the Lie group G by the Lie group N *equivalent* if there exists a Lie group morphism $\varphi\colon \widehat{G}_1 \to \widehat{G}_2$ such that the following diagram commutes:

$$\begin{array}{ccccc} N & \hookrightarrow & \widehat{G}_1 & \twoheadrightarrow & G \\ \big\downarrow \mathrm{id}_N & & \big\downarrow \varphi & & \big\downarrow \mathrm{id}_G \\ N & \hookrightarrow & \widehat{G}_2 & \twoheadrightarrow & G. \end{array}$$

It is easy to see that any such φ is an isomorphism of groups and that its inverse is smooth. Thus φ is an isomorphism of Lie groups, and we obtain indeed an equivalence relation. We write $\mathrm{Ext}(G,N)$ for the set of equivalence classes of Lie group extensions of G by N. ■

Lemma 2.3. *If $A \hookrightarrow \widehat{G}_1 \xrightarrow{q_1} G$ and $A \hookrightarrow \widehat{G}_2 \xrightarrow{q_2} G$ are equivalent abelian extensions of G by the Lie group A, then the induced actions of G on A coincide.*

Proof. An equivalence of extensions yields a morphism of Lie groups $\varphi\colon \widehat{G}_1 \to \widehat{G}_2$ with $\varphi\mid_A = \mathrm{id}_A$ and $q_2 \circ \varphi = q_1$. For $g \in G$ and $a \in A$ the extension $\widehat{G}_1$ defines an action of G on A by $g *_1 a := g_1 a g_1^{-1}$, where $q_1(g_1) = g$. We likewise obtain from the extension $\widehat{G}_2$ an action of G on A by $g *_2 a := g_2 a g_2^{-1}$ for $q_2(g_2) = g$. We then have

$$g *_1 a = g_1 a g_1^{-1} = \varphi(g_1 a g_1^{-1}) = \varphi(g_1) a \varphi(g_1)^{-1} = q_2(\varphi(g_1)) *_2 a = q_1(g_1) *_2 a = g *_2 a.$$

■

Definition 2.4. If (A, ρ_A) is a smooth G-module, then an *extension of G by A* is always understood to be an abelian Lie group extension $q\colon \widehat{G} \to G$ with kernel A for which the natural action of G on A induced by the conjugation action coincides with ρ_A. In view of Lemma 2.3, it makes sense to write $\mathrm{Ext}_{\rho_A}(G,A) \subseteq \mathrm{Ext}(G,A)$ for the subset of equivalence classes of those extensions of G by A for which the induced action of G on A coincides with ρ_A. ■

Definition 2.5. Let G be a Lie group and A a smooth G-module. For $f \in Z^2_s(G,A)$ (cf. Definition B.2) and $g \in G$ we consider the function

$$f_g\colon G \to A, \quad f_g(g') := f(g,g') - f(gg'g^{-1}, g)$$

and write

$$Z^2_{ss}(G,A) := \{f \in Z^2_s(G,A)\colon (\forall g \in G)\ f_g \in C^1_s(G,A)\}$$

for those $f \in Z^2_s(G,A)$ for which, in addition, all functions f_g are smooth in an identity neighborhood of G.

If $\ell \in C^1_s(G, A)$ and $f(g, g') = (d_G\ell)(g, g') = \ell(g) + g.\ell(g') - \ell(gg')$, then

$$\begin{aligned} f_g(g') &= \ell(g) + g.\ell(g') - \ell(gg') - \big(\ell(gg'g^{-1}) + (gg'g^{-1}).\ell(g) - \ell(gg')\big) \\ &= \ell(g) + g.\ell(g') - \ell(gg'g^{-1}) - (gg'g^{-1}).\ell(g) \end{aligned}$$

is smooth in an identity neighborhood of G for each $g \in G$. Therefore $B^2_s(G, A) \subseteq Z^2_{ss}(G, A)$ and

$$H^2_{ss}(G, A) := Z^2_{ss}(G, A)/B^2_s(G, A)$$

is a subgroup of $H^2_s(G, A)$. ■

Proposition 2.6. *Let G be a Lie group and (A, ρ_A) a smooth G-module. Then for each $f \in Z^2_{ss}(G, A)$ the group $A \times_f G$ carries the structure of a Lie group such that the map $q\colon A \times_f G \to G, (a, g) \mapsto g$ is a Lie group extension of G by the smooth G-module A. Conversely, every Lie group extension of G by the smooth G-module A is equivalent to one of this form. The assignment*

$$Z^2_{ss}(G, A) \to \mathrm{Ext}_{\rho_A}(G, A), \quad f \mapsto [A \times_f G]$$

factors through a bijection

$$H^2_{ss}(G, A) \to \mathrm{Ext}_{\rho_A}(G, A).$$

If G is connected, then $Z^2_{ss}(G, A) = Z^2_s(G, A)$ and we obtain a bijection

$$H^2_s(G, A) \to \mathrm{Ext}_{\rho_A}(G, A).$$

Proof. (1) Let $f \in Z^2_{ss}(G, A)$ and form the group $\widehat{G} := A \times_f G$ (Lemma 2.1). First we construct the Lie group structure on $\widehat{G}$. Let $U_G \subseteq G$ be an open symmetric $\mathbf{1}$-neighborhood such that f is smooth on $U_G \times U_G$, and consider the subset

$$U := A \times U_G = q^{-1}(U_G) \subseteq \widehat{G} = A \times_f G.$$

Then $U = U^{-1}$. We endow U with the product manifold structure from $A \times U_G$. Since the multiplication $m_G|_{U_G \times U_G}\colon U_G \times U_G \to G$ is continuous, there exists an open identity neighborhood $V_G \subseteq U_G$ with $V_G V_G \subseteq U_G$. Then the set $V := A \times V_G$ is an open subset of U such that the multiplication map

$$V \times V \to U, \quad \big((a, x), (a', x')\big) \to (a + x.a' + f(x, x'), xx')$$

is smooth. The inversion

$$U \to U, \quad (a, x) \mapsto \big(-x^{-1}.(a + f(x, x^{-1})), x^{-1}\big)$$

(Lemma 2.1) is also smooth.

For $(a, g) \in \widehat{G}$ let $V_g \subseteq U_G$ be an open identity neighborhood such that the conjugation map $c_g(x) = gxg^{-1}$ satisfies $c_g(V_g) \subseteq U_G$. Then $c_{(a,g)}(q^{-1}(V_g)) \subseteq U$ and the conjugation map

$$c_{(a,g)} \colon q^{-1}(V_g) \to U, \quad (a', g') \mapsto (a + g.a' - gg'g^{-1}.a + f_g(g'), gg'g^{-1})$$

(Lemma 2.1) is smooth in an identity neighborhood because $f \in Z^2_{ss}(G, A)$.

Now Theorem C.2 implies that $\widehat{G}$ carries a unique Lie group structure for which the inclusion map $U = A \times U_G \hookrightarrow \widehat{G}$ restricts to a diffeomorphism of some open $\mathbf{1}$-neighborhood in $A \times G$ to an open $\mathbf{1}$-neighborhood in $\widehat{G}$. It is clear that with respect to this Lie group structure on $\widehat{G}$, the map $q \colon \widehat{G} \to G$ defines a smooth A-principal bundle because the map $V_G \to \widehat{G}, g \mapsto (0, g)$ defines a section of q which is smooth on an identity neighborhood in G which might be smaller than V_G.

(2) Assume, conversely, that $q \colon \widehat{G} \to G$ is an extension of G by the smooth G-module A. Then there exists an open $\mathbf{1}$-neighborhood $U_G \subseteq G$ and a smooth section $\sigma \colon U_G \to \widehat{G}$ of the map $q \colon \widehat{G} \to G$. We extend σ to a global section $G \to \widehat{G}$ which need neither be continuous nor smooth. Then

$$f(x, y) := \sigma(x)\sigma(y)\sigma(xy)^{-1}$$

defines a 2-cocycle $G \times G \to A$ which is smooth in a neighborhood of $(\mathbf{1}, \mathbf{1})$, and the map

$$A \times_f G \to \widehat{G}, \quad (a, g) \mapsto a\sigma(g)$$

is an isomorphism of abstract groups. The functions $f_g \colon G \to A$ are given by

$$\begin{aligned} f_g(g') &= f(g, g') - f(gg'g^{-1}, g) = \sigma(g)\sigma(g')\sigma(gg')^{-1} - \sigma(gg'g^{-1})\sigma(g)\sigma(gg')^{-1} \\ &= \sigma(g)\sigma(g')\sigma(gg')^{-1}\sigma(gg')\sigma(g)^{-1}\sigma(gg'g^{-1})^{-1} = \sigma(g)\sigma(g')\sigma(g)^{-1}\sigma(gg'g^{-1})^{-1}, \end{aligned}$$

hence smooth near $\mathbf{1}$. This shows that $f \in Z^2_{ss}(G, A)$. In view of (1), the group $A \times_f G$ carries a Lie group structure for which there exists an identity neighborhood $V_G \subseteq G$ for which the product map

$$A \times V_G \to A \times_f G, \quad (a, v) \mapsto (a, \mathbf{1})(0, v) = (a, v)$$

is smooth. This implies that the group isomorphism $A \times_f G \to \widehat{G}$ is a local diffeomorphism, hence an isomorphism of Lie groups.

(3) Step (1) provides a map

$$Z^2_{ss}(G, A) \to \mathrm{Ext}_{\rho_A}(G, A), \quad f \mapsto [A \times_f G],$$

and (2) shows that it is surjective. Assume that two extensions of the form $A \times_{f_i} G$ for $f_1, f_2 \in Z^2_{ss}(G, A)$ are equivalent as Lie group extensions. An isomorphism $A \times_{f_1} G \to A \times_{f_2} G$ inducing an equivalence of abelian extensions must be of the form

$$(a, g) \mapsto (a + h(g), g), \tag{2.4}$$

where $h \in C^1_s(G, A)$. The condition that (2.4) is a group homomorphism implies that

$$(h(gg') + f_1(g, g'), gg') = (h(g), g)(h(g'), g') = (h(g) + g.h(g') + f_2(g, g'), gg'),$$

which means that

$$(f_1 - f_2)(g.g') = g.h(g') - h(gg') + h(g) = (d_G h)(g, g'), \tag{2.5}$$

so that $f_1 - f_2 \in B^2_s(G, A)$.

If, conversely, $h \in C^1_s(G, A)$ and $f_1 - f_2 = d_G h$, then it is easily verified that (2.4) defines a group isomorphism for which there exists an open identity neighborhood mapped diffeomorphically onto its image. Hence (2.5) is an isomorphism of Lie groups. We conclude that the map $Z^2_{ss}(G, A) \to \mathrm{Ext}_{\rho_A}(G, A)$ factors through a bijection $H^2_{ss}(G, A) \to \mathrm{Ext}_{\rho_A}(G, A)$.

(4) Assume now that G is connected and that $f \in Z^2_s(G, A)$. In the context of (1), the conjugation map $c_{(a,g)} : q^{-1}(V_g) \to U$ is smooth in an identity neighborhood if and only if the function f_g is smooth in an identity neighborhood. As $f \in Z^2_s(G, A)$, the set W of all $g \in G$ for which this condition is satisfied is an identity neighborhood. On the other hand, the set W is closed under multiplication. In view of the connectedness of G, we have $G = \bigcup_{n \in \mathbb{N}} W^n = W$. This means that $f \in Z^2_{ss}(G, A)$, and therefore that $Z^2_s(G, A) = Z^2_{ss}(G, A)$. ■

Problem 2. Do the two spaces $Z^2_s(G, A)$ and $Z^2_{ss}(G, A)$ also coincide if G is not connected? ■

The following lemma shows that the derivation map

$$D: Z^2_s(G, A) \to Z^2_c(\mathfrak{g}, \mathfrak{a}), \quad (Df)(x, y) = d^2 f(\mathbf{1}, \mathbf{1})(x, y) - d^2 f(\mathbf{1}, \mathbf{1})(y, x)$$

from Theorem B.6 and Lemma B.7 is compatible with the construction in Proposition 2.6. In the following proof we use the notation $d^2 f$ introduced in Appendix A.

Lemma 2.7. *Let $A \cong \mathfrak{a}/\Gamma_A$, where $\Gamma_A \subseteq \mathfrak{a}$ is a discrete subgroup, $f \in Z^2_{ss}(G, A)$ and $\widehat{G} = A \times_f G$ the corresponding extension of G by A. Then the Lie algebra cocycle Df satisfies $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_{Df} \mathfrak{a}$.*

Proof. Let $U_\mathfrak{a} \subseteq \mathfrak{a}$ be an open 0-neighborhood such that the restriction $\varphi_A : U_\mathfrak{a} \to U_\mathfrak{a} + \Gamma_A \subseteq A$ of the quotient map $q_A : \mathfrak{a} \to A$ is a diffeomorphism onto an open identity neighborhood in A and $\varphi_G : U_\mathfrak{g} \to G$ a local chart of G, where $U_\mathfrak{g} \subseteq \mathfrak{g}$ is an open 0-neighborhood, $\varphi_G(0) = \mathbf{1}$ and $d\varphi_G(0) = \mathrm{id}_\mathfrak{g}$. After shrinking $U_\mathfrak{g}$ further, we obtain a chart of $A \times_f G$ by the map

$$\varphi : U_\mathfrak{a} \times U_\mathfrak{g} \to A \times_f G, \quad (a, x) \mapsto (\varphi_A(a), \varphi_G(x)).$$

Moreover, we may assume that $U_\mathfrak{g}$ is so small that $f(\varphi_G(U_\mathfrak{g}) \times \varphi_G(U_\mathfrak{g})) \subseteq \varphi_A(U_\mathfrak{a}))$, which implies that there exists a smooth function $f_\mathfrak{a} : U_\mathfrak{g} \times U_\mathfrak{g} \to U_\mathfrak{a}$ with $\varphi_A \circ f_\mathfrak{a} = f \circ (\varphi_G \times \varphi_G)$.

Writing $x * x' := \varphi_G^{-1}(\varphi_G(x)\varphi_G(x'))$ for $x, x' \in U_{\mathfrak{g}}$ with $\varphi_G(x)\varphi_G(x') \in \varphi_G(U_{\mathfrak{g}})$, the multiplication

$$(a,g)(a',g') = (a + g.a' + f(g,g'), gg')$$

in $A \times_f G$ can be expressed in local coordinates for sufficiently small $a, a' \in \mathfrak{a}, x, x' \in \mathfrak{g}$ by

$$\begin{aligned}\varphi(a,x)\varphi(a',x') &= (\varphi_A(a) + \varphi_G(x).\varphi_A(a') + f(\varphi_G(x), \varphi_G(x')), \varphi_G(x)\varphi_G(x')) \\ &= (\varphi_A(a + \varphi_G(x).a' + f_{\mathfrak{a}}(x,x')), \varphi_G(x * x')) \\ &= \varphi(a + \varphi_G(x).a' + f_{\mathfrak{a}}(x,x'), x * x').\end{aligned}$$

Here the identity element has the coordinates $(0,0) \in \mathfrak{a} \times \mathfrak{g}$.

For the multiplication in G we have

$$x * x' = x + x' + b(x,x') + \cdots$$

where $\cdots$ stands for the terms of order at least three in the Taylor expansion of the product map and the quadratic term $b(x,x')$ is bilinear. The Lie bracket in $\mathfrak{g}$ is given by

$$[x,x'] = b(x,x') - b(x',x)$$

([Mil83, p.1036]). Therefore the Lie bracket in the Lie algebra $\mathbf{L}(A \times_f G)$ of $A \times_f G$ can be obtained from

$$\begin{aligned}&(a + \varphi_G(x).a' + f_{\mathfrak{a}}(x,x'), x * x') \\ &= (a + a' + x.a' + d^2 f_{\mathfrak{a}}(0,0)(x,x') + \cdots, x + x' + b(x,x') + \cdots) \\ &= (a + a' + x.a' + d^2 f(\mathbf{1},\mathbf{1})(x,x') + \cdots, x + x' + b(x,x') + \cdots),\end{aligned}$$

which leads to

$$[(a,x),(a',x')] = (x.a' - x'.a + Df(x,x'), [x,x']). \qquad \blacksquare$$

3. Locally smooth 1-cocycles

Let G be a Lie group and A a smooth G-module. In this section we take a closer look at the space $Z^1_s(G,A)$ of locally smooth A-valued 1-cocycles on G. We know from Appendix B that there is a natural map

$$D_1 \colon Z^1_s(G,A) \to Z^1_c(\mathfrak{g},\mathfrak{a}), \quad D_1(f)(x) := df(\mathbf{1})(x).$$

If $A \cong \mathfrak{a}/\Gamma_A$ holds for a discrete subgroup Γ_A of $\mathfrak{a}$ and $q_A \colon \mathfrak{a} \to A$ is the quotient map, then we have for $a \in \mathfrak{a}$ the relation

$$D_1(d_G(q_A(a))) = d_{\mathfrak{g}}(a)$$

and thus $D_1(B^1_s(G,A)) = B^1_c(\mathfrak{g},\mathfrak{a})$. Hence D_1 induces a map

$$D_1 \colon H^1_s(G,A) \to H^1_c(\mathfrak{g},\mathfrak{a}),$$

and it is of fundamental importance to have a good description of kernel and cokernel of D_1 on the level of cocycles and cohomology classes.

We shall see that the integration problem for Lie algebra 1-cocycles has a rather simple solution, the only obstruction coming from $\pi_1(G)$.

Lemma 3.1. *Each $f \in Z^1_s(G,A)$ is a smooth function and its differential $df \in \Omega^1(G,\mathfrak{a})$ is an equivariant 1-form.*

Proof. Let $g \in G$. In view of

$$f(gh) = g.f(h) + f(g), \tag{3.1}$$

the smoothness of f in an identity neighborhood implies the smoothness in a neighborhood of g.

Formula (3.1) means that $f \circ \lambda_g = \rho_A(g) \circ f + f(g)$, so that df satisfies $\lambda_g^* df = \rho_A(g) \circ df$, i.e., df is equivariant. ■

Lemma 3.2. *Let G be a Lie group with identity component G_0 and A a smooth G-module. Then for a smooth function $f \colon G \to A$ with $f(\mathbf{1}) = 0$ the following are equivalent:*

(1) *df is an equivariant $\mathfrak{a}$-valued 1-form on G.*

(2) *$f(gn) = f(g) + g.f(n)$ for $g \in G$ and $n \in G_0$.*

If, in addition, G is connected, then df is equivariant if and only if f is a cocycle.

Proof. We write $g.a = \rho_{\mathfrak{a}}(g).a$ for the action of G on $\mathfrak{a}$ and $g.a = \rho_A(g).a$ for the action of G on A.

(1) $\Rightarrow$ (2): Let $g \in G$. In view of $d(\rho_A(g) \circ f) = \rho_{\mathfrak{a}}(g) \circ df$, we have

$$d(f \circ \lambda_g - \rho_A(g) \circ f - f(g)) = \lambda_g^* df - \rho_{\mathfrak{a}}(g) \circ df.$$

Hence (1) implies that all the functions $f \circ \lambda_g - \rho_A(g) \circ f - f(g)$ are locally constant. Since the value of these functions in $\mathbf{1}$ is 0, they are are constant 0 on G_0, which is (2).

(2) $\Rightarrow$ (1): If (2) is satisfied, then $df(g) d\lambda_g(\mathbf{1}) = \rho_{\mathfrak{a}}(g) \circ df(\mathbf{1})$ holds for each $g \in G$, and this means that df is equivariant. ■

Definition 3.3. Suppose that $\mathfrak{a}$ is Mackey complete. If $\alpha \in Z^1_c(\mathfrak{g},\mathfrak{a})$ and $\alpha^{\rm eq}$ is the corresponding closed equivariant 1-form on G (cf. Definition B.4), then we obtain a morphism of abelian groups, called the *period map of α:*

$$\mathrm{per}_\alpha \colon \pi_1(G) \to \mathfrak{a}, \quad [\gamma] \mapsto \int_\gamma \alpha^{\rm eq} = \int_0^1 \alpha^{\rm eq}_{\gamma(t)}(\gamma'(t))\, dt = \int_0^1 \gamma(t).\alpha(\gamma(t)^{-1}.\gamma'(t))\, dt,$$

where $\gamma\colon [0,1] \to G$ is a piecewise smooth loop based in $\mathbf{1}$. The map

$$C^\infty(\mathbb{S}^1, G) \to \mathfrak{a}, \quad \gamma \mapsto \int_\gamma \alpha^{\rm eq}$$

is locally constant, so that the connectedness of G implies in particular that for $g \in G$ the curves γ and $\lambda_g \circ \gamma$ are homotopic, and we get

$$\int_\gamma \alpha^{\rm eq} = \int_{\lambda_g \circ \gamma} \alpha^{\rm eq} = \int_\gamma \lambda_g^* \alpha^{\rm eq} = \rho_{\mathfrak{a}}(g).\int_\gamma \alpha^{\rm eq}$$

which leads to

$$\mathrm{im}(\mathrm{per}_\alpha) \subseteq \mathfrak{a}^G.$$

If $\Gamma_A \subseteq \mathfrak{a}^G$ is a discrete subgroup, then $A := \mathfrak{a}/\Gamma_A$ is a smooth G-module with respect to the induced action. Let $q_A\colon \mathfrak{a} \to A$ denote the quotient map. We then obtain a group homomorphism

$$P_1\colon Z^1_c(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Hom}(\pi_1(G), A^G), \quad P_1(\alpha) := q_A \circ \mathrm{per}_\alpha. \qquad \blacksquare$$

Proposition 3.4. *If G is a connected Lie group and $A_0 \cong \mathfrak{a}/\Gamma_A$, where $\Gamma_A \subseteq \mathfrak{a}^G$ is a discrete subgroup and $\mathfrak{a}$ is Mackey complete, then the sequence*

$$\mathbf{0} \to Z^1_s(G, A) \xrightarrow{D_1} Z^1_c(\mathfrak{g}, \mathfrak{a}) \xrightarrow{P_1} \mathrm{Hom}(\pi_1(G), A^G) \tag{3.2}$$

is exact. If, in addition, $A \cong \mathfrak{a}/\Gamma_A$, then it induces an exact sequence

$$\mathbf{0} \to H^1_s(G, A) \xrightarrow{D_1} H^1_c(\mathfrak{g}, \mathfrak{a}) \xrightarrow{P_1} \mathrm{Hom}(\pi_1(G), A^G). \tag{3.3}$$

Proof. If $f \in Z^1_s(G, A)$ satisfies $D_1 f = 0$, then Lemma 3.2 implies that $df = 0$ because df is equivariant, and hence that f is constant, so that $f(g) = f(\mathbf{1}) = 0$ for each $g \in G$. Therefore D_1 is injective on $Z^1_s(G, A)$. The kernel of $P_1\colon Z^1_c(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Hom}(\pi_1(G), A)$ consists of those 1-cocycles α for which $\alpha^{\rm eq}$ is the differential of a smooth function $f\colon G \to A$ with $f(\mathbf{1}) = 0$ ([Ne02, Prop. 3.9]), which means that $\alpha = D_1 f$ for some $f \in Z^1_s(G, A)$ (Lemma 3.2). This proves the exactness of the first sequence.

Now we assume that $A \cong \mathfrak{a}/\Gamma_A$. If $\alpha \in B^1_c(\mathfrak{g}, \mathfrak{a})$, then $\alpha^{\rm eq}$ is exact (Lemma B.5), so that $P_1(\alpha) = 0$. Therefore P_1 factors through a map $H^1_c(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Hom}(\pi_1(G), \mathfrak{a})$. The exactness of (3.3) now follows from $D_1(B^1_s(G, A)) = D_1 d_G A = d_{\mathfrak{g}} \mathfrak{a} = B^1_c(\mathfrak{g}, \mathfrak{a})$ and the exactness of (3.2). $\blacksquare$

Remark 3.5. For each $\alpha \in Z^1_c(\mathfrak{g}, \mathfrak{a})$ the corresponding equivariant 1-form $\alpha^{\rm eq}$ is closed, and it is exact if $\alpha \in B^1_c(\mathfrak{g}, \mathfrak{a})$, so that we obtain a map

$$H^1_c(\mathfrak{g}, \mathfrak{a}) \to H^1_{\rm dR}(G, \mathfrak{a}), \quad [\alpha] \mapsto [\alpha^{\rm eq}].$$

Proposition 3.4, applied to $A := \mathfrak{a}$ now means that the sequence

$$\mathbf{0} \to H^1_s(G,\mathfrak{a}) \xrightarrow{D_1} H^1_c(\mathfrak{g},\mathfrak{a}) \longrightarrow H^1_{\rm dR}(G,\mathfrak{a})$$

is exact. Let $\Gamma_A \subseteq \mathfrak{a}$ be a discrete subgroup and consider $A := \mathfrak{a}/\Gamma_A$. For

$$H^1_{\rm dR}(G,\Gamma_A) := \Big\{[\alpha] \in H^1_{\rm dR}(G,\mathfrak{a}) \colon (\forall \gamma \in C^\infty(\mathbb{S}^1, G)) \int_\gamma \alpha \in \Gamma_A\Big\},$$

we then have

$$H^1_{\rm dR}(G,\Gamma_A) = dC^\infty(G,A)/dC^\infty(G,\mathfrak{a})$$

([Ne02, Prop. 3.9]), and we obtain an exact sequence

$$\begin{aligned}\mathbf{0} \to H^1_s(G,A) \xrightarrow{D_1} H^1_c(\mathfrak{g},\mathfrak{a}) &\longrightarrow \Omega^1(G,\mathfrak{a})/dC^\infty(G,A) \\ &\cong (\Omega^1(G,\mathfrak{a})/dC^\infty(G,\mathfrak{a}))/H^1_{\rm dR}(G,\Gamma_A),\end{aligned} \tag{3.4}$$

because for $\alpha \in Z^1_c(\mathfrak{g},\mathfrak{a})$ the condition $[\alpha^{\rm eq}] \in dC^\infty(G,A)$ is equivalent to $P_1([\alpha]) = 0$ (Proposition 3.4). ■

In the remainder of this section we address the question how to classify the smooth G-modules A with a given identity component (as G-module). We shall see that the crucial data is given by a homomorphism $\overline{\theta}_A \colon \pi_0(A) \to H^1_c(\mathfrak{g},\mathfrak{a})$.

Definition 3.6. Let A be a smooth G-module for the connected Lie group G and assume that $A_0 \cong \mathfrak{a}/\Gamma_A$ holds for the identity component of A. Then for each $a \in A$ we obtain a smooth cocycle

$$d_G'(a) \in Z^1_s(G,A_0), \quad d_G'(a)(g) := g.a - a.$$

Taking derivatives in $\mathbf{1}$ leads to homomorphisms

$$\theta_A := D_1 \circ d_G' \colon A \to Z^1_c(\mathfrak{g},\mathfrak{a}) \quad \text{and} \quad \overline{\theta}_A \colon \pi_0(A) \to H^1_c(\mathfrak{g},\mathfrak{a}).$$

The map $\overline{\theta}_A$ is called the *characteristic homomorphism of the G-module A*. It can be viewed as an obstruction for the existence of a derivation map $H^1_s(G,A) \to H^1_c(\mathfrak{g},\mathfrak{a})$. ■

Remark 3.7. The characteristic homomorphism clearly defines an action of $\pi_0(A)$ on $H^1_c(\mathfrak{g},\mathfrak{a})$, and we have in this sense

$$H^1_c(\mathfrak{g},\mathfrak{a})/\pi_0(A) \cong \operatorname{coker} \overline{\theta}_A = \operatorname{coker} \theta_A.$$

If $\widetilde{G}$ is the simply connected covering group of G, then Proposition 3.4 shows that the map $D_1 \colon Z^1_s(\widetilde{G},A) \to Z^1_c(\mathfrak{g},\mathfrak{a})$ is bijective, so that

$$H^1_s(\widetilde{G},A) = Z^1_s(\widetilde{G},A)/d_{\widetilde{G}}(A) \cong H^1_c(\mathfrak{g},\mathfrak{a})/\pi_0(A).$$ ■

The following lemma shows how the characteristic homomorphism classifies all smooth G-modules for which the module structure on the identity component is given.

Lemma 3.8. *Let A and B be smooth modules of the connected Lie group G and assume that $A_0 = B_0 \cong \mathfrak{a}/\Gamma_A$ as G-modules, where $\Gamma_A \subseteq \mathfrak{a}$ is a discrete subgroup. Then there exists an isomorphism $\psi\colon A \to B$ of G-modules with $\psi\,|_{A_0} = \mathrm{id}_{A_0}$ if and only if there exists an isomorphism $\gamma\colon \pi_0(A) \to \pi_0(B)$ such that the characteristic homomorphisms of A and B are related by*

$$\overline{\theta}_B \circ \gamma = \overline{\theta}_A.$$

Proof. If $\psi\colon A \to B$ is an isomorphism of G-modules restricting to the identity on A_0, then ψ induces an isomorphism $\gamma := \pi_0(\psi)\colon \pi_0(A) \to \pi_0(B)$, and it follows directly from the definitions that $\overline{\theta}_B \circ \gamma = \overline{\theta}_A$.

Suppose, conversely, that $\gamma\colon \pi_0(A) \to \pi_0(B)$ is an isomorphism with $\overline{\theta}_B \circ \gamma = \overline{\theta}_A$. Since A_0 is an open divisible subgroup of A, we have $A \cong A_0 \times \pi_0(A)$ as abelian Lie groups, and likewise $B \cong A_0 \times \pi_0(B)$. For each homomorphism $\varphi_0\colon \pi_0(A) \to A_0$ we then obtain a Lie group isomorphism

$$\varphi\colon A \to B, \quad (a_0, a_1) \mapsto (a_0 + \varphi_0(a_1), \gamma(a_1)). \tag{3.5}$$

Since G acts on $A \cong A_0 \times \pi_0(A)$ by

$$g.(a_0, a_1) = (g.a_0 + d_G'(a_1)(g), a_1),$$

the isomorphism φ is G-equivariant if and only if

$$\varphi_0(a_1) + d_G'(a_1)(g) = g.\varphi_0(a_1) + d_G'(\gamma(a_1))(g) \tag{3.6}$$

for $g \in G$, $a_1 \in \pi_0(A)$, which means that

$$d_G(\varphi_0(a_1)) = d_G'(a_1) - d_G'(\gamma(a_1)) =: \beta(a_1).$$

To see that a homomorphism φ_0 with the required properties exists, we first observe that our assumption implies that β is a homomorphism $\pi_0(A) \to Z^1_s(G, A_0)$ with $\mathrm{im}(D_1 \circ \beta) \subseteq d_{\mathfrak{g}}\mathfrak{a}$. In view of the divisibility of $\mathfrak{a}$, there exists a homomorphism $\delta\colon \pi_0(A) \to \mathfrak{a}$ with $D_1 \circ \beta = d_{\mathfrak{g}} \circ \delta = D_1 \circ d_G \circ q_A \circ \delta$. Since D_1 is injective on cocycles (Proposition 3.4), we obtain $\beta = d_G \circ q_A \circ \delta$. We may therefore put $\varphi_0 := q_A \circ \delta$ to obtain an isomorphism φ of G-modules as in (3.5). ∎

4. The period homomorphism

In this section G denotes a connected Lie group, $\mathfrak{a}$ is a smooth Mackey complete G-module, and $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$ is a continuous Lie algebra cocycle. We shall define the period homomorphism

$$\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{a}, \quad \mathrm{per}_\omega([\sigma]) := \int_\sigma \omega^{\mathrm{eq}},$$

where σ is a (piecewise) smooth representative in the homotopy class.

If $q\colon \widehat{G} \to G$ is an extension of G by the smooth G-module A whose Lie algebra is isomorphic to $\mathfrak{a} \oplus_\omega \mathfrak{g}$ and $A_0 \cong \mathfrak{a}/\Gamma_A$ holds for a discrete subgroup Γ_A of $\mathfrak{a}$. Then we show that the period map is, up to sign, the connecting map $\pi_2(G) \to \pi_1(A) \cong \Gamma_A$ of the long exact homotopy sequence of the principal A-bundle $A \hookrightarrow \widehat{G} \twoheadrightarrow G$.

Definition 4.1. In the following $\Delta_p = \{(x_1, \ldots, x_p) \in \mathbb{R}^p \colon x_i \geq 0, \sum_j x_j \leq 1\}$ denotes the *p-dimensional standard simplex* in $\mathbb{R}^p$. We also write $\langle v_0, \ldots, v_p\rangle$ for the affine simplex in a vector space spanned by the points $v_0, \ldots, v_p$. In this sense $\Delta_p = \langle 0, e_1, \ldots, e_p\rangle$, where e_i denotes the i-th canonical basis vector in $\mathbb{R}^p$.

Let Y be a smooth manifold. A continuous map $f\colon \Delta_p \to Y$ is called *a C^1-map* if it is differentiable in the interior $\mathrm{int}(\Delta_p)$ and in each local chart of Y all directional derivatives $x \mapsto df(x)(v)$ of f extend continuously to the boundary $\partial\Delta_p$ of Δ_p. For $k \geq 2$ we call f a C^k-map if it is C^1 and all maps $x \mapsto df(x)(v)$ are C^{k-1}. We say that f is *smooth* if f is C^k for every $k \in \mathbb{N}$. We write $C^\infty(\Delta_p, Y)$ for the set of smooth maps $\Delta_p \to Y$.

If Σ is a simplicial complex, then we call a map $f\colon \Sigma \to Y$ *piecewise smooth* if it is continuous and its restrictions to all simplices in Σ are smooth. We write $C^\infty_{pw}(\Sigma, Y)$ for the set of piecewise smooth maps $\Sigma \to Y$. There is a natural topology on this space inherited from the natural embedding of $C^\infty_{pw}(\Sigma, Y)$ into the space $\prod_{S\subseteq\Sigma} C^\infty_{pw}(S, Y)$, where S runs through all simplices of Σ and the topology on $C^\infty_{pw}(S, Y)$ is defined as in [Ne02, Def. A.3.5] as the topology of uniform convergence of all directional derivatives of arbitrarily high order. ■

The equivariant form ω^{eq} is a closed $\mathfrak{a}$-valued 2-form on G, and we obtain with [Ne02, Lemma 5.7] a period map

$$\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{a}$$

which is given on piecewise smooth representatives $\sigma\colon \mathbb{S}^2 \to G$ of free homotopy classes by the integral

$$\mathrm{per}_\omega([\sigma]) = \int_{\mathbb{S}^2} \sigma^*\omega^{\mathrm{eq}} = \int_\sigma \omega^{\mathrm{eq}}.$$

If ω is a coboundary, then Lemma B.5 implies that ω^{eq} is exact, so that the period map is trivial by Stoke's Theorem. We therefore obtain a homomorphism

$$H^2_c(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Hom}(\pi_2(G), \mathfrak{a}), \quad [\omega] \mapsto \mathrm{per}_\omega.$$

The image $\Pi_\omega := \mathrm{per}_\omega(\pi_2(G))$ is called the *period group* of ω. Since the group G is connected, the group $\pi_0(C^\infty(\mathbb{S}^2, G))$ of connected components of the Lie group $C^\infty(\mathbb{S}^2, G)$ is isomorphic to $\pi_2(G)$, and we may think of per_ω as the map on $\pi_2(G)$ obtained by factorization of the locally constant map

$$C^\infty(\mathbb{S}^2, G) \to \mathfrak{a}, \quad \sigma \mapsto \int_\sigma \omega^{\mathrm{eq}}$$

to $\pi_0(C^\infty(\mathbb{S}^2, G)) \cong \pi_2(G)$ ([Ne02, Lemma 5.7]).

Lemma 4.2. *The image of the period map is fixed pointwise by G, i.e., $\Pi_\omega \subseteq \mathfrak{a}^G$.*

Proof. In view of [Ne02, Th. A.3.7], each homotopy class in $\pi_2(G)$ has a smooth representative $\sigma\colon \mathbb{S}^2 \to G$. Since G is connected, and the map $G \to C^\infty(\mathbb{S}^2, G), g \mapsto \lambda_g \circ \sigma$ is continuous, we have for each $g \in G$:

$$\mathrm{per}_\omega([\sigma]) = \int_{\mathbb{S}^2} \sigma^* \omega^{\mathrm{eq}} = \int_{\mathbb{S}^2} \sigma^* \lambda_g^* \omega^{\mathrm{eq}} = \int_{\mathbb{S}^2} \sigma^*(\rho_{\mathfrak{a}}(g) \circ \omega^{\mathrm{eq}}) = \rho_{\mathfrak{a}}(g).\mathrm{per}_\omega([\sigma]).$$

We conclude that the image of per_ω is fixed pointwise by G. ■

Let $A \hookrightarrow \widehat{G} \xrightarrow{q} G$ be an abelian Lie group extension of A. Then the Lie algebra $\widehat{\mathfrak{g}}$ of $\widehat{G}$ has the form $\mathfrak{a} \oplus_\omega \mathfrak{g}$ because the existence of a smooth local section implies that $\widehat{\mathfrak{g}} \to \mathfrak{g}$ has a continuous linear section (Proposition 1.3). In this subsection we show that the period homomorphism per_ω coincides up to sign with the connecting homomorphism $\delta\colon \pi_2(G) \to \pi_1(A)$ from the long exact homotopy sequence of the bundle $A \hookrightarrow \widehat{G} \xrightarrow{q} G$.

Definition 4.3. We recall the definition of *relative homotopy groups.* Let $I^n := [0,1]^n$ denote the n-dimensional cube. Then the boundary ∂I^n of I^n can be written as $I^{n-1} \cup J^{n-1}$, where I^{n-1} is called the *initial face* and J^{n-1} is the union of all other faces of I^n.

Let X be a topological space, $Y \subseteq X$ a subspace, and $x_0 \in Y$. A *map*

$$f\colon (I^n, I^{n-1}, J^{n-1}) \to (X, Y, x_0)$$

of space triples is a continuous map $f\colon I^n \to X$ satisfying $f(I^{n-1}) \subseteq Y$ and $f(J^{n-1}) = \{x_0\}$. We write $F^n(X, Y, x_0)$ for the set of all such maps and $\pi_n(X, Y, x_0)$ for the homotopy classes of such maps, i.e., the arc-components of the topological space $F^n(X, Y, x_0)$ endowed with the compact open topology (cf. [Ste51]). We define $F^n(X, x_0) := F^n(X, \{x_0\}, x_0)$ and $\pi_n(X, x_0) := \pi_n(X, \{x_0\}, x_0)$ and observe that we have a canonical map

$$\partial\colon \pi_n(X, Y, x_0) \to \pi_{n-1}(Y, x_0), \quad [f] \mapsto [f|_{I^{n-1}}].$$ ■

Remark 4.4. Let $q\colon P \to M$ be a (locally trivial) H-principal bundle, $y_0 \in P$ a base point, $x_0 := q(y_0)$, and identity H with the fiber $q^{-1}(x_0)$. Then the maps

$$q_*\colon \pi_k(P, H) := \pi_k(P, H, y_0) \to \pi_k(M) := \pi_k(M, x_0), \quad [f] \mapsto [q \circ f]$$

are isomorphisms ([Ste51, Cor. 17.2]), so that we obtain connecting homomorphisms

$$\delta := \partial \circ (q_*)^{-1}\colon \pi_k(M) \to \pi_{k-1}(H).$$

The so obtained sequence

$$\ldots \to \pi_k(P) \to \pi_k(M) \to \pi_{k-1}(H) \to \ldots \to \pi_1(P) \to \pi_1(M) \to \pi_0(H) \to \pi_0(P) \twoheadrightarrow \pi_0(M)$$

is exact, where the last two maps cannot be considered as group homomorphisms. This is the *long exact homotopy sequence of the principal bundle* $P \to M$. ■

Proposition 4.5. *Let* $q\colon \widehat{G} \to G$ *be an abelian extension of not necessarily connected Lie groups with kernel* A *satisfying* $A_0 \cong \mathfrak{a}/\Gamma_A$, *where* $\mathfrak{a}$ *is a Mackey complete locally convex space. Then* q *defines in particular the structure of an* A*-principal bundle on* $\widehat{G}$. *If* $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$ *is a Lie algebra* 2*-cocycle with* $\widehat{\mathfrak{g}} \cong \mathfrak{a} \oplus_\omega \mathfrak{g}$, *then* $\delta\colon \pi_2(G) \to \pi_1(A)$ *and the period map* $\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{a}$ *are related by*

$$\delta = -\,\mathrm{per}_\omega\colon \pi_2(G) \to \pi_1(A) \subseteq \mathfrak{a}.$$

Proof. We consider the action of $\widehat{G}$ on A given by $g.a := q(g).a$. Then $q^*\omega^{\mathrm{eq}}$ is an equivariant closed 2-form on $\widehat{G}$ with $(q^*\omega^{\mathrm{eq}})_{\mathbf{1}} = \mathbf{L}(q)^*\omega$. Let $p_{\mathfrak{a}}\colon \widehat{\mathfrak{g}} \cong \mathfrak{a} \oplus_\omega \mathfrak{g} \to \mathfrak{a}, (a,x) \mapsto x$ denote the projection onto $\mathfrak{a}$. Then

$$\begin{aligned} d_{\mathfrak{g}} p_{\mathfrak{a}}((a,x),(a',x')) &= (a,x).p_{\mathfrak{a}}(a',x') - (a',x').p_{\mathfrak{a}}(a,x) - p_{\mathfrak{a}}([(a,x),(a',x')]) \\ &= x.a' - x'.a - (x.a' - x'.a + \omega(x,x')) = -\omega(x,x') \\ &= -(\mathbf{L}(q)^*\omega)((a,x),(a',x')). \end{aligned}$$

In view of Lemma B.5, this implies

$$d(p_{\mathfrak{a}}^{\mathrm{eq}}) = (d_{\mathfrak{g}} p_{\mathfrak{a}})^{\mathrm{eq}} = -(\mathbf{L}(q)^*\omega)^{\mathrm{eq}} = -q^*\omega^{\mathrm{eq}}.$$

We conclude $\Theta := p_{\mathfrak{a}}^{\mathrm{eq}} \in \Omega^1(\widehat{G}, \mathfrak{a})$ is a 1-form for which $\Theta|_A$ is the Maurer-Cartan form on A. Therefore [Ne02, Prop. 5.11] and $q^*\omega^{\mathrm{eq}} = -d\Theta$ imply that $\delta = -\,\mathrm{per}_\Omega$. ■

Remark 4.6. Let $A \hookrightarrow \widehat{G} \twoheadrightarrow G$ be an abelian extension of connected Lie groups and assume that $A \cong \mathfrak{a}/\Gamma_A$ holds for a discrete subgroup $\Gamma_A \subseteq \mathfrak{a}$ that we identify with $\pi_1(A)$. In view of $\pi_2(A) \cong \pi_2(\mathfrak{a}) = \mathbf{0}$, the long exact homotopy sequence of the bundle $\widehat{G} \to G$ leads to an exact sequence

$$\mathbf{0} \to \pi_2(\widehat{G}) \hookrightarrow \pi_2(G) \xrightarrow{\mathrm{per}_\omega} \pi_1(A) \to \pi_1(\widehat{G}) \twoheadrightarrow \pi_1(G) \to \mathbf{0}.$$

This implies that

$$\pi_2(\widehat{G}) \cong \ker \mathrm{per}_\omega \subseteq \pi_2(G) \quad \text{and} \quad \pi_1(G) \cong \pi_1(\widehat{G})/\operatorname{coker} \mathrm{per}_\omega.$$

These relations show how the period homomorphism controls how the first two homotopy groups of G and $\widehat{G}$ are related. ■

5. From Lie algebra cocycles to group cocycles

In Sections V and VI we describe the image of the map

$$D := D_2 \colon H^2_s(G,A) \to H^2_c(\mathfrak{g},\mathfrak{a}), \quad [f] \mapsto [Df], \quad Df(x,y) = d^2f(\mathbf{1},\mathbf{1})(x,y) - d^2f(\mathbf{1},\mathbf{1})(y,x)$$

for a connected Lie group G and an abelian Lie group A of the form $\mathfrak{a}/\Gamma_A$. In the present section we deal with the special case where, in addition, G is simply connected.

Let G be a connected simply connected Lie group and $\mathfrak{a}$ a Mackey complete locally convex smooth G-module. Further let $\Gamma_A \subseteq \mathfrak{a}^G$ be a subgroup and write $A := \mathfrak{a}/\Gamma_A$ for the quotient group, that carries a natural G-module structure. We write $q_A \colon \mathfrak{a} \to A$ for the quotient map. If, in addition, Γ_A is discrete, then A carries a natural Lie group structure and the action of G on A is smooth, but we won't make this assumption a priori.

Let $\omega \in Z^2_c(\mathfrak{g},\mathfrak{a})$ and $\Pi_\omega \subseteq \mathfrak{a}^G$ be the corresponding period group (Lemma 4.2). In the following we shall assume that

$$\Pi_\omega \subseteq \Gamma_A.$$

The main result of the present section is the existence of a locally smooth group cocycle $f \in Z^2_s(G,A)$ with $Df = \omega$ if Γ_A is discrete (Corollary 5.3).

A special case of the following construction has also been used in [Ne02] in the context of central extensions. For $g \in G$ we choose a smooth path $\alpha_{\mathbf{1},g} \colon [0,1] \to G$ from $\mathbf{1}$ to g. We thus obtain a left invariant system of smooth arcs $\alpha_{g,h} := \lambda_g \circ \alpha_{\mathbf{1},g^{-1}h}$ from g to h, where $\lambda_g(x) = gx$ denotes left translation. For $g,h,u \in G$ we then obtain a singular smooth cycle

$$\alpha_{g,h,u} := \alpha_{g,h} + \alpha_{h,u} - \alpha_{g,u},$$

that corresponds to the piecewise smooth map $\alpha_{g,h,u} \in C^\infty_{pw}(\partial\Delta_2, G)$ with

$$\alpha_{g,h,u}(s,t) = \begin{cases} \alpha_{g,h}(s), & \text{for } t = 0 \\ \alpha_{h,u}(1-s), & \text{for } s+t = 1 \\ \alpha_{g,u}(t), & \text{for } s = 0. \end{cases}$$

For a simplicial complex Σ we write $\Sigma_{(j)}$ for the j-th *barycentric subdivision of* Σ. According to [Ne02, Prop. 5.6], each map $\alpha_{g,h,u}$ is the restriction of a piecewise smooth map $\sigma \colon (\Delta_2)_{(1)} \to G$. Let $\sigma' \colon (\Delta_2)_{(1)} \to G$ be another piecewise smooth map with the same boundary values as σ. We claim that $\int_\sigma \omega^{\mathrm{eq}} - \int_{\sigma'} \omega^{\mathrm{eq}} \in \Pi_\omega$. In fact, we consider the sphere $\mathbb{S}^2$ as an oriented simplicial complex Σ obtained by gluing two copies D and D' of Δ_2 along their boundary, where the inclusion of D is orientation preserving and the inclusion on D' reverses orientation. Then σ and σ' combine to a piecewise smooth map $\gamma\colon \Sigma \to G$ with $\gamma|_D = \sigma$ and $\gamma|_{D'} = \sigma'$, and we get with [Ne02, Lemma 5.7]

$$\int_\sigma \omega^{\rm eq} - \int_{\sigma'} \omega^{\rm eq} = \int_\gamma \omega^{\rm eq} \in \Pi_\omega \subseteq \Gamma_A.$$

We thus obtain a well-defined map

$$F\colon G^3 \to A, \quad (g,h,u) \mapsto q_A\Big(\int_\sigma \omega^{\rm eq}\Big),$$

where $\sigma \in C^\infty_{pw}((\Delta_2)_{(1)}, G)$ is a piecewise smooth map whose boundary values coincide with $\alpha_{g,h,u}$.

Lemma 5.1. *The function*

$$f\colon G^2 \to A, \quad (g,h) \mapsto F(\mathbf{1}, g, gh)$$

is a group cocycle with respect to the action of G on A.

Proof. First we show that for $g, h \in G$ we have

$$f(g,\mathbf{1}) = F(\mathbf{1},g,g) = 0 \quad \text{and} \quad f(\mathbf{1},h) = F(\mathbf{1},\mathbf{1},h) = 0.$$

If $g = h$ or $h = u$, then we can choose the map $\sigma\colon \Delta_2 \to G$ extending $\alpha_{g,h,u}$ in such a way that $\mathrm{rk}(d\sigma) \leq 1$ in every point, so that $\sigma^*\omega^{\rm eq} = 0$. In particular, we obtain $F(g,h,u) = 0$ in these cases.

From $\alpha_{g,h,u} = \lambda_g \circ \alpha_{\mathbf{1}, g^{-1}h, g^{-1}u}$ we see that for every extensions $\sigma\colon (\Delta_2)_{(1)} \to G$ of $\alpha_{\mathbf{1},g^{-1}h,g^{-1}u}$, the map $\lambda_g \circ \sigma$ is an extension of $\alpha_{g,h,u}$. In view of $\lambda_g^*\omega^{\rm eq} = \rho_{\mathfrak a}(g) \circ \omega^{\rm eq}$, we obtain

$$\int_{\mathbb{S}^2} (\lambda_g \circ \sigma)^*\omega^{\rm eq} = \int_{\mathbb{S}^2} \sigma^*\lambda_g^*\omega^{\rm eq} = \rho_{\mathfrak a}(g).\int_{\mathbb{S}^2} \sigma^*\omega^{\rm eq},$$

and therefore

$$F(g,h,u) = \rho_A(g).F(\mathbf{1}, g^{-1}h, g^{-1}u). \tag{5.1}$$

Let $\Delta_3 \subseteq \mathbb{R}^3$ be the standard 3-simplex. Then we define a piecewise smooth map γ of its 1-skeleton to G by

$$\gamma(t,0,0) = \alpha_{\mathbf{1},g}(t), \quad \gamma(0,t,0) = \alpha_{\mathbf{1},gh}(t), \quad \gamma(0,0,t) = \alpha_{\mathbf{1},ghu}(t)$$

and

$$\gamma(1-t,t,0) = \alpha_{g,gh}(t), \quad \gamma(0,1-t,t) = \alpha_{gh,ghu}(t), \quad \gamma(1-t,0,t) = \alpha_{g,ghu}(t).$$

As G is simply connected, we obtain with [Ne02, Prop. 5.6] for each face Δ_3^j, $j = 0,\ldots,3$, of Δ_3 a piecewise smooth map γ_j of the first barycentric subdivision to G, extending the given map on the 1-skeleton. These maps combine to a piecewise smooth map $\gamma\colon (\partial\Delta_3)_{(1)} \to G$. Modulo the period group Π_ω, we now have

$$\begin{aligned}\int_\gamma \omega^{\rm eq} &= \int_{\partial\Delta_3} \gamma^*\omega^{\rm eq} = \sum_{i=0}^3 \int_{\gamma_i} \omega^{\rm eq}\\ &= F(g, gh, ghu) - F(\mathbf{1}, gh, ghu) + F(\mathbf{1}, g, ghu) - F(\mathbf{1}, g, gh)\\ &= \rho_A(g).f(h,u) - f(gh,u) + f(g,hu) - f(g,h).\end{aligned}$$

Since $\int_\gamma \omega^{\rm eq} \in \Pi_\omega$, this proves that f is a group cocycle. ∎

In the next lemma we show that for an appropriate choice of paths from $\mathbf{1}$ to group elements close to $\mathbf{1}$ the cocycle f will be smooth in an identity neighborhood. The following lemma is a slight generalization of Lemma 6.2 in [Ne02].

Lemma 5.2. *Let $U \subseteq \mathfrak{g}$ be an open convex 0-neighborhood and $\varphi\colon U \to G$ a chart of G with $\varphi(0) = \mathbf{1}$ and $d\varphi(0) = \mathrm{id}_{\mathfrak{g}}$. We define the arcs $\alpha_{\varphi(x)}(t) := \varphi(tx)$. Let $V \subseteq U$ be an open convex 0-neighborhood with $\varphi(V)\varphi(V) \subseteq \varphi(U)$ and define $x * y := \varphi^{-1}(\varphi(x)\varphi(y))$ for $x, y \in V$. If we define $\sigma_{x,y} := \varphi \circ \gamma_{x,y}$ with*

$$\gamma_{x,y}\colon \Delta_2 \to U, \quad (t,s) \mapsto t(x * sy) + s(x * (1-t)y),$$

then for any closed 2-form $\Omega \in \Omega^2(G, \mathfrak{a})$, $\mathfrak{a}$ a Mackey complete locally convex space, the function

$$f_V\colon V \times V \to \mathfrak{a}, \quad (x,y) \mapsto \int_{\sigma_{x,y}} \Omega$$

is smooth with $d^2 f_V(0,0)(x,y) = \frac{1}{2}\Omega_{\mathbf{1}}(x,y)$ (see the end of Appendix B for the notation).

Proof. First we note that the function $V \times V \to U, (x,y) \mapsto x * y$ is smooth. We consider the cycle

$$\alpha_{\mathbf{1},\varphi(x),\varphi(x)\varphi(y)} = \alpha_{\mathbf{1},\varphi(x),\varphi(x*y)} = \alpha_{\mathbf{1},\varphi(x)} + \alpha_{\varphi(x),\varphi(x*y)} - \alpha_{\mathbf{1},\varphi(x*y)}.$$

The arc connecting x to $x * y$ is given by $s \mapsto x * sy$, so that we may define $\sigma_{x,y} := \varphi \circ \gamma_{x,y}$ with $\gamma_{x,y}$ as above. Then

$$f_V\colon V \times V \to \mathfrak{a}, \quad (x,y) \mapsto \int_{\varphi\circ\gamma_{x,y}} \Omega = \int_{\Delta_2} \gamma_{x,y}^*\varphi^*\Omega,$$

and

$$f_V(x,y) = \int_{\Delta_2} (\varphi^*\Omega)\big(\varphi(\gamma_{x,y}(t,s))\big)\Big(\frac{\partial}{\partial t}\gamma_{x,y}(t,s), \frac{\partial}{\partial s}\gamma_{x,y}(t,s)\Big)\, dt\, ds \tag{5.2}$$

implies that f_V is a smooth function in $V \times V$.

The map $\gamma\colon (x,y) \mapsto \gamma_{x,y}$ satisfies

(1) $\gamma_{0,y}(t,s) = sy$ and $\gamma_{x,0}(t,s) = (t+s)x$.

(2) $\frac{\partial}{\partial t}\gamma_{x,y} \wedge \frac{\partial}{\partial s}\gamma_{x,y} = 0$ for $x = 0$ or $y = 0$.

In particular we obtain $f_V(x,0) = f_V(0,y) = 0$. Therefore the second order Taylor polynomial

$$T_2(f_V)(x,y) = f_V(0,0) + df_V(0,0)(x,0) + df_V(0,0)(0,y) + \frac{1}{2}d^{[2]}f_V(0,0)\big((x,y),(x,y)\big)$$

of f_V in $(0,0)$ is bilinear and given by

$$\begin{aligned} T_2(f_V)(x,y) &= \frac{1}{2}d^{[2]}f_V(0,0)\big((x,0),(0,y)\big) + \frac{1}{2}d^{[2]}f_V(0,0)\big((0,y),(x,0)\big) \\ &= d^2 f_V(0,0)(x,y) \end{aligned}$$

(see the end of Appendix B).

Next we observe that (1) implies that $\frac{\partial}{\partial t}\gamma_{x,y}$ and $\frac{\partial}{\partial s}\gamma_{x,y}$ vanish in $(0,0)$. Therefore the chain rule for Taylor expansions and (1) imply that for each pair (t,s) the second order term of

$$(\varphi^*\Omega)(\gamma_{x,y}(t,s))\Big(\frac{\partial}{\partial t}\gamma_{x,y}(t,s), \frac{\partial}{\partial s}\gamma_{x,y}(t,s)\Big)$$

is given by

$$(\varphi^*\Omega)(\gamma_{0,0}(t,s))(x,y) = (d\varphi(0)^*\Omega_{\mathbf{1}})(x,y) = \Omega_{\mathbf{1}}(x,y),$$

and eventually

$$d^2 f_V(0,0)(x,y) = T_2(f_V)(x,y) = \int_{\Delta_2} dt\,ds \cdot \Omega_{\mathbf{1}}(x,y) = \frac{1}{2}\Omega_{\mathbf{1}}(x,y). \qquad \blacksquare$$

Corollary 5.3. *Suppose that Γ_A is discrete with $\Pi_\omega \subseteq \Gamma_A$ and construct for $\omega \in Z^2_c(\mathfrak{g},\mathfrak{a})$ the group cocycle $f \in Z^2(G,A)$ as above from the closed 2-form $\omega^{\rm eq} \in \Omega^2(G,\mathfrak{a})$. If the paths $\alpha_{\mathbf{1},g}$ for $g \in \varphi(U)$ are chosen as in* Lemma 5.2*, then $f \in Z^2_s(G,A)$ with $D(f) = \omega$.*

Proof. In the notation of Lemma 5.2 we have for $x, y \in V$ the relation

$$f(\varphi(x),\varphi(y)) = q_A(f_V(x,y)),$$

so that f is smooth on $\varphi(V) \times \varphi(V)$, and further

$$Df(x,y) = d^2 f_V(\mathbf{1},\mathbf{1})(x,y) - d^2 f_V(\mathbf{1},\mathbf{1})(y,x) = \omega(x,y). \qquad \blacksquare$$

The outcome of this section is the following result:

Theorem 5.4. *Let G be a connected simply connected Lie group and A a smooth G-module of the form $\mathfrak{a}/\Gamma_A$, where $\Gamma_A \subseteq \mathfrak{a}$ is a discrete subgroup of the Mackey complete space $\mathfrak{a}$. Let $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$ be a continuous 2-cocycle and $\Pi_\omega \subseteq \mathfrak{a}^G$ its period group. Then the following assertions are equivalent:*

(1) *The Lie algebra extension $\mathfrak{a} \hookrightarrow \widehat{\mathfrak{g}} := \mathfrak{a} \oplus_\omega \mathfrak{g} \twoheadrightarrow \mathfrak{g}$ can be integrated to a Lie group extension $A \hookrightarrow \widehat{G} \twoheadrightarrow G$.*
(2) $[\omega] \in \operatorname{im}(D)$.
(3) $\omega \in \operatorname{im}(D)$.
(4) $\Pi_\omega \subseteq \Gamma_A$.
(5) *If $q_A \colon \mathfrak{a} \to A$ is the quotient map, then $q_A \circ \operatorname{per}_\omega = 0$.*

Proof. (1) $\Rightarrow$ (2): If $\widehat{G}$ is an extension of G by A corresponding to the Lie algebra extension $\widehat{\mathfrak{g}} = \mathfrak{a} \oplus_\omega \mathfrak{g}$, then we can write $\widehat{G}$ as $A \times_f G$ (Proposition 2.6), and Lemma 2.7 implies that $D[f] = [Df] = [\omega]$.

(2) $\Rightarrow$ (3): If $[\omega] = D[f] = [Df]$ for some $f \in Z^2_s(G, A)$, then there exists an $\alpha \in C^1_c(\mathfrak{g}, \mathfrak{a})$ with $Df - \omega = d_{\mathfrak{g}}\alpha$. The 2-form $(d_{\mathfrak{g}}\alpha)^{\rm eq} = d\alpha^{\rm eq} \in \Omega^2(G, \mathfrak{a})$ is exact (Lemma B.5), so that its period group is trivial, and Corollary 5.3 implies the existence of $h \in Z^2_s(G, A)$ with $Dh = d_{\mathfrak{g}}\alpha$. Then $f_1 := f - h \in Z^2_s(G, A)$ satisfies $D(f - h) = Df - Dh = \omega$.

(3) $\Rightarrow$ (1): If $Df = \omega$, then the Lie group extension $A \times_f G \to G$ (Proposition 2.6) corresponds to the Lie algebra extension $\mathfrak{a} \oplus_{Df} \mathfrak{g} = \mathfrak{a} \oplus_\omega \mathfrak{g} \to \mathfrak{g}$ (Lemma 2.7).

(1) $\Rightarrow$ (4) follows from Proposition 4.5, which implies that if $\widehat{G}$ exists, then the period map coincides up to sign with the connecting homomorphism $\delta \colon \pi_2(G) \to \pi_1(A) \cong \Gamma_A \subseteq \mathfrak{a}$ in the long exact homotopy sequence of the principal A-bundle $\widehat{G}$.

(4) $\Rightarrow$ (3) follows from Corollary 5.3.

(4) $\Leftrightarrow$ (5) is a trivial consequence of the definitions. ∎

6. Abelian extensions of non-simply connected groups

We have seen in the preceding section that for a simply connected Lie group G and a smooth G-module of the form $A = \mathfrak{a}/\Gamma_A$ the image of the map $D \colon H^2_s(G, A) \to H^2_c(\mathfrak{g}, \mathfrak{a})$ consists of the classes $[\omega]$ of those cocycles $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$ for which $\Pi_\omega \subseteq \Gamma_A$.

In this section we drop the assumption that G is simply connected. We write $q_G \colon \widetilde{G} \to G$ for the simply connected covering group of G and identify $\pi_1(G)$ with the discrete central subgroup $\ker q_G$ of $\widetilde{G}$.

Let $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$. In the following we write ρ_A for the action of G on A, $\rho_{\mathfrak{a}}$ for the action of G on $\mathfrak{a}$ and $\dot\rho_{\mathfrak{a}}$ for the derived representation of $\mathfrak{g}$ on $\mathfrak{a}$.

Remark 6.1. (a) To a 2-cocycle $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$ we associate the linear map

$$\widetilde{f}_\omega \colon \mathfrak{g} \to C^1_c(\mathfrak{g}, \mathfrak{a}) = \mathrm{Lin}(\mathfrak{g}, \mathfrak{a}), \quad x \mapsto i_x\omega.$$

We consider $\mathrm{Lin}(\mathfrak{g}, \mathfrak{a})$ as a $\mathfrak{g}$-module with respect to the action

$$(x.\alpha)(y) := \dot{\rho}_{\mathfrak{a}}(x).\alpha(y) - \alpha([x, y]).$$

We do not consider any topology on this space of maps. The corresponding Lie algebra differential $d_{\mathfrak{g}} \colon C^1(\mathfrak{g}, \mathrm{Lin}(\mathfrak{g}, \mathfrak{a})) \to C^2(\mathfrak{g}, \mathrm{Lin}(\mathfrak{g}, \mathfrak{a}))$ then satisfies

$$\begin{aligned}(d_{\mathfrak{g}}\widetilde{f}_\omega)(x, y)(z) &= (x.i_y\omega - y.i_x\omega - i_{[x,y]}\omega)(z)\\ &= x.\omega(y, z) - \omega(y, [x, z]) - y.\omega(x, z) + \omega(x, [y, z]) - \omega([x, y], z)\\ &= -z.\omega(x, y) = -d_{\mathfrak{g}}(\omega(x, y))(z).\end{aligned}$$

Since the subspace $B^1_c(\mathfrak{g}, \mathfrak{a}) = d_{\mathfrak{g}}\mathfrak{a} \subseteq C^1_c(\mathfrak{g}, \mathfrak{a})$ is $\mathfrak{g}$-invariant, we can also form the quotient $\mathfrak{g}$-module

$$\widehat{H}^1_c(\mathfrak{g}, \mathfrak{a}) := C^1_c(\mathfrak{g}, \mathfrak{a})/B^1_c(\mathfrak{g}, \mathfrak{a}).$$

We then obtain a linear map

$$f_\omega \colon \mathfrak{g} \to \widehat{H}^1_c(\mathfrak{g}, \mathfrak{a}), \quad x \mapsto [i_x\omega],$$

and the preceding calculation shows that this map is a 1-cocycle. We call f_ω the *infinitesimal flux cocycle*. In the following we are concerned with integrating this cocycle to a group cocycle

$$F_\omega \colon \widetilde{G} \to \widehat{H}^1_c(\mathfrak{g}, \mathfrak{a}).$$

This is problematic because the right hand side does not carry a natural topology, so that we cannot directly apply Proposition 3.4.

(b) The injective map

$$\mathrm{Eq} \colon C^p(\mathfrak{g}, \mathfrak{a}) \to \Omega^p(G, \mathfrak{a}), \quad \alpha \mapsto \alpha^{\mathrm{eq}}$$

satisfies with respect to the natural action of G on $C^p(\mathfrak{g}, \mathfrak{a})$ by

$$(g.\alpha)(x_1, \ldots, x_p) := g.\alpha(\mathrm{Ad}(g)^{-1}.x_1, \ldots, \mathrm{Ad}(g)^{-1}.x_p)$$

for $h \in G$ and $y \in \mathfrak{g} = T_{\mathbf{1}}(G)$ the relation

$$\begin{aligned}(g.\alpha)^{\mathrm{eq}}(h.y_1, \ldots, h.y_p) &= h.((g.\alpha)(y_1, \ldots, y_p)) = hg.\alpha(\mathrm{Ad}(g)^{-1}.y_1, \ldots, \mathrm{Ad}(g)^{-1}.y_p)\\ &= \alpha^{\mathrm{eq}}(h.y_1.g, \ldots, h.y_p.g) = (\rho_g^*\alpha^{\mathrm{eq}})(h.y_1, \ldots, h.y_p).\end{aligned}$$

This means that Eq is equivariant with respect to the action of G on $\Omega^\bullet(G, \mathfrak{a})$ by $g.\alpha := \rho_g^*\alpha$. The corresponding derived action of the Lie algebra $\mathfrak{g}$ on $\Omega^\bullet(G, \mathfrak{a})$ is given by $X.\alpha := \mathcal{L}_{x_l}.\alpha$, where $\mathcal{L}_{x_l} = d \circ i_{x_l} + i_{x_l} \circ d$ denotes the Lie derivative as an operator on differential forms.

For the linear map

$$\mathrm{Eq} \circ \widetilde{f}_\omega \colon \mathfrak{g} \to \Omega^1(G, \mathfrak{a}), \quad x \mapsto (i_x\omega)^{\mathrm{eq}}$$

we obtain for $h \in G$ and $y \in \mathfrak{g}$:

$$(i_x\omega)^{\mathrm{eq}}(h.y) = h.(i_x\omega(y)) = h.\omega(x,y) = \omega^{\mathrm{eq}}(h.x, h.y) = \omega^{\mathrm{eq}}(x_l(h), h.y),$$

which means that $(i_x\omega)^{\mathrm{eq}} = i_{x_l}\omega^{\mathrm{eq}}$. With respect to the natural action of $\mathfrak{g}$ on $\Omega^1(G,\mathfrak{a})$ by $\mathcal{L}_{x_l}$ we then obtain a Lie algebra cocycle

$$\widehat{f}_\omega := \mathrm{Eq} \circ f_\omega \colon \mathfrak{g} \to \widehat{H}^1_{\mathrm{dR}}(G,\mathfrak{a}) := \Omega^1(G,\mathfrak{a})/dC^\infty(G,\mathfrak{a}), \quad x \mapsto [i_{x_l}\omega^{\mathrm{eq}}]$$

(cf. Lemma 9.8).

(c) Next we derive some formulas that will be useful in the following. The equivariance of ω^{eq} leads to

$$\mathcal{L}_{x_r}.\omega^{\mathrm{eq}} = \dot\rho_{\mathfrak{a}}(x) \circ \omega^{\mathrm{eq}}$$

([Ne02, Lemma A.2.4]). In view of the closedness of ω^{eq}, this leads to

$$d(i_{x_r}\omega^{\mathrm{eq}}) = \mathcal{L}_{x_r}.\omega^{\mathrm{eq}} - i_{x_r}d\omega^{\mathrm{eq}} = \dot\rho_{\mathfrak{a}}(x) \circ \omega^{\mathrm{eq}}. \tag{6.1}$$

Further $[\mathcal{L}_{x_r}, i_{y_r}] = i_{[x_r,y_r]} = -i_{[x,y]_r}$ implies

$$\begin{aligned} i_{[x,y]_r}\omega^{\mathrm{eq}} &= i_{y_r}\mathcal{L}_{x_r}\omega^{\mathrm{eq}} - \mathcal{L}_{x_r}i_{y_r}\omega^{\mathrm{eq}} = i_{y_r}(\dot\rho_{\mathfrak{a}}(x)\circ\omega^{\mathrm{eq}}) - (i_{x_r}\circ d + d \circ i_{x_r})i_{y_r}\omega^{\mathrm{eq}} \\ &= \dot\rho_{\mathfrak{a}}(x) \circ i_{y_r}\omega^{\mathrm{eq}} - \dot\rho_{\mathfrak{a}}(y) \circ i_{x_r}\omega^{\mathrm{eq}} - d(i_{x_r}i_{y_r}\omega^{\mathrm{eq}}). \end{aligned}$$

This means that the $\mathfrak{a}$-valued 1-form

$$\dot\rho_{\mathfrak{a}}(x) \circ i_{y_r}\omega^{\mathrm{eq}} - \dot\rho_{\mathfrak{a}}(y) \circ i_{x_r}\omega^{\mathrm{eq}} - i_{[x,y]_r}\omega^{\mathrm{eq}} = d(i_{x_r}i_{y_r}\omega^{\mathrm{eq}}) \tag{6.2}$$

is exact. ■

A first step to integrate the Lie algebra cocycle f_ω is to translate matters from the Lie algebra to vector fields and differential forms on G. On the formal level, without worrying about topologies on the target space, the linear map $\widetilde{f}_\omega \colon \mathfrak{g} \to \mathrm{Lin}(\mathfrak{g},\mathfrak{a}), x \mapsto i_x\omega$ defines an equivariant $\mathrm{Lin}(\mathfrak{g},\mathfrak{a})$-valued 1-form $\widetilde{f}_\omega^{\mathrm{eq}}$ on G. For each $x \in \mathfrak{g}$, evaluation ev_x in x is a linear map $\mathrm{ev}_x \colon \mathrm{Lin}(\mathfrak{g},\mathfrak{a}) \to \mathfrak{a}, \alpha \mapsto \alpha(x)$ and $\mathrm{ev}_x \circ \widetilde{f}_\omega^{\mathrm{eq}} \in \Omega^1(G,\mathfrak{a})$ is an $\mathfrak{a}$-valued smooth 1-form on G, a well-defined object, satisfying for $g \in G$ and $y \in \mathfrak{g}$:

$$\begin{aligned} (\mathrm{ev}_x \circ \widetilde{f}_\omega^{\mathrm{eq}})(g.y) = \mathrm{ev}_x(g.\widetilde{f}_\omega(y)) = g.\widetilde{f}_\omega(y)(\mathrm{Ad}(g)^{-1}.x) = g.\omega(y, g^{-1}.x.g) \\ = \omega^{\mathrm{eq}}(g.y, x_r(g)), \end{aligned}$$

which leads to

$$\mathrm{ev}_x \circ \widetilde{f}_\omega^{\mathrm{eq}} = -i_{x_r}\omega^{\mathrm{eq}}.$$

Having this formula in mind, the definition in Lemma 6.2 below is natural.

Lemma 6.2. *Let $\gamma\colon [0,1] \to G$ be a piecewise smooth path. Then we obtain a continuous linear map*

$$\widetilde{F}_\omega(\gamma) \in \mathrm{Lin}(\mathfrak{g}, \mathfrak{a}), \quad \widetilde{F}_\omega(\gamma)(x) := -\int_\gamma i_{x_r}\omega^{\mathrm{eq}} = \int_0^1 \gamma(t).\omega(\gamma(t)^{-1}\gamma'(t), \mathrm{Ad}(\gamma(t))^{-1}.x)\, dt$$

with the following properties:

(1) *If $\gamma(1)^{-1}\gamma(0)$ is contained in $Z(G)$ and acts trivially on $\mathfrak{a}$, then $\widetilde{F}_\omega(\gamma) \in Z^1_c(\mathfrak{g}, \mathfrak{a})$.*

(2) *If γ_1 and γ_2 are homotopic with fixed endpoints, then $\widetilde{F}_\omega(\gamma_1) - \widetilde{F}_\omega(\gamma_2)$ is a coboundary.*

(3) *For a piecewise smooth curve $\eta\colon [0,1] \to G$ we have*

$$\int_\eta \widetilde{F}_\omega(\gamma)^{\mathrm{eq}} = \int_H \omega^{\mathrm{eq}}$$

for the piecewise smooth map $H\colon [0,1]^2 \to G, (t,s) \mapsto \eta(s)\cdot\gamma(t)$.

(4) *For a differentiable curve $\gamma\colon [0,1] \to G$ with $\gamma(0) = \mathbf{1}$ and $\gamma'(0) = y$ we have pointwise in $\mathrm{Lin}(\mathfrak{g}, \mathfrak{a})$:*

$$\frac{d}{dt}\Big|_{t=0} \widetilde{F}_\omega(\gamma|_{[0,t]}) = i_y\omega = \widetilde{f}_\omega(y).$$

Proof. In view of formula (6.2) above, we find for $x, y \in \mathfrak{g}$ the relation

$$\begin{aligned}
&d_{\mathfrak{g}}(\widetilde{F}_\omega(\gamma))(x,y)\\
&= x.\widetilde{F}_\omega(\gamma)(y) - y.\widetilde{F}_\omega(\gamma)(x) - \widetilde{F}_\omega(\gamma)([x,y])\\
&= -\int_\gamma \dot\rho_{\mathfrak{a}}(x)\circ i_{y_r}\omega^{\mathrm{eq}} - \dot\rho_{\mathfrak{a}}(y)\circ i_{x_r}\omega^{\mathrm{eq}} - i_{[x,y]_r}\omega^{\mathrm{eq}} = -\int_\gamma d(i_{x_r}i_{y_r}\omega^{\mathrm{eq}})\\
&= \omega^{\mathrm{eq}}(\gamma(0))\big(y_r(\gamma(0)), x_r(\gamma(0))\big) - \omega^{\mathrm{eq}}(\gamma(1))\big(y_r(\gamma(1)), x_r(\gamma(1))\big)\\
&= \gamma(0).\omega(\mathrm{Ad}(\gamma(0))^{-1}.y, \mathrm{Ad}(\gamma(0))^{-1}.x) - \gamma(1).\omega(\mathrm{Ad}(\gamma(1))^{-1}.y, \mathrm{Ad}(\gamma(1))^{-1}.x).
\end{aligned}$$

(1) If $\gamma(1)^{-1}\gamma(0) \in Z(G) = \ker\mathrm{Ad}$ acts trivially on $\mathfrak{a}$, then the above formula implies that $d_{\mathfrak{g}}\big(\widetilde{F}_\omega(\gamma)\big) = 0$, i.e., that $\widetilde{F}_\omega(\gamma) \in Z^1_c(\mathfrak{g}, \mathfrak{a})$.

(2) For $g \in G$ we first observe that

$$\begin{aligned}
\widetilde{F}_\omega(g\cdot\gamma)(x) &= -\int_{\lambda_g\circ\gamma} i_{x_r}.\omega^{\mathrm{eq}} = \int_0^1 g\gamma(t).\omega(\gamma(t)^{-1}.\gamma'(t), \mathrm{Ad}(g\gamma(t))^{-1}.x)\, dt\\
&= g.\int_0^1 \gamma(t).\omega(\gamma(t)^{-1}.\gamma'(t), \mathrm{Ad}(\gamma(t))^{-1}\,\mathrm{Ad}(g)^{-1}.x)\, dt\\
&= g.\widetilde{F}_\omega(\gamma)(\mathrm{Ad}(g)^{-1}.x) = (g.\widetilde{F}_\omega(\gamma))(x).
\end{aligned}$$

For the natural action of G on $\mathrm{Lin}(\mathfrak{g},\mathfrak{a})$ by $(g.\varphi)(x) := g.\varphi(\mathrm{Ad}(g)^{-1}.x)$ and the left translation action on the space $C^1_{pw}(I,G)$ of piecewise smooth maps $I := [0,1] \to G$, the preceding calculation implies that the map

$$\widetilde{F}_\omega \colon C^1_{pw}(I,G) \to \mathrm{Lin}(\mathfrak{g},\mathfrak{a}) = C^1_c(\mathfrak{g},\mathfrak{a})$$

is equivariant.

For the composition

$$(\gamma_1\sharp\gamma_2)(t) := \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_1(1)\gamma_2(0)^{-1}\gamma_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

of paths we thus obtain the composition formula

$$\widetilde{F}_\omega(\gamma_1\sharp\gamma_2) = \widetilde{F}_\omega(\gamma_1) + \widetilde{F}_\omega(\gamma_1(1)\gamma_2(0)^{-1}\gamma_2) = \widetilde{F}_\omega(\gamma_1) + \gamma_1(1)\gamma_2(0)^{-1}.\widetilde{F}_\omega(\gamma_2). \tag{6.3}$$

For the inverse path $\gamma^-(t) := \gamma(1-t)$ we trivially get $\widetilde{F}_\omega(\gamma^-) = -\widetilde{F}_\omega(\gamma)$ from the transformation formula for one-dimensional integrals. If the two paths γ_1 and γ_2 have the same start and endpoints, then the path $\gamma_1\sharp\gamma_2^-$ is closed, and we derive with (1) that

$$\widetilde{F}_\omega(\gamma_1) - \widetilde{F}_\omega(\gamma_2) = \widetilde{F}_\omega(\gamma_1) + \gamma_1(1)\gamma_2^-(0)^{-1}.\widetilde{F}_\omega(\gamma_2^-) = \widetilde{F}_\omega(\gamma_1\sharp\gamma_2^-) \in Z^1_c(\mathfrak{g},\mathfrak{a}).$$

That two paths γ_1 and γ_2 with the same endpoints are homotopic with fixed endpoints implies that the loop $\gamma := \gamma_1\sharp\gamma_2^-$ is contractible. It therefore has a closed piecewise smooth lift $\widetilde{\gamma} \colon [0,1] \cong \partial\Delta_2 \to \widetilde{G}$ with $q_G \circ \widetilde{\gamma} = \gamma$. Using Proposition 4.6 in [Ne02], we find a piecewise smooth map $\widetilde{\sigma} \colon \Delta_2 \to \widetilde{G}$ such that $\widetilde{\sigma}|_{\partial\Delta_2} = \widetilde{\gamma}$. Let $\sigma := q_G \circ \widetilde{\sigma}$. Then $\sigma|_{\partial\Delta_2} = \gamma$, so that Stoke's Theorem and formula (6.1) lead to

$$\begin{aligned} -\widetilde{F}_\omega(\gamma)(x) &= \int_\gamma i_{x_r}\omega^{\mathrm{eq}} = \int_{\partial\Delta_2} \sigma^*(i_{x_r}\omega^{\mathrm{eq}}) = \int_{\Delta_2} \sigma^* d(i_{x_r}\omega^{\mathrm{eq}}) \\ &= \int_\sigma d(i_{x_r}\omega^{\mathrm{eq}}) = \int_\sigma \dot\rho_{\mathfrak{a}}(x) \circ \omega^{\mathrm{eq}} = \dot\rho_{\mathfrak{a}}(x).\int_\sigma \omega^{\mathrm{eq}}. \end{aligned}$$

Therefore $\widetilde{F}_\omega(\gamma) \in B^1_c(\mathfrak{g},\mathfrak{a})$, and (2) follows.

(3) We have

$$\begin{aligned} \int_\eta \widetilde{F}_\omega(\gamma)^{\mathrm{eq}} &= \int_0^1 \eta(s).\widetilde{F}_\omega(\gamma)(\eta(s)^{-1}.\eta'(s))\,ds \\ &= \int_0^1\int_0^1 \eta(s)\gamma(t).\omega(\gamma(t)^{-1}.\gamma'(t), \mathrm{Ad}(\gamma(t)^{-1}) \circ \eta(s)^{-1}.\eta'(s))\,dt\,ds \\ &= \int_0^1\int_0^1 H(t,s).\omega(H(t,s)^{-1}\eta(s).\gamma'(t), H(t,s)^{-1}.(\eta'(s).\gamma(t)))\,dt\,ds \\ &= \int_0^1\int_0^1 H(t,s).\omega\Big(H(t,s)^{-1}.\frac{\partial H(t,s)}{\partial t}, H(t,s)^{-1}.\frac{\partial H(t,s)}{\partial s}\Big)\,dt\,ds \end{aligned}$$

$$= \int_{[0,1]^2} H^*\omega^{\mathrm{eq}} = \int_H \omega^{\mathrm{eq}}.$$

(4) For $\eta_t(s) := \gamma(ts)$ we have

$$\begin{aligned}\widetilde{F}_\omega(\gamma|_{[0,t]})(x) &= \int_0^t \gamma(s).\omega(\gamma(s)^{-1}.\gamma'(s), \mathrm{Ad}(\gamma(s))^{-1}.x)\, ds \\ &= \int_0^1 \gamma(st).\omega(\gamma(st)^{-1}.\gamma'(st), \mathrm{Ad}(\gamma(st))^{-1}.x)\, tds = t\widetilde{F}_\omega(\eta_t)(x).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \widetilde{F}_\omega(\gamma|_{[0,t]})(x) &= \lim_{t\to 0} \int_0^1 \gamma(st).\omega(\gamma(st)^{-1}.\gamma'(st), \mathrm{Ad}(\gamma(st))^{-1}.x)\, ds \\ &= \int_0^1 \gamma(0).\omega(\gamma(0)^{-1}.\gamma'(0), \mathrm{Ad}(\gamma(0))^{-1}.x)\, ds \\ &= \omega(y,x) = (i_y\omega)(x).\end{aligned}$$

■

Proposition 6.3. *We have a well-defined map*

$$F_\omega \colon \widetilde{G} \to \widehat{H}^1_c(\mathfrak{g},\mathfrak{a}) = \mathrm{Lin}(\mathfrak{g},\mathfrak{a})/B^1_c(\mathfrak{g},\mathfrak{a}), \quad g \mapsto [\widetilde{F}_\omega(q_G \circ \gamma_g)] := \widetilde{F}_\omega(q_G \circ \gamma_g) + B^1_c(\mathfrak{g},\mathfrak{a}),$$

where $\gamma_g \colon [0,1] \to \widetilde{G}$ is piecewise smooth with $\gamma_g(0) = \mathbf{1}$ and $\gamma_g(1) = g$. The map F_ω is a 1-cocycle with respect to the natural action of $\widetilde{G}$ on $\widehat{H}^1_c(\mathfrak{g},\mathfrak{a})$. Moreover, we obtain by restriction a group homomorphism $Z(\widetilde{G}) \cap \ker \rho_{\mathfrak{a}} \to H^1_c(\mathfrak{g},\mathfrak{a}), [\gamma] \mapsto [\widetilde{F}_\omega(\gamma)]$ and further by restriction to $\pi_1(G)$ a homomorphism

$$F_\omega \colon \pi_1(G) \to H^1_c(\mathfrak{g},\mathfrak{a}).$$

Proof. That F_ω is well-defined follows from Lemma 6.1(2) because two different choices of paths γ_g and η_g lead to paths $q_G \circ \gamma_g$ and $q_G \circ \eta_g$ in G which are homotopic with fixed endpoints. Next we note that for paths γ_{g_i}, $i = 1,2$, from $\mathbf{1}$ to g_i in $\widetilde{G}$ the composed path $\gamma_{g_1} \sharp \gamma_{g_2}$ connects $\mathbf{1}$ to g_1g_2. Hence the composition formula (6.3) leads to

$$F_\omega(g_1g_2) = \widetilde{F}_\omega(\gamma_{g_1}\sharp\gamma_{g_2}) = \widetilde{F}_\omega(\gamma_{g_1}) + g_1.\widetilde{F}_\omega(\gamma_{g_2}) = F_\omega(g_1) + g_1.F_\omega(g_2),$$

showing that the map F_ω is a 1-cocycle.

Since $Z(\widetilde{G}) \cap \ker \rho_{\mathfrak{a}}$ acts trivially on $\mathfrak{g}$ and $\mathfrak{a}$, hence on $\mathrm{Lin}(\mathfrak{g},\mathfrak{a})$, the restriction of F_ω to this subgroup is a group homomorphism, and Lemma 6.2(1) shows that its values lie in the subspace $H^1_c(\mathfrak{g},\mathfrak{a})$ of $\widehat{H}^1_c(\mathfrak{g},\mathfrak{a})$. ■

We call $F_\omega \colon \widetilde{G} \to \widehat{H}^1_c(\mathfrak{g}, \mathfrak{a})$ the *flux cocycle* and its restriction to $\pi_1(G)$ the *flux homomorphism* for reasons that will become clear in Definition 9.9 below. Composing with the map

$$\mathrm{Eq} \colon \widehat{H}^1_c(\mathfrak{g}, \mathfrak{a}) \to \widehat{H}^1_{\mathrm{dR}}(G, \mathfrak{a}),$$

we obtain a group cocycle

$$\widehat{F}_\omega := \mathrm{Eq} \circ F_\omega \colon \widetilde{G} \to \widehat{H}^1_{\mathrm{dR}}(G, \mathfrak{a}), \quad \text{i.e.,} \quad \widehat{F}_\omega(g_1 g_2) = \widehat{F}_\omega(g_1) + \rho^*_{g_1} \widehat{F}_\omega(g_2).$$

Since the elements of the target space are uniquely determined by their integrals over loops, Lemma 6.2(3) completely determines $\widehat{F}_\omega$.

We now relate the flux homomorphism to group extensions. Although the following proposition is quite technical, it contains a lot of interesting information, even for non-connected groups A.

Proposition 6.4. *Let A be an abelian Lie group whose identity component satisfies $A_0 \cong \mathfrak{a}/\Gamma_A$, where $\Gamma_A \subseteq \mathfrak{a}$ is a discrete subgroup. Further let $q \colon \widehat{G} \to G$ be a Lie group extension of the connected Lie group G by A corresponding to the Lie algebra cocycle $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$, so that its Lie algebra is $\widehat{\mathfrak{g}} \cong \mathfrak{a} \oplus_\omega \mathfrak{g}$. In these terms we write the adjoint action of $\widehat{G}$ on $\widehat{\mathfrak{g}}$ as*

$$\mathrm{Ad}(g).(a, x) = (g.a - \theta(g)(g.x), g.x), \quad g \in \widehat{G}, a \in \mathfrak{a}, x \in \mathfrak{g}, \tag{6.4}$$

where $g.x = \mathrm{Ad}(q(g)).x$ and

$$\theta \colon \widehat{G} \to C^1_c(\mathfrak{g}, \mathfrak{a}) = \mathrm{Lin}(\mathfrak{g}, \mathfrak{a})$$

is a 1-cocycle with respect to the action of $\widehat{G}$ on $\mathrm{Lin}(\mathfrak{g}, \mathfrak{a})$ by $(g.\alpha)(x) := g.\alpha(g^{-1}.x)$. Its restriction $\theta_A := \theta|_A$ is a homomorphism given by

$$\theta_A(a) = D(d_G(a)) \text{ with } (d_G a)(g) := g.a - a \text{ and } D(d_G a)(x) := x.a := \big(d(d_G a)(\mathbf{1})\big)(x).$$

This 1-cocycle maps A_0 to $B^1_c(\mathfrak{g}, \mathfrak{a})$ and factors through a 1-cocycle

$$\overline{\theta} \colon \widehat{G}/A_0 \to \widehat{H}^1_c(\mathfrak{g}, \mathfrak{a}) = \mathrm{Lin}(\mathfrak{g}, \mathfrak{a})/B^1_c(\mathfrak{g}, \mathfrak{a}), \quad q(g) \mapsto [\theta(g)].$$

The map $\overline{q} \colon \widehat{G}/A_0 \to G, gA_0 \mapsto q(g)$ is a covering of G, so that there is a unique covering morphism $\widehat{q}_G \colon \widetilde{G} \to \widehat{G}/A_0$ with $\overline{q} \circ \widehat{q}_G = q_G$, and the following assertions hold:

1. *The coadjoint action of $\widehat{G}$ on $\widehat{\mathfrak{g}}$ and the flux cocycle are related by $F_\omega = -\overline{\theta} \circ \widehat{q}_G$.*
2. *If $\delta \colon \pi_1(G) \to \pi_0(A) \subseteq \widehat{G}/A_0$ is the connecting homomorphism from the long exact homotopy sequence of the principal A-bundle $q \colon \widehat{G} \to G$, then*

$$F_\omega = -\overline{\theta}_A \circ \delta \colon \pi_1(G) \to H^1_c(\mathfrak{g}, \mathfrak{a}),$$

where $\theta_A \colon \pi_0(A) \to H^1_c(\mathfrak{g}, \mathfrak{a})$ *is the characteristic homomorphism of the smooth* G*-module* A*.*

(3) *The induced map*

$$\overline{F}_\omega \colon \pi_1(G) \to H^1_c(\mathfrak{g}, \mathfrak{a})/\pi_0(A)$$

vanishes, and if A *is connected, then* $F_\omega(\pi_1(G)) = \{0\}$*.*

Proof. From the description of the Lie algebra $\widehat{\mathfrak{g}}$ as $\mathfrak{a} \oplus_\omega \mathfrak{g}$, it is clear that there exists a function $\theta \colon \widehat{G} \to \mathrm{Lin}(\mathfrak{g}, \mathfrak{a})$ for which the map $(g, x) \mapsto \theta(g)(x)$ is smooth and the adjoint action of $\widehat{G}$ on $\mathfrak{g}$ is given by (6.4). Since Ad is a representation of G, we have $\theta(\mathbf{1}, x) = 0$ and

$$\theta(g_1 g_2)(g_1 g_2.x) = g_1.\theta(g_2)(g_2.x) + \theta(g_1)(g_1 g_2.x), \quad g_1, g_2 \in \widehat{G}, x \in \mathfrak{g}, \tag{6.5}$$

which means that

$$\theta(g_1 g_2) = g_1.\theta(g_2) + \theta(g_1),$$

i.e., θ is a 1-cocycle. As A acts trivially on $\mathfrak{a}$ and $\mathfrak{g}$, the restriction $\theta_A = \theta|_A$ is a homomorphism

$$\theta_A \colon A \to Z^1_c(\mathfrak{g}, \mathfrak{a}) \quad \text{with} \quad \mathrm{Ad}(b).(a, x) = (a - \theta_A(b)(x), x), \quad b \in A, a \in \mathfrak{a}, x \in \mathfrak{g}.$$

The relation $\theta(b) \in Z^1_c(\mathfrak{g}, \mathfrak{a})$ follows directly from $\mathrm{Ad}(b) \in \mathrm{Aut}(\widehat{\mathfrak{g}})$.

For $\widehat{g} \in \widehat{G}$ with $q(\widehat{g}) = g$ and $b \in A$ we have $b\widehat{g}b^{-1} = (b\widehat{g}b^{-1}\widehat{g}^{-1})\widehat{g} = (b - g.b) \cdot \widehat{g}$, which leads to

$$\mathrm{Ad}(b).(a, x) = (a - x.b, x)$$

and therefore to $\theta_A(b)(x) = x.b$. For $a \in \mathfrak{a}$ and $b = q_A(a)$ we have $x.b = x.a$, so that $\theta(A_0) = B^1_c(\mathfrak{g}, \mathfrak{a})$. Hence θ factors through a 1-cocycle $\overline{\theta} \colon \widehat{G}/A_0 \to \widehat{H}^1_c(\mathfrak{g}, \mathfrak{a})$ whose restriction $\overline{\theta}_A$ to $\pi_0(A) = A/A_0$ is given by

$$\overline{\theta}_A \colon \pi_0(A) \cong A/A_0 \to H^1_c(\mathfrak{g}, \mathfrak{a}), \quad [a] \mapsto [\theta_A(a)] = [D(d_G a)].$$

(1) For a fixed $x \in \mathfrak{g}$ the cocycle condition (6.5) implies for the smooth functions $\theta_x \colon \widehat{G} \to \mathfrak{a}, g \mapsto \theta(g)(x)$ the relation

$$\theta_x(gh) = g.\theta_{g^{-1}.x}(h) + \theta_x(g).$$

For the differentials we thus obtain

$$d\theta_x(g) d\lambda_g(\mathbf{1}) = \rho_{\mathfrak{a}}(g) \circ d\theta_{g^{-1}.x}(\mathbf{1}). \tag{6.6}$$

From formula (6.4) for the adjoint action, we get in view of $\theta(\mathbf{1}) = 0$ the formula

$$(x'.a - x.a' + \omega(x', x), [x', x]) = \mathrm{ad}(a', x')(a, x) = (x'.a - d\theta_x(\mathbf{1})(a', x'), [x', x]),$$

so that θ and the corresponding Lie algebra cocycle are related by

$$d\theta_x(\mathbf{1})(a', x') = \omega(x, x') + x.a'.$$

With (6.6) this further leads to

$$d\theta_x(g)d\lambda_g(\mathbf{1})(a',x') = g.\big(\omega(g^{-1}.x,x') + (g^{-1}.x).a'\big) = \omega^{\rm eq}(x_r(q(g)), d\lambda_{q(g)}(\mathbf{1}).x') + x.(g.a').$$

In $\Omega^1(\widehat{G},\mathfrak{a})$ we therefore have the relation

$$d\theta_x = \dot\rho_{\mathfrak{a}}(x)\circ p_{\mathfrak{a}}^{\rm eq} + q^*(i_{x_r}\omega^{\rm eq}),$$

where $p_{\mathfrak{a}}(a',x') = a'$ is the projection of $\widehat{\mathfrak{g}}$ onto $\mathfrak{a}$ and $p_{\mathfrak{a}}^{\rm eq}$ the corresponding equivariant 1-form on $\widehat{G}$.

Let $\gamma\colon [0,1]\to G$ be any piecewise smooth loop based in $\mathbf{1}$. Then there exists a piecewise smooth map $\widehat{\gamma}\colon [0,1]\to\widehat{G}$ with $q\circ\widehat{\gamma} = \gamma$ and $\widehat{\gamma}(0) = \mathbf{1}$, so that $\widetilde{\gamma} := \widehat{q}_G\circ\widehat{\gamma}\colon [0,1]\to\widetilde{G}$ is the unique lift of γ to a piecewise smooth path in $\widetilde{G}$ starting in $\mathbf{1}$. We now have

$$\begin{aligned}
-\widetilde{F}_\omega(\gamma)(x) &= \int_\gamma i_{x_r}\omega^{\rm eq} = \int_{[0,1]}\gamma^*(i_{x_r}\omega^{\rm eq}) = \int_{[0,1]}\widehat{\gamma}^*q^*(i_{x_r}\omega^{\rm eq})\\
&= \int_{\widehat{\gamma}} q^*(i_{x_r}\omega^{\rm eq}) = \int_{\widehat{\gamma}} d\theta_x - \rho_{\mathfrak{a}}(x)\circ p_{\mathfrak{a}}^{\rm eq}\\
&= \theta_x(\widehat{\gamma}(1)) - \theta_x(\widehat{\gamma}(0)) - \rho_{\mathfrak{a}}(x).\int_{\widehat{\gamma}} p_{\mathfrak{a}}^{\rm eq} = \theta(\widehat{\gamma}(1))(x) - \rho_{\mathfrak{a}}(x).\int_{\widehat{\gamma}} p_{\mathfrak{a}}^{\rm eq}.
\end{aligned}$$

This means that

$$F_\omega(\widetilde{\gamma}(1)) = [\widetilde{F}_\omega(\gamma)] = -[\theta(\widehat{\gamma}(1))] = -\overline{\theta}(\widehat{q}_G(\widetilde{\gamma}(1)))$$

and therefore that $F_\omega = -\overline{\theta}\circ\widehat{q}_G$ because γ was arbitrary.

(2) If $\gamma\colon [0,1]\to G$ is a piecewise smooth loop based in $\mathbf{1}$, then $\widehat{\gamma}(1)\in\ker q = A$ and $\delta([\gamma]) = [\widehat{\gamma}(1)]$, as an element of $\pi_0(A)$. This means that δ can be considered as the restriction of $\widehat{q}_G\colon\widetilde{G}\to\widehat{G}/A_0$ to the subgroup $\pi_1(G) = \ker q_G$. Therefore (2) follows from (1) by restriction.

(3) This follows directly from (2) because $H^1_c(\mathfrak{g},\mathfrak{a})/\pi_0(A) = \operatorname{coker}\overline{\theta}_A$ (Definition 3.6). ■

Corollary 6.5. *If, in addition to the assumptions of* Proposition 6.4*, the group G is simply connected, then G is isomorphic to the identity component of the group $\widehat{G}/A_0$, and in this sense*

$$F_\omega = -\overline{\theta}\colon G\to\widehat{H}^1_c(\mathfrak{g},\mathfrak{a}).$$

On the subgroup $A^\sharp := q^{-1}(Z(G)\cap\ker\rho_A)$ of $\widehat{G}$ the cocycle θ restricts to a homomorphism

$$\theta^\sharp\colon A^\sharp\to Z^1_c(\mathfrak{g},\mathfrak{a}),\quad a\mapsto D(d_G(a)), \tag{6.7}$$

where for each $a \in A^\sharp$ the smooth cocycle $d_G(a) \in Z^1_s(G,A)$ is defined by $d_G(a)(q(g)) := gag^{-1}a^{-1}$. For two piecewise smooth curves $\gamma, \eta\colon [0,1] \to G$ with $\gamma(0) = \eta(0) = \mathbf{1}$ and $\gamma(1), \eta(1) \in A^\sharp$ we have for $H\colon I^2 \to G, H(t,s) = \gamma(t)\eta(s)$ the formula

$$\gamma(1)\eta(1)\gamma(1)^{-1}\eta(1)^{-1} = -q_A\Big(\int_\gamma \widetilde{F}_\omega(\eta)^{\mathrm{eq}}\Big) = q_A\Big(\int_H \omega^{\mathrm{eq}}\Big). \tag{6.8}$$

Proof. To derive the first part from Propositions 6.3 and 6.4, we only have to observe that for $a \in A^\sharp$ the condition $\rho_A(a) = \mathrm{id}_A$ implies that $d_G(a)$ is well-defined on G by $d_G(a)(q(g)) = gag^{-1}a^{-1}$, and that this is an element of A because $q(a) \in Z(G)$ implies $d_G(a) \in \ker q$.

For (6.8) we first observe that for $x \in \mathfrak{a}$ and $q_A(x) = x + \Gamma_A \in A$ the map $d_G q_A(x)\colon G \to A$ satisfies

$$\begin{aligned} 0 &= \rho_A(\gamma(1))(q_A(x)) - q_A(x) = (d_G q_A(x))(\gamma(1)) = \int_\gamma d(d_G(q_A(x))) + \Gamma_A \\ &= \int_\gamma (D(d_G q_A(x)))^{\mathrm{eq}} + \Gamma_A = \int_\gamma (d_{\mathfrak{g}} x)^{\mathrm{eq}} + \Gamma_A, \end{aligned}$$

so that the integration along γ yields a well-defined map $\widehat{H}^1_c(\mathfrak{g},\mathfrak{a}) \to A, [\alpha] \mapsto q_A\Big(\int_\gamma \alpha^{\mathrm{eq}}\Big)$. We therefore get with Proposition 6.4, Lemma 6.2(3) (note the sign change) and $-\overline{\theta} = F_\omega$ on the identity component G of $\widehat{G}/A_0$:

$$\begin{aligned} \gamma(1)\eta(1)\gamma(1)^{-1}\eta(1)^{-1} = d_G(\eta(1))(\gamma(1)) &= \int_\gamma d(d_G(\eta(1))) + \Gamma_A \\ &= \int_\gamma D(d_G(\eta(1)))^{\mathrm{eq}} + \Gamma_A \\ = \int_\gamma \theta^\sharp(\eta(1))^{\mathrm{eq}} + \Gamma_A = -\int_\gamma F_\omega(\eta(1))^{\mathrm{eq}} + \Gamma_A &= -\int_\gamma \widetilde{F}_\omega(\eta)^{\mathrm{eq}} + \Gamma_A \\ &= \int_H \omega^{\mathrm{eq}} + \Gamma_A. \end{aligned}$$

■

Corollary 6.6. *Suppose that $A \cong \mathfrak{a}/\Gamma_A$, that $q_G\colon \widetilde{G} \to G$ is a universal covering homomorphism, let $q\colon \widehat{G} \to \widetilde{G}$ be an A-extension of $\widetilde{G}$ corresponding to $\omega \in Z^2_c(\mathfrak{g},\mathfrak{a})$, and $\widehat{\pi}_1(G) := q^{-1}(\pi_1(G))$. Then the following are equivalent:*

(1) $F_\omega(\pi_1(G)) = \{0\}$.
(2) $\theta(\widehat{\pi}_1(G)) \subseteq B^1_c(\mathfrak{g},\mathfrak{a}) = \theta(A)$.
(3) $\widehat{\pi}_1(G) = A + \ker(\theta|_{\widehat{\pi}_1(G)})$.
(4) $q(\ker(\theta|_{\widehat{\pi}_1(G)})) = \pi_1(G)$.

(5) *There exists a group homomorphism* $\sigma\colon \pi_1(G) \to \ker(\theta|_{\widehat{\pi}_1(G)}) = \widehat{\pi}_1(G) \cap Z(\widehat{G})$ *with* $q \circ \sigma = \mathrm{id}_{\pi_1(G)}$.

Proof. The equivalence of (1) and (2) follows from Corollary 6.5, and (2) is clearly equivalent to (3), which in turn is equivalent to (4) because $\ker q = A$.

That (5) implies (4) is trivial. If (4) is satisfied, then we first observe that $\ker(\theta|_{\widehat{\pi}_1(G)}) = \widehat{\pi}_1(G) \cap Z(\widehat{G})$, so that (3) implies that $\widehat{\pi}_1(G)$ is abelian. Further (6.7) in Corollary 6.5 leads to

$$\ker(\theta|_{\widehat{\pi}_1(G)}) \cap \ker q = \ker(\theta|_A) = q_A(\mathfrak{a}^{\mathfrak{g}}),$$

which is a divisible group. Hence the extension $q_A(\mathfrak{a}^{\mathfrak{g}}) \hookrightarrow \ker(\theta|_{\widehat{\pi}_1(G)}) \twoheadrightarrow \pi_1(G)$ splits, which is (5). ■

The following theorem is a central result of this paper.

Theorem 6.7. (Integrability Criterion) *Let* G *be a connected Lie group and* A *be a smooth* G*-module with* $A_0 \cong \mathfrak{a}/\Gamma_A$, *where* Γ_A *is a discrete subgroup of the Mackey complete locally convex space* $\mathfrak{a}$. *For* $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$ *the abelian Lie algebra extension* $\mathfrak{a} \hookrightarrow \widehat{\mathfrak{g}} := \mathfrak{a} \times_\omega \mathfrak{g} \twoheadrightarrow \mathfrak{g}$ *integrates to a Lie group extension* $A \hookrightarrow \widehat{G} \twoheadrightarrow G$ *if and only if*

(1) $\Pi_\omega \subseteq \Gamma_A$, *and*

(2) *there exists a homomorphism* $\gamma\colon \pi_1(G) \to \pi_0(A)$ *such that the flux homomorphism*

$$F_\omega\colon \pi_1(G) \to H^1_c(\mathfrak{g}, \mathfrak{a})$$

is related to the characteristic homomorphism $\overline{\theta}_A\colon \pi_0(A) \to H^1_c(\mathfrak{g}, \mathfrak{a})$ *by*

$$F_\omega = \overline{\theta}_A \circ \gamma.$$

If, in addition, A *is connected, then* (2) *is equivalent to* $F_\omega = 0$.

Proof. Suppose first that $\widehat{G}$ is a Lie group extension of G by A corresponding to the Lie algebra cocycle ω. According to Proposition 4.5, $-\mathrm{per}_\omega$ is the connecting map $\pi_2(G) \to \pi_1(A) \cong \Gamma_A$. This implies (1). That (2) is satisfied follows from Proposition 6.4(2) with $\gamma = -\delta$.

Conversely, suppose that (1) and (2) hold. Let $q_G\colon \widetilde{G} \to G$ denote the simply connected covering group of G and recall that $\pi_2(q_G)$ is an isomorphism $\pi_2(\widetilde{G}) \to \pi_2(G)$. We may therefore identify the period maps per_ω of G and $\widetilde{G}$ and likewise for all quotients of $\widetilde{G}$ by subgroups of $\pi_1(G)$.

From the case of simply connected groups (Proposition 5.3), we know that there exists an A_0-extension $q^\sharp\colon G^\sharp \to \widetilde{G}$, where A carries the natural $\widetilde{G}$-module structure induced by the G-module structure. The Lie algebra of $G^\sharp$ is $\widehat{\mathfrak{g}} = \mathfrak{a} \oplus_\omega \mathfrak{g}$. Let $G_1 := \widetilde{G}/\ker\gamma$ and observe that $\pi_1(G_1) \cong \ker\gamma$. Condition (2) implies $\pi_1(G_1) = \ker\gamma \subseteq \ker F_\omega$, so that Corollary 6.6 implies that there exists a homomorphism

$$\sigma\colon \pi_1(G_1) \to \ker(\theta|_{\widehat{\pi}_1(G)}) \subseteq Z(G^\sharp)$$

with $q^\sharp \circ \sigma = \mathrm{id}_{\pi_1(G_1)}$. Then the image of σ is a discrete central subgroup of $G^\sharp$, and therefore

$$\widehat{G}_1 := G^\sharp/\sigma(\pi_1(G_1))$$

defines an abelian extension $A_0 \hookrightarrow \widehat{G}_1 \xrightarrow{q_1} G_1$ corresponding to the given Lie algebra extension $\mathfrak{a} \oplus_\omega \mathfrak{g} \to \mathfrak{g}$.

If $q_1 \colon G_1 \to G$ is the quotient map with kernel $\pi_1(G)/\ker\gamma \cong \operatorname{im}\gamma \subseteq \pi_0(A)$, then $B := q_1^{-1}(\pi_1(G)/\ker\gamma)$ is a subgroup of $\widehat{G}_1$ with $B_0 = A_0$ and $\pi_0(B) = B/B_0 \cong \operatorname{im}(\gamma) \subseteq \pi_0(A)$. Let $A_1 \subseteq A$ denote the open subgroup whose image in $\pi_0(A)$ is $\operatorname{im}(\gamma)$. Then $B \cong B_0 \times \pi_0(B) \cong A_0 \times \operatorname{im}(\gamma) \cong A_1$ as abelian Lie groups. As γ factors through an isomorphism $\overline{\gamma} \colon \pi_0(B) \to \operatorname{im}\gamma \subseteq \pi_0(A_1)$ and the characteristic maps $\overline{\theta}_A \colon \pi_0(A) \to H^1_c(\mathfrak{g},\mathfrak{a})$ and $\overline{\theta}_B \colon \pi_0(B) \to H^1_c(\mathfrak{g},\mathfrak{a})$ satisfy

$$\overline{\theta}_A \circ \overline{\gamma} = \overline{\theta}_B$$

(Proposition 6.4, Corollary 6.5), Lemma 3.8 implies that $A_1 \cong B$ as smooth G-modules. Therefore $\widehat{G}_1$ is an A_1-extension of G. Write $\widehat{G}_1 = A_1 \times_f G$ for some $f \in Z^2_{ss}(G, A_1) \subseteq Z^2_{ss}(G,A)$ (Proposition 2.6). Then $\widehat{G} := A \times_f G$ is an extension of G by A containing $\widehat{G}_1$ as an open subgroup. ■

Remark 6.8. The condition (2) in the preceding theorem reduces to the simple condition $F_\omega = 0$ if A is connected, but if A is not connected, it can become quite involved. From the short exact sequence of abelian groups

$$\mathbf{0} \to \ker\overline{\theta}_A \to \pi_0(A) \twoheadrightarrow \operatorname{im}(\overline{\theta}_A) \to \mathbf{0}$$

and the corresponding long exact cohomology sequence we obtain an exact sequence

$$\operatorname{Hom}(\pi_1(G), \pi_0(A)) \to \operatorname{Hom}(\pi_1(G), \operatorname{im}\overline{\theta}_A) \to H^2(\pi_1(G), \ker\overline{\theta}_A).$$

Clearly $\operatorname{im}(F_\omega) \subseteq \operatorname{im}(\overline{\theta}_A)$ is necessary for (2), but if this condition is satisfied, then the obstruction for the existence of $\gamma \colon \pi_1(G) \to \pi_0(A)$ as in (2) is the image of F_ω in $H^2(\pi_1(G), \ker\overline{\theta}_A)$. This cohomology class can be interpreted as a central extension of $\pi_1(G)$ by the discrete group $\operatorname{im}(\overline{\theta}_A)$ (see also the discussion in Example D.11). ■

Remark 6.9. (a) Suppose that only (1) in Theorem 6.7 is satisfied. Consider the corresponding extension $q^\sharp \colon G^\sharp \to \widetilde{G}$ of $\widetilde{G}$ by $A_0 \cong \mathfrak{a}/\Gamma_A$. Then $G \cong G^\sharp/\widehat{\pi}_1(G)$, where $\widehat{\pi}_1(G) := (q^\sharp)^{-1}(\pi_1(G))$ is a central A_0-extension of $\pi_1(G)$, hence 2-step nilpotent Lie group with Lie algebra $\mathfrak{a}$.

If $\widehat{\pi}_1(G)$ is abelian, then we have an abelian Lie group extension

$$\mathbf{1} \to \widehat{\pi}_1(G) \hookrightarrow G^\sharp \to G \to \mathbf{1}$$

of G by the abelian group $\widehat{\pi}_1(G)$, and the corresponding Lie algebra extension is

$$\mathbf{0} \to \mathfrak{a} \hookrightarrow \widehat{\mathfrak{g}} \cong \mathfrak{a} \oplus_\omega \mathfrak{g} \twoheadrightarrow \mathfrak{g} \to \mathbf{0}.$$

(b) We have seen in the proof of Theorem 6.7 that whenever an A-extension $\widehat{G}$ of G corresponding to $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$ exists, then its identity component is a quotient of $G^\sharp$ by a subgroup $\sigma(\pi_1(G))$, where $\sigma \colon \pi_1(G) \to Z(G^\sharp) \cap \widehat{\pi}_1(G)$ is a splitting homomorphism for $\widehat{\pi}_1(G)$. This implies in particular that $\widehat{\pi}_1(G)$ is abelian.

Let us take a closer look at the nilpotent group $\widehat{\pi}_1(G)$. If this group is abelian, then the divisibility of $A_0 \cong \mathfrak{a}/\Gamma_A$ implies that $\widehat{\pi}_1(G)$ splits as an A_0-extension of $\pi_1(G)$. Clearly this condition is weaker than the requirement that it splits by a homomorphism $\sigma \colon \pi_1(G) \to \widehat{\pi}_1(G) \cap Z(G^\sharp)$ with values in the center of $G^\sharp$.

That $\widehat{\pi}_1(G)$ is abelian is equivalent to the triviality of the induced commutator map

$$C^A_\omega \colon \pi_1(G) \times \pi_1(G) \to A_0 \subseteq A.$$

According to Corollary 6.5,

$$C^A_\omega([\gamma], [\eta]) = -q_A\Big(\int_\gamma \widetilde{F}_\omega(\eta)^{\mathrm{eq}}\Big) = -P_1(F_\omega([\eta]))([\gamma]) = -\int_\gamma \widehat{F}_\omega([\eta]), \tag{6.9}$$

where $P_1 \colon H^1_c(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Hom}(\pi_1(G), A), P_1([\alpha])([\gamma]) := q_A\Big(\int_\gamma \alpha^{\mathrm{eq}}\Big)$ as in Proposition 3.4. Therefore the commutator map vanishes if and only if

$$P_1 \circ F_\omega(\pi_1(G)) = \{0\}, \tag{6.10}$$

which means that $\widehat{F}_\omega(\pi_1(G)) = \{0\}$. This means that for all smooth loops $\gamma, \eta \colon \mathbb{S}^1 \to G$ and $H \colon \mathbb{T}^2 \to G, (t, s) \mapsto \gamma(t)\eta(s)$ we have

$$q_A\Big(\int_{\mathbb{T}^2} H^*\omega^{\mathrm{eq}}\Big) = P_1(F_\omega([\eta]))([\gamma]) = 0.$$

In view of Proposition 3.4, Condition (6.10) is equivalent to

$$F_\omega(\pi_1(G)) \subseteq \ker P_1 = \mathrm{im}(D_1) \cong H^1_s(G, A_0) \subseteq H^1_c(\mathfrak{g}, \mathfrak{a}), \tag{6.11}$$

i.e., that the image of the flux homomorphism consists of classes of integrable 1-cocycles, so that we may view F_ω as a homomorphism

$$F_\omega \colon \pi_1(G) \to H^1_s(G, A_0).$$

In Corollary 6.5 we have seen that we have a homomorphism

$$\theta^\sharp = D_1 \circ d_{\widetilde{G}} \colon \widehat{\pi}_1(G) \to Z^1_c(\mathfrak{g}, \mathfrak{a})$$

which factors through the (negative) flux homomorphism $-F_\omega\colon \pi_1(G) \to H^1_c(\mathfrak{g}, \mathfrak{a})$. The group $\widehat{\pi}_1(G)$ is a smooth $\widetilde{G}$-module which is abelian if and only $\pi_1(G)$ acts trivially, which in turn is (6.11). If this is the case, then

$$-F_\omega\colon \pi_0(\widehat{\pi}_1(G)) \cong \pi_1(G) \to H^1_c(\mathfrak{g}, \mathfrak{a})$$

is the characteristic homomorphism of the smooth G-module $\widehat{\pi}_1(G)$. In view of Lemma 3.8, it vanishes if and only if the identity component $\widehat{\pi}_1(G)_0 \cong A_0$ has a G-invariant complement in $\widehat{\pi}_1(G)$. ■

Below we describe a typical example where the commutator map C^A_ω vanishes and the flux homomorphism $F_\omega\colon \pi_1(G) \to H^1_c(\mathfrak{g}, \mathfrak{a})$ is non-zero (see also Example D.11(b)).

Example 6.10. Let $G := \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with universal covering group $q_G\colon \widetilde{G} \cong \mathbb{R}^2 \to G$. We consider $Z := \mathbb{T} = \mathbb{R}/\mathbb{Z}$ as a trivial G-module and write $q_Z\colon \mathfrak{z} \cong \mathbb{R} \to Z$ for the quotient map. Then

$$\omega(x, y) := x_1 y_2 - x_2 y_1$$

defines a Lie algebra cocycle in $Z^2_c(\mathfrak{g}, \mathfrak{z})$. Since $\pi_2(G) \cong \pi_2(\mathbb{R}^2)$ vanishes, we have $\mathrm{per}_\omega = 0$. The corresponding central extension of $\widetilde{G}$ is given by

$$G^\sharp := Z \times_f \widetilde{G} = \mathbb{T} \times_f \mathbb{R}^2, \quad f(x, y) = q_Z(x_1 y_2) = x_1 y_2 + \mathbb{Z}.$$

Note that the biadditivity of f implies that it is a group cocycle, and clearly $D_2 f = \omega$.

Since the differential $d_\mathfrak{g}$ on $C^\bullet(\mathfrak{g}, \mathfrak{z})$ vanishes, we have $\widehat{H}^1_c(\mathfrak{g}, \mathfrak{z}) = \mathrm{Lin}(\mathfrak{g}, \mathfrak{z}) = H^1_c(\mathfrak{g}, \mathfrak{z})$. The flux cocycle, which actually is a homomorphism, is given by

$$F_\omega\colon \widetilde{G} \cong \mathbb{R}^2 \to \widehat{H}^1_c(\mathfrak{g}, \mathfrak{z}) \cong \mathrm{Lin}(\mathfrak{g}, \mathbb{R}) \cong \mathfrak{g}^*, \quad x \mapsto i_x\omega.$$

It is a bijective linear map and in particular non-zero.

The kernel of the map

$$\begin{aligned} P_1\colon H^1_c(\mathfrak{g}, \mathfrak{z}) \cong \mathrm{Lin}(\mathfrak{g}, \mathfrak{z}) &\to \mathrm{Hom}(\pi_1(G), Z) = \mathrm{Hom}(\mathbb{Z}^2, \mathbb{T}) \cong \mathbb{T}^2, \\ \alpha &\mapsto (q_Z(\alpha(e_1)), q_Z(\alpha(e_2))) \end{aligned}$$

is the additive subgroup $\mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})$ of $\mathrm{Lin}(\mathbb{R}^2, \mathbb{R})$. Therefore $\omega(\mathbb{Z}^2 \times \mathbb{Z}^2) \subseteq \mathbb{Z}$ leads to

$$P_1 \circ F_\omega(\pi_1(G)) = P_1(i_{\mathbb{Z}^2}\omega) = \{0\},$$

which corresponds to the triviality of the extension $\widehat{\pi}_1(G) := Z \times_f \mathbb{Z}^2 = Z \times \mathbb{Z}^2$.

We conclude that there is no Lie group extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ corresponding to the Lie algebra cocycle ω. Another reason for such an extension not to exist is that any such extension would be central, but for all central extensions of tori by connected Lie groups the corresponding Lie algebra cocycle vanishes (cf. [Ne02]).

We now consider the smooth G-module $A := C^\infty(G,Z)$ with the action $(g.f)(x) := f(g+x)$ and note that $\mathfrak{a} = C^\infty(G,\mathfrak{z})$. We identify $Z \subseteq A$ with the subgroup consisting of constant functions. Then ω is an element of $Z^2_c(\mathfrak{g},\mathfrak{a})$. A linear functional $\alpha\colon \mathfrak{g} \to \mathfrak{a}$ is determined by the pair $(f_1,f_2) := (\alpha(e_1),\alpha(e_2))$, and the cocycle condition means that

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1},$$

i.e., $f_1 dx_1 + f_2 dx_2 \in \Omega^1(G,\mathfrak{z})$ is a closed 1-form. This implies that

$$H^1_c(\mathfrak{g},\mathfrak{a}) \cong Z^1_{\rm dR}(G,\mathfrak{z})/dC^\infty(G,\mathfrak{z}) \cong H^1_{\rm dR}(G,\mathfrak{z}) = \mathbb{R}[dx_1] \oplus \mathbb{R}[dx_2].$$

In this sense the flux homomorphism is given by

$$F_\omega\colon \mathbb{Z}^2 \cong \pi_1(G) \to H^1_c(\mathfrak{g},\mathfrak{a}), \quad F_\omega(n,m) = n[i_{e_1}\omega] + m[i_{e_2}\omega] = n[dx_2] - m[dx_1].$$

For a smooth function $f\colon \mathbb{T}^2 \to \mathbb{T}$ we have $d'_G(f)(g) = (g.f)f^{-1}$ and therefore $\theta_A(f) = D_1(d'_G f) = f^{-1}df$, so that the characteristic homomorphism

$$\overline{\theta}_A\colon \pi_0(A) \to H^1_c(\mathfrak{g},\mathfrak{a}) \cong H^1_{\rm dR}(\mathbb{T}^2,\mathbb{R}), \quad [f] \mapsto [f^{-1}df]$$

is an injective homomorphism onto the discrete subgroup of integral cohomology classes. The map

$$\gamma\colon \pi_1(G) \cong \mathbb{Z}^2 \to \pi_0(A), \quad (n,m) \mapsto [q_G(x_1,x_2) \mapsto q_Z(nx_2 - mx_1)]$$

is an isomorphism of groups satisfying

$$\overline{\theta}_A \circ \gamma(n,m) = [ndx_2 - mdx_1] = F_\omega(n,m),$$

so that the assumptions of the integrability criterion Theorem 6.7 are satisfied.

With the cocycle $f \in Z^2_s(\widetilde{G},Z) \subseteq Z^2_s(\widetilde{G},A)$ from above, we obtain a group extension

$$\widehat{G} := A_0 \times_f \widetilde{G} = A_0 \times_f \mathbb{R}^2, \quad f(x,y) = q_Z(x_1 y_2),$$

whose restriction to $\pi_1(G)$ is the abelian group

$$\widehat{\pi}_1(G) := A_0 \times_f \mathbb{Z}^2 \cong A_0 \times \mathbb{Z}^2 \cong A.$$

For $(n,m) \in \mathbb{Z}^2$ we have in $\widehat{G}$

$$(0,(x,y))(0,(n,m))(0,(-x,-y)) = (f((x,y),(n,m)),(n,m)) = (q_Z(xm),(n,m)),$$

showing that $Z(\widehat{G}) = Z$, and there is no section $\pi_1(G) \to \widehat{\pi}_1(G)$ with central values. Nevertheless, the existence of γ implies that $\widehat{\pi}_1(G) \cong A$ as smooth G-modules, so that

$$A \cong \widehat{\pi}_1(G) \hookrightarrow \widehat{G} \twoheadrightarrow G$$

is an A-extension of G whose Lie algebra cocycle is ω. ■

In Example 9.17 we shall discuss a generalization of the setting used in Example 6.10 for the two-dimensional torus.

If G is a smoothly paracompact group, then each closed $\mathfrak{a}$-valued differential form defines a singular A-valued cohomology class, and it is instructive to compare the condition that the period map $q_A \circ \mathrm{per}_\omega$ and the flux homomorphism F_ω vanish with the condition that the corresponding cohomology class in

$$H^2_{\mathrm{sing}}(G, A) \cong \mathrm{Hom}(H_2(G), A)$$

vanishes (the equality follows from the Universal Coefficient Theorem). To evaluate this cohomology classes, it is crucial to have a good description of the generators of the group $H_2(G)$.

Proposition 6.11. *Let G be a topological group, $S_2(G) \subseteq H_2(G)$ the subgroup of spherical 2-cycles, i.e., the image of $\pi_2(G)$ under the Hurewicz homomorphism $\pi_2(G) \to H_2(G)$, and $\Lambda_2(G) := H_2(G)/S_2(G)$ the quotient group. Then $\Lambda_2(G)$ is generated by the images of cycles defined by maps of the form*

$$\alpha * \beta \colon \mathbb{T}^2 \to G, \quad (t, s) \mapsto \alpha(t)\beta(s),$$

where $\alpha, \beta \colon \mathbb{T} \to G$ are loops in G.

Proof. First we recall that the group $H_2(G)$ is generated by the cycles defined by continuous maps $F \colon \Sigma \to G$, where Σ is a compact orientable surface of genus $g \in \mathbb{N}_0$ (this comes from the fact that the cone over each connected compact 1-manifold is a disc). We may assume that $g > 0$, otherwise $\Sigma \cong \mathbb{S}^2$, and there is nothing to show.

We recall that Σ can be described as a CW-complex by starting with a bouquet

$$A_{2g} \cong \underbrace{\mathbb{S}^1 \vee \mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1}_{2g}$$

of $2g$-circles. Let $x_0 \in \mathbb{S}^1$ be the base point in $\mathbb{S}^1$ and a_0 the base point in A_{2g}. We write $a_1, a_2, \ldots, a_{2g-1}, a_{2g} \colon \mathbb{S}^1 \to A_{2g}$ for the corresponding generators of the fundamental group of A_{2g} which is a free group on $2g$ generators. Then we consider the continuous map $\gamma \colon \mathbb{S}^1 \to A_{2g}$ corresponding to

$$[a_1, a_2] \cdots [a_{2g-1}, a_{2g}] \in \pi_1(A_{2g}), \tag{6.12}$$

where $[x, y] := xyx^{-1}y^{-1}$ denotes a commutator. Now Σ is homeomorphic to the space obtained by identifying the points in $\partial\mathbb{B}^2 \cong \mathbb{S}^1$ (the boundary of the 2-dimensional unit disc $\mathbb{B}^2$) with their images in A_{2g} under γ, i.e.,

$$\Sigma \cong A_{2g} \cup_\gamma \mathbb{B}^2.$$

In this sense we can identify A_{2g} with a subset of Σ. Let $f_j \colon A_{2g} \to \mathbb{S}^1$ be the pointed map for which $f_j \circ \alpha_j = \mathrm{id}_{\mathbb{S}^1}$ and for $i \neq j$ the map $f_j \circ \alpha_i$ is constant x_0.

Then f_j extends to a continuous map $f_j : \Sigma \to \mathbb{S}^1$ because it maps the commutator (6.12) to a contractible loop in $\mathbb{S}^1$.

For a continuous map $F\colon \Sigma \to G$ we now consider the continuous map

$$F_1 := (F \circ \alpha_1 \circ f_1) \cdots (F \circ \alpha_{2g} \circ f_{2g})$$

and observe that for each j we have $F \circ \alpha_j = F_1 \circ \alpha_j$, i.e., F and F_1 coincide on the subset A_{2g} of Σ. Therefore the map $F_2 := F_1^{-1} \cdot F\colon \Sigma \to G$ is a continuous map mapping A_{2g} to $\mathbf{1}$, so that it induces a map $\Sigma/A_{2g} \cong \mathbb{S}^2 \to G$. Hence $F = F_1 \cdot F_2$, where F_1 factors through the continuous map $f = (f_1, \ldots, f_{2g})\colon \Sigma \to \mathbb{T}^{2g}$ and F_2 factors through the quotient map $\Sigma \to \mathbb{S}^2$ collapsing A_{2g} to a point. We thus obtain a factorization of F into maps

$$\Sigma \to \mathbb{T}^{2g} \times \mathbb{S}^2 \to G \times G \xrightarrow{m_G} G,$$

where m_G is the group multiplication. Since the homology groups of $\mathbb{S}^2$ and $\mathbb{T}^{2g}$ are free, the Künneth Theorem yields

$$H_\bullet(\mathbb{S}^2 \times \mathbb{T}^{2g}) \cong H_\bullet(\mathbb{S}^2) \otimes H_\bullet(\mathbb{T}^{2g})$$

as graded abelian groups, and in particular

$$H_2(\mathbb{S}^2 \times \mathbb{T}^{2g}) \cong H_2(\mathbb{S}^2) \oplus H_2(\mathbb{T}^{2g}).$$

This implies that $H_2(F)\colon H_2(\Sigma) \cong \mathbb{Z} \to H_2(G)$ maps the fundamental class $[\Sigma] \in H_2(\Sigma)$ to the sum of two homology classes in the image of $H_2(F_1)$ and $H_2(F_2)$. Since $\operatorname{im}(H_2(F_1)) \in S_2(G)$, it remains to consider the image of $H_2(F)$ of maps $F\colon \mathbb{T}^{2g} \to G$. As $H_2(\mathbb{T}^{2g})$ is generated by the classes of the $\binom{2g}{2}$ 2-dimensional sub-tori obtained by the coordinatewise inclusions $\mathbb{T}^2 \to \mathbb{T}^{2g}$, everything reduces to maps $F\colon \mathbb{T}^2 \to G$. Writing F, as above, as $F_1 \cdot F_2$, we obtain a factorization of F into maps

$$\mathbb{T}^2 \to (\mathbb{T} \times \mathbb{T}) \times \mathbb{S}^2 \xrightarrow{(F\circ\alpha_1\circ f_1, F\circ\alpha_2\circ f_2, F_2)} G^3 \xrightarrow{m_G\circ(m_G\times \mathrm{id}_G)} G.$$

Now $H_2(\mathbb{T}^2 \times \mathbb{S}^2) \cong H_2(\mathbb{T}^2) \oplus H_2(\mathbb{S}^2)$ permits us to reduce matters to maps $F_1\colon \mathbb{T}^2 \to G$ of the form $\alpha * \beta$. This completes the proof. ∎

Remark 6.12. We apply the preceding proposition to Lie groups. Let G be a smoothly paracompact Lie group. Then de Rham's Theorem ([KM97, Thm. 34.7]) implies that the map

$$H^2_{\mathrm{dR}}(G, \mathfrak{a}) \to \operatorname{Hom}(H_2(G), \mathfrak{a}) \cong H^2_{\mathrm{sing}}(G, \mathfrak{a})$$

is an isomorphism of vector spaces. According to Proposition 6.11, $H_2(G)$ is generated by $S_2(G)$ and the classes of the maps $\alpha * \beta$. Therefore we obtain an injective map

$$\Phi\colon H^2_{\mathrm{dR}}(G, \mathfrak{a}) \to \operatorname{Hom}(\pi_2(G), \mathfrak{a}) \oplus \operatorname{Hom}(\pi_1(G) \otimes \pi_1(G), \mathfrak{a}),$$

with first component $\Phi_1([\Omega]) = \mathrm{per}_\Omega$ and

$$\Phi_2([\Omega])([\alpha] \otimes [\beta]) = \int_{\alpha * \beta} \Omega,$$

where $\alpha, \beta \colon \mathbb{S}^1 \to G$ are piecewise smooth representatives of their homotopy classes. That the second component is well-defined follows from the fact that m_G induces a map

$$\pi_1(G) \otimes \pi_1(G) \to H_1(G) \otimes H_1(G) \to H_2(G \times G) \xrightarrow{H_2(m_G)} H_2(G)$$

mapping $[\alpha] \otimes [\beta]$ onto $H_2(\alpha * \beta)([\mathbb{T}^2]) \in H_2(G)$.

For an equivariant 2-form ω^{eq} we conclude in particular that the corresponding cohomology class in $H^2_{\mathrm{sing}}(G, A)$ vanishes if and only if

$$q_A \circ \mathrm{per}_\omega = 0 \quad \text{and} \quad C^A_\omega = 0$$

(cf. Remark 6.9). This condition is weaker than $P_1([\omega]) = (q_A \circ \mathrm{per}_\omega, F_\omega) = 0$ (Example 6.10), but it already implies the existence of an abelian extension of G by the abelian group $\widehat{\pi}_1(G)$, even though A might not be isomorphic to $\widehat{\pi}_1(G)$ as G-modules (cf. Remark 6.9). ■

Remark 6.13. With similar arguments as in Section 4, resp. Section 5 of [Ne02], we can define a *toroidal period map* by observing that the integration map

$$\widetilde{\mathrm{per}}^{\mathbb{T}}_\omega \colon C^\infty(\mathbb{T}^2, G) \to \mathfrak{a}^G, \quad [\sigma] \mapsto \int_\sigma \omega^{\mathrm{eq}}$$

is constant on the connected components, hence defines a map

$$\mathrm{per}^{\mathbb{T}}_\omega \colon \pi_0(C^\infty(\mathbb{T}^2, G)) \cong [\mathbb{T}^2, G] \cong \pi_1(G) \times \pi_1(G) \times \pi_2(G) \to \mathfrak{a}$$

(cf. [MN03, Remark 1.11(b)], [Ne02, Th. A.3.7]). The restriction to $\pi_2(G)$, which corresponds to homotopy classes of maps vanishing on $(\mathbb{T} \times \{\mathbf{1}\}) \cup (\{\mathbf{1}\} \times \mathbb{T})$, is the period map $\mathrm{per}_\omega \colon \pi_2(G) \to \mathfrak{a}$. The elements of the subgroup

$$\pi_1(G) \times \pi_1(G) \subseteq \pi_0(C^\infty(\mathbb{T}^2, G))$$

are represented by maps of the form $\alpha * \beta$ with $\alpha, \beta \in C^\infty(\mathbb{T}, G)$, and from the proof of Proposition 6.11 we know that

$$\pi_1(G) \times \pi_1(G) \to \mathfrak{a}, \quad ([\alpha], [\beta]) \mapsto \mathrm{per}^{\mathbb{T}}_\omega([\alpha * \beta])$$

is biadditive. This implies in particular that, in general, $\mathrm{per}^{\mathbb{T}}_\omega$ is *not* a group homomorphism. The condition $q_A \circ \mathrm{per}^{\mathbb{T}}_\omega = 0$ means that $q_A \circ \mathrm{per}_\omega = 0$ and the commutator map C^A_ω vanish, which, for smoothly paracompact groups G, is equivalent to the vanishing of the cohomology class in $H^2_{\mathrm{sing}}(G, A)$ defined by the closed 2-form ω^{eq}. ■

Remark 6.14. If $A \cong \mathfrak{a}/\Gamma_A$, then $\mathfrak{a}^G = \mathfrak{a}^{\mathfrak{g}}$ is a closed subspace of $\mathfrak{a}$ containing Γ_A. Therefore

$$A/A^G \cong \mathfrak{b} := \mathfrak{a}/\mathfrak{a}^G$$

is a locally convex space which carries a natural smooth G-module structure. Note that the quotient space $\mathfrak{b}$ need not be sequentially complete if $\mathfrak{a}$ has this property. Nevertheless the construction in Section 5 leads to a group cocycle $f \in Z^2_s(G, \mathfrak{a}/\Pi_\omega)$ and since Π_ω is always contained in $\mathfrak{a}^G$ (Lemma 4.2), we obtain a group cocycle

$$f_1 \in Z^2_s(G, \mathfrak{b}) \quad \text{with} \quad Df_1 = \omega^{\mathfrak{b}} := q_{\mathfrak{b}} \circ \omega,$$

where $q_{\mathfrak{b}} \colon \mathfrak{a} \to \mathfrak{b}$ is the quotient map (Corollary 5.3). This leads to a Lie group extension

$$\mathfrak{b} \hookrightarrow \widehat{G} \twoheadrightarrow \widetilde{G}$$

with $\widehat{\mathfrak{g}} \cong \mathfrak{b} \oplus_{\omega^{\mathfrak{b}}} \mathfrak{g}$. Note that

$$\mathfrak{b} = \mathfrak{a}/\mathfrak{a}^G \cong B^1_c(\mathfrak{g}, \mathfrak{a}) \subseteq Z^1_c(\mathfrak{g}, \mathfrak{a}),$$

so that we may identify the quotient map $q_{\mathfrak{b}}$ with the coboundary map $d_{\mathfrak{g}} \colon \mathfrak{a} \to B^1_c(\mathfrak{g}, \mathfrak{a})$. This makes it easier to identify the corresponding flux cocycle.

In Proposition 10.4 we shall encounter examples of modules $\mathfrak{a}$ with $\mathfrak{a}^{\mathfrak{g}} = \{0\}$ for which the flux cocycle is non-trivial (this is the case for the module $\mathcal{F}_1$ of $\mathrm{Diff}(\mathbb{S}^1)_0$). Therefore one cannot expect $F_{\omega_{\mathfrak{b}}}$ to vanish. ■

7. An exact sequence for abelian Lie group extensions

Let G be a connected Lie group and A a smooth G-module of the form $A \cong \mathfrak{a}/\Gamma_A$, where $\Gamma_A \subseteq \mathfrak{a}$ is a discrete subgroup. The main result of the present section is an exact sequence relating the group homomorphism

$$D := D_2 \colon H^2_s(G, A) \to H^2_c(\mathfrak{g}, \mathfrak{a})$$

to the exact Inflation-Restriction Sequence associated to the normal subgroup $\pi_1(G) \cong \ker q_G$ of $\widetilde{G}$, where $q_G \colon \widetilde{G} \to G$ is the universal covering map (cf. Appendix D). The crucial information on $\mathrm{im}(D)$ has already been obtained in Theorem 6.7, so that it essentially remains to show that $\ker D$ coincides with the image of the connecting homomorphism $\delta \colon \mathrm{Hom}(\pi_1(G), A^G) \to H^2_s(G, A)$.

In the following we shall always consider A as a $\widetilde{G}$-module, where $g \in \widetilde{G}$ acts on A by $g.a := q_G(g).a$, so that $\pi_1(G)$ acts trivially.

Proposition 7.1. *Let G be a connected Lie group. For an abelian Lie group extension $A \hookrightarrow \widehat{G} \xrightarrow{q} G$ the following conditions are equivalent:*

(1) *There exists an open identity neighborhood $U \subseteq G$ and a smooth section $\sigma_U \colon U \to \widehat{G}$ of q with $\sigma_U(xy) = \sigma_U(x)\sigma_U(y)$ for $x, y, xy \in U$.*

(2) *$\widehat{G} \cong A \times_f G$, where $f \in Z^2_s(G, A)$ is constant 0 on an identity neighborhood in $G \times G$.*

(3) *There exists a homomorphism $\gamma \colon \pi_1(G) \to A^G$ and an isomorphism $\Phi \colon (A \rtimes \widetilde{G})/\Gamma(\gamma) \to \widehat{G}$ with $q\big(\Phi([\mathbf{1}, x])\big) = q_G(x)$ for $x \in \widetilde{G}$, where $\Gamma(\gamma) = \{(\gamma(d), d) \colon d \in \pi_1(G)\}$ is the graph of γ.*

Proof. (1) $\Leftrightarrow$ (2) follows directly from the definitions and Proposition 2.6.

(1) $\Rightarrow$ (3): We may w.l.o.g. assume that U is connected, $U = U^{-1}$, and that there exists a smooth section $\widetilde{\sigma} \colon U \to \widetilde{G}$ of the universal covering map q_G. Then

$$\sigma_U \circ q_G\,|_{\widetilde{\sigma}(U)} \colon \widetilde{\sigma}(U) \to \widehat{G}$$

extends uniquely to a smooth homomorphism $\varphi \colon \widetilde{G} \to \widehat{G}$ with $\varphi \circ \widetilde{\sigma} = \sigma_U$ and $q \circ \varphi = q_G$ ([Ne02, Lemma 2.1]; see also [HofMo98, Cor. A.2.26]). We define $\psi \colon A \rtimes \widetilde{G} \to \widehat{G}, (a, g) \mapsto a\varphi(g)$. Then ψ is a smooth group homomorphism which is a local diffeomorphism because

$$\psi(a, \widetilde{\sigma}(x)) = a\varphi(\widetilde{\sigma}(x)) = a\sigma_U(x) \quad \text{for} \quad x \in U, a \in A.$$

We conclude that ψ is a covering homomorphism. Moreover, ψ is surjective because its range is a subgroup of $\widehat{G}$ containing A and mapped surjectively by q onto G. This proves that

$$\widehat{G} \cong (A \rtimes \widetilde{G})/\ker\psi, \quad \ker\psi = \{(-\varphi(g), g) \colon g \in \varphi^{-1}(A)\}.$$

On the other hand, $\varphi^{-1}(A) = \ker(q \circ \varphi) = \ker q_G = \pi_1(G)$, so that

$$\ker\psi = \{(\gamma(d), d) \colon d \in \pi_1(G)\} = \Gamma(\gamma) \quad \text{for} \quad \gamma := -\varphi|_{\pi_1(G)}.$$

(3) $\Rightarrow$ (1) follows directly from the fact that the map $A \rtimes \widetilde{G} \to \widehat{G}$ is a covering morphism. ∎

For the following theorem we recall the definition of the period map per_ω (Section 4) and the flux homomorphism $F_\omega \colon \pi_1(G) \to H^1_c(\mathfrak{g}, \mathfrak{a})$ associated to $\omega \in Z^2_s(\mathfrak{g}, \mathfrak{a})$ (Proposition 6.3).

Theorem 7.2. *Let G be a connected Lie group, A a smooth G-module of the form $A \cong \mathfrak{a}/\Gamma_A$, where $\Gamma_A \subseteq \mathfrak{a}$ is a discrete subgroup of the Mackey complete locally convex space $\mathfrak{a}$ and $q_A \colon \mathfrak{a} \to A$ the quotient map. Then the map*

$$\widetilde{P} \colon Z^2_c(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Hom}\,\big(\pi_2(G), A\big) \times \mathrm{Hom}\,\big(\pi_1(G), H^1_c(\mathfrak{g}, \mathfrak{a})\big), \quad \widetilde{P}(\omega) = (q_A \circ \mathrm{per}_\omega, F_\omega)$$

factors through a homomorphism

$$P \colon H^2_c(\mathfrak{g}, \mathfrak{a}) \to \mathrm{Hom}\,\big(\pi_2(G), A\big) \times \mathrm{Hom}\,\big(\pi_1(G), H^1_c(\mathfrak{g}, \mathfrak{a})\big), \quad P([\omega]) = (q_A \circ \mathrm{per}_\omega, F_\omega)$$

and the following sequence is exact:

$$\mathbf{0} \to H^1_s(G,A) \xrightarrow{I} H^1_s(\widetilde{G},A) \xrightarrow{R} H^1\big(\pi_1(G),A\big)^G \cong \mathrm{Hom}\big(\pi_1(G),A^G\big) \xrightarrow{\delta}$$
$$\xrightarrow{\delta} H^2_s(G,A) \xrightarrow{D} H^2_c(\mathfrak{g},\mathfrak{a}) \xrightarrow{P} \mathrm{Hom}\big(\pi_2(G),A\big) \times \mathrm{Hom}\big(\pi_1(G),H^1_c(\mathfrak{g},\mathfrak{a})\big).$$

Here the map δ assigns to a group homomorphism $\gamma\colon \pi_1(G) \to A^G$ the quotient of the semi-direct product $A \rtimes \widetilde{G}$ by the graph $\{(\gamma(d),d)\colon d \in \pi_1(G)\}$ of γ, which is a discrete central subgroup.

Proof. First we verify that $\widetilde{P}$ vanishes on $B^2_c(\mathfrak{g},\mathfrak{a})$, so that the map P is well-defined. In Theorem 6.7 we have seen that $[\omega] \in \mathrm{im}(D)$ is equivalent to $\widetilde{P}(\omega) = 0$. If $[\omega] = 0$, then $\mathfrak{a}\oplus_\omega\mathfrak{g} \cong \mathfrak{a}\rtimes\mathfrak{g}$ and the semi-direct product $A\rtimes G$ is a corresponding extension of G by A, so that Theorem 6.7 leads to $\widetilde{P}(\omega) = 0$. As $\widetilde{P}$ is a group homomorphism, it factors to a homomorphism P on $H^2_c(\mathfrak{g},\mathfrak{a})$.

The exactness of the sequence in $H^1_s(G,A)$, $H^1_s(\widetilde{G},A)$ and $\mathrm{Hom}(\pi_1(G),A^G)$ follows from Example D.11(b) and the exactness in $H^2_c(\mathfrak{g},\mathfrak{a})$ from Theorem 6.7. It therefore remains to verify the exactness in $H^2_s(G,A)$.

First we need a more concrete interpretation of the map δ in terms of abelian extensions. Let $\gamma \in \mathrm{Hom}(\pi_1(G),A^G)$ and $f \in C^1_s(\widetilde{G},A)$ as in Lemma D.7, applied with $N = \pi_1(G)$ with $f(gd) = f(g) + \gamma(d)$ for $g \in \widetilde{G}, d \in \pi_1(G)$. Then the arguments in Remark D.10 show that the map

$$\Phi\colon A \times_{d_{\widetilde{G}}f} \widetilde{G} \to A \rtimes \widetilde{G}, \quad (a,g) \mapsto (a + f(g), g)$$

is a bijective group homomorphism. Since, in addition, Φ is a local diffeomorphism, it also is an isomorphism of Lie groups, and therefore the cocycle $\delta(f) := \overline{d_{\widetilde{G}}f} \in Z^2_s(G,A)$ satisfies

$$\begin{aligned} A \times_{\delta(f)} G &\cong (A \times_{d_{\widetilde{G}}f} \widetilde{G})/(\{\mathbf{0}\} \times \pi_1(G)) \cong (A \rtimes \widetilde{G})/\Phi(\{\mathbf{0}\} \times \pi_1(G)) \\ &\cong (A \rtimes \widetilde{G})/\{(d,\gamma(d))\colon d \in \pi_1(G)\}. \end{aligned}$$

Now the inclusion $\mathrm{im}(\delta) \subseteq \ker(D)$ follows from Proposition 7.1 because for a cocycle $f \in Z^2_s(G,A)$ vanishing in an identity neighborhood we clearly have $Df = 0$.

Conversely, let $f \in Z^2_s(G,A)$ be a locally smooth group cocycle for which $\omega := Df$ is a coboundary and let $q\colon \widehat{G} = A \times_f G \to G$ be a corresponding Lie group extension (Proposition 2.6). Then the Lie algebra extension $\widehat{\mathfrak{g}} \cong \mathfrak{a} \oplus_\omega \mathfrak{g} \to \mathfrak{g}$ splits, and there exists a continuous projection $p_{\mathfrak{a}}\colon \widehat{\mathfrak{g}} \to \mathfrak{a}$ whose kernel is a closed subalgebra isomorphic to $\mathfrak{g}$. Considering $p_{\mathfrak{a}}$ as an element of $C^1_c(\widehat{\mathfrak{g}},\mathfrak{a})$, we have

$$(d_{\mathfrak{g}}p_{\mathfrak{a}})(x,y) = x.p_{\mathfrak{a}}(y) - y.p_{\mathfrak{a}}(x) - p_{\mathfrak{a}}([x,y]) = p_{\mathfrak{a}}([x - p_{\mathfrak{a}}(x), p_{\mathfrak{a}}(y) - y]) = 0,$$

for $x,y \in \widehat{\mathfrak{g}}$, so that $p_{\mathfrak{a}} \in Z^1_c(\widehat{\mathfrak{g}},\mathfrak{a})$. Let $q_{\widehat{G}}\colon G^\sharp \to \widehat{G}$ denote the universal covering group of $\widehat{G}$. Then the corresponding equivariant 1-form $p^{\mathrm{eq}}_{\mathfrak{a}}$ on $G^\sharp$ is closed (Lemma B.5), so that we find with [Ne02, Prop. 3.9] a smooth function

$$\varphi\colon G^\sharp \to \mathfrak{a} \quad \text{with} \quad \varphi(\mathbf{1}) = 0 \quad \text{and} \quad d\varphi = p^{\mathrm{eq}}_{\mathfrak{a}},$$

and Lemma 3.2 implies that $\varphi \in Z^1_s(\widehat{G}, \mathfrak{a})$ is a group cocycle.

Using the local description of $\widehat{G}$, resp., $G^\sharp$ by a 2-cocycle, we see that the inclusion map $A_0 \hookrightarrow \widehat{G}$ of the identity component of A lifts to a Lie group morphism $\eta_\mathfrak{a} \colon \mathfrak{a} \to G^\sharp$ whose differential is the inclusion $\mathfrak{a} \hookrightarrow \widehat{\mathfrak{g}}$. Since $p_\mathfrak{a}|_\mathfrak{a} = \mathrm{id}_\mathfrak{a}$ and the image of $\eta_\mathfrak{a}$ acts trivially on $\mathfrak{a}$, the composition $\varphi \circ \eta_\mathfrak{a} \colon \mathfrak{a} \to \mathfrak{a}$ is a morphism of Lie groups whose differential is $\mathrm{id}_\mathfrak{a}$, which implies that $\varphi \circ \eta_\mathfrak{a} = \mathrm{id}_\mathfrak{a}$. Moreover, the cocycle condition implies that

$$\varphi(ag) = \varphi(a) + \varphi(g), \quad a \in \eta_\mathfrak{a}(\mathfrak{a}), g \in G^\sharp. \tag{7.1}$$

Let $U \subseteq G$ be a connected open identity neighborhood on which there exists a smooth section $\sigma \colon U \to G^\sharp$ of the quotient map $q^\sharp := q \circ q_{\widehat{G}} \colon G^\sharp \to G$. We then obtain another smooth map by

$$\sigma_1 \colon U \to G^\sharp, \quad x \mapsto \eta_\mathfrak{a}(\varphi(\sigma(x))^{-1})\sigma(x).$$

In view of (7.1), this map is also a section of $q^\sharp$. Moreover, $\mathrm{im}(\sigma_1) \subseteq \varphi^{-1}(0)$.

From the description of $\widehat{G}$ with the cocycle f it follows that there exists an open $\mathbf{1}$-neighborhood in $G^\sharp$ of the form

$$U^\sharp := \eta_\mathfrak{a}(U_\mathfrak{a})\sigma_1(U),$$

where $U_\mathfrak{a} \subseteq \mathfrak{a}$ is an open 0-neighborhood. Restricting φ to $U^\sharp$, we see that $\sigma_1(U) = \varphi^{-1}(0) \cap U^\sharp$. Since $\varphi^{-1}(0)$ is a subgroup of $G^\sharp$, we have

$$(\sigma_1(U)\sigma_1(U)) \cap U^\sharp \subseteq \sigma_1(U).$$

Let $V \subseteq U$ be an open connected symmetric $\mathbf{1}$-neighborhood in G such that there exists a smooth section $\sigma_V \colon V \to \widetilde{G}$ of the universal covering map $q_G \colon \widetilde{G} \to G$ and, in addition, $VV \subseteq U$ and $\sigma_1(V)\sigma_1(V) \subseteq U^\sharp$. For $x, y \in V$ we then have $xy \in U$, and $\sigma_1(x)\sigma_1(y) \in U^\sharp$ implies the existence of $z \in U$ with $\sigma_1(z) = \sigma_1(x)\sigma_1(y)$. Applying $q^\sharp$ to both sides leads to

$$z = q^\sharp\sigma_1(z) = q^\sharp(\sigma_1(x)\sigma_1(y)) = xy.$$

We therefore have

$$\sigma_1(xy) = \sigma_1(x)\sigma_1(y) \quad \text{for} \quad x, y \in V.$$

Hence there exists a unique group homomorphism $f \colon \widetilde{G} \to G^\sharp$ with $f \circ \sigma_V = \sigma_1$ ([HofMo98, Cor. A.2.26]). Composing f with the covering map $q_{\widehat{G}} \colon G^\sharp \to \widehat{G}$, we obtain a smooth homomorphism $\widehat{f} \colon \widetilde{G} \to \widehat{G}$ with $q \circ \widehat{f} = q_G$. According to Proposition 7.1, this implies that $\widehat{G}$ is isomorphic to a group of the type $(A \rtimes \widetilde{G})/\Gamma(\gamma)$, where $\gamma \colon \pi_1(G) \to A^G$ is a group homomorphism. ∎

Since the fundamental group $\pi_1(\widetilde{G})$ vanishes, we obtain in particular:

Corollary 7.3. *The map* $D_2\colon H^2_s(\widetilde{G},A)\to H^2_s(\mathfrak{g},\mathfrak{a})$ *is injective.* ∎

Remark 7.4. In view of Corollary 7.3, we may identify $H^2_s(\widetilde{G},A)$ with a subgroup of $H^2_c(\mathfrak{g},\mathfrak{a})$. Then the inflation map

$$I\colon H^2_s(G,A)\to H^2_s(\widetilde{G},A) \quad \text{satisfies} \quad \widetilde{D}_2^{\widetilde{G}}\circ I = D_2^G\colon H^2_s(G,A)\to H^2_c(\mathfrak{g},\mathfrak{a}).$$ ∎

Remark 7.5. At first sight, the following argument seems to be more natural to show in the proof of Theorem 7.2 that $\ker D\subseteq \operatorname{im}\delta$: If the group $\widehat{G}$ is regular (cf. [Mil83]), then the Lie algebra morphism $\sigma\colon\mathfrak{g}\to\widehat{\mathfrak{g}}$ whose existence is guaranteed by $[Df]=0$ can be integrated to a Lie group morphism $\widetilde{G}\to\widehat{G}$, and we can argue as above. Unfortunately this argument requires the regularity of the group $\widehat{G}$, which is not needed for the argument given above. ∎

Although the following example is concerned with a finite-dimensional group G, it demonstrates quite nicely the difficulties arising for smooth modules with are neither connected nor simply connected.

Example 7.6. Let G be a connected finite-dimensional Lie group, $\mathfrak{z}$ a Fréchet space, $\Gamma_Z\subseteq\mathfrak{z}$ a discrete subgroup and $Z:=\mathfrak{z}/\Gamma_Z$. Then $\mathfrak{a}:=C^\infty(G,\mathfrak{z})$ also is a Fréchet space and $(g.f)(x):=f(xg)$ defines a smooth action of G on $\mathfrak{a}$ ([Ne01, Th. III.5]; note that $C^\infty(G,\mathfrak{z})$ is Fréchet and therefore metrizable, which is needed for the proof in loc. cit., although it is not stated there explicitly). We endow the abelian group $A:=C^\infty(M,Z)$ with the Lie group structure for which $A_0:=q_Z\circ C^\infty(M,\mathfrak{z})$ is an open subgroup isomorphic to the quotient group $\mathfrak{a}/\Gamma_Z=C^\infty(M,\mathfrak{z})/\Gamma_Z$, which is a Lie group because Γ_Z is discrete in the closed subspace $\mathfrak{z}$ of $\mathfrak{a}$, hence also discrete in $\mathfrak{a}$. It is clear from the construction that G acts smoothly on the identity component A_0 of A and further that each element of G acts as a smooth automorphism on the whole group A. To see that G acts smoothly on A, it remains to show that all orbit maps are smooth in the identity. For $f\in A$ and $g\in G$ the connectedness of G implies that $g.f-f=f\circ\rho_g-f\in A_0$, so that we have to verify that the map $G\to A_0, g\mapsto g.f-f$ is smooth. Let $\delta^l(f):=f^{-1}.df$ denote the left logarithmic derivative of f, which is a closed $\mathfrak{z}$-valued 1-form on G with periods in Γ_Z, so that we may write

$$(g.f-f)(x)=f(xg)-f(x)=q_Z\Big(\int_x^{gx}\delta^l(f)\Big)=q_Z\Big(\int_0^1\delta^l(f)(x\gamma_g(t))(x.\gamma_g'(t))\,dt\Big)$$

where $\gamma_g\colon[0,1]\to G$ is a smooth path from $\mathbf{1}$ to g. Locally we may choose γ_g in such a way that it depends smoothly on g (cf. Lemma 5.2), which implies that the orbit map of f is smooth in $\mathbf{1}$ and hence that G acts smoothly on A.

Since $\mathfrak{g}\to\mathcal{V}(G), x\mapsto x_l$ $(x_l(g)=g.x)$ is a homomorphism of Lie algebras, the map

$$\Phi\colon\Omega^p(G,\mathfrak{z})\to C^p_c(\mathfrak{g},\mathfrak{a}),\quad \Phi(\alpha)(x_1,\dots,x_p):=\alpha(x_{1,l},\dots,x_{p,l}),\quad x_l(g)=g.x$$

is a morphism of chain complexes, i.e., $d_{\mathfrak g} \circ \Phi = \Phi \circ d$. On the other hand, we obtain from each $\mathfrak a$-valued p-form α on G a $\mathfrak z$-valued p-form $\mathrm{ev}_{\mathbf 1}^{\mathfrak a} \circ \alpha$ by composing it with the evaluation map $\mathrm{ev}_{\mathbf 1}^{\mathfrak a} \colon \mathfrak a \to \mathfrak z, f \mapsto f(\mathbf 1)$. Then

$$\Psi := \mathrm{ev}_{\mathbf 1}^{\mathfrak a} \circ \mathrm{Eq} \colon C^p_c(\mathfrak g, \mathfrak a) \to \Omega^p(G, \mathfrak z)$$

satisfies

$$\begin{aligned}\Psi(\alpha)(x_{1,l}, \ldots, x_{p,l})(h) &= \mathrm{ev}_{\mathbf 1}^{\mathfrak a} \circ \alpha^{\mathrm{eq}}(x_{1,l}, \ldots, x_{p,l})(h) = \mathrm{ev}_{\mathbf 1}^{\mathfrak a}\big(h.\alpha(x_1, \ldots, x_p)\big) \\ &= \mathrm{ev}_{\mathbf 1}^{\mathfrak a}\big(\alpha(x_1, \ldots, x_p) \circ \rho_h\big) = \alpha(x_1, \ldots, x_p)(h),\end{aligned}$$

which leads to $\Psi \circ \Phi = \mathrm{id}$ and $\Phi \circ \Psi = \mathrm{id}$. Therefore Ψ induces isomorphisms

$$\Psi \colon H^p_c(\mathfrak g, \mathfrak a) \to H^p_{\mathrm{dR}}(G, \mathfrak z), \tag{7.2}$$

indentifying $\mathfrak a$-valued Lie algebra cohomology with $\mathfrak z$-valued de Rham cohomology.

For $g \in G$, $\alpha \in C^p_c(\mathfrak g, \mathfrak a)$ and $\alpha_G := \Psi(\alpha) \in \Omega^p(G, \mathfrak z)$ we have the relation

$$\lambda_g^* \alpha_G = \mathrm{ev}_g^{\mathfrak a} \circ \alpha^{\mathrm{eq}},$$

which follows directly from

$$\begin{aligned}(\lambda_g^* \alpha_G)(x_{1,l}, \ldots, x_{p,l})(h) &= (\alpha_G(x_{1,l}, \ldots, x_{p,l}))(gh) = \mathrm{ev}_g^{\mathfrak a}(h.\alpha(x_1, \ldots, x_p)) \\ &= \mathrm{ev}_g^{\mathfrak a} \circ \alpha^{\mathrm{eq}}(x_{1,l}, \ldots, x_{p,l})(h).\end{aligned}$$

If M is a smooth oriented p-dimensional compact manifold, then we thus obtain for a smooth map $\gamma \colon M \to G$ and $\alpha \in Z^p_c(\mathfrak g, \mathfrak a)$:

$$\Big(\int_\gamma \alpha^{\mathrm{eq}}\Big)(g) = \int_M \mathrm{ev}_g^{\mathfrak a} \circ \gamma^* \alpha^{\mathrm{eq}} = \int_M (\lambda_g \circ \gamma)^* \alpha_G = \int_{\lambda_g \circ \gamma} \alpha_G = \int_\gamma \alpha_G \in \mathfrak z = \mathfrak a^G.$$

This shows that the period map $\mathrm{per}_\alpha \colon \pi_p(G) \to \mathfrak a^G$ of α coincides with the period map per_{α_G} of the closed $\mathfrak z$-valued p-form α_G. With Proposition 3.4 we now obtain

$$H^1_s(G, A_0) \cong \ker P_1 \cong H^1_{\mathrm{dR}}(G, \Gamma_Z) \cong \mathrm{Hom}(\pi_1(G), \Gamma_Z) \cong H^1_{\mathrm{sing}}(G, \Gamma_Z). \tag{7.3}$$

The characteristic homomorphism of A is given by

$$\overline{\theta}_A \colon \pi_0(A) = [G, Z] \to H^1_c(\mathfrak g, \mathfrak a) \cong H^1_{\mathrm{dR}}(G, \mathfrak z), \quad [f] \mapsto D_1[d_G f] = [\delta^l(f)] = [f^{-1} df].$$

It is injective with image $\ker P_1 = H^1_{\mathrm{dR}}(G, \Gamma_Z)$. This implies that $Z^1_s(G, A_0) = Z^1_s(G, A) \subseteq d_G A$, and therefore

$$H^1_s(G, A) = \mathbf 0. \tag{7.4}$$

We now turn to $H^2_s(G, A)$. Since $\pi_2(G)$ vanishes ([Car52]), Proposition 6.11 shows that $H_2(G)$ is generated by homology classes defined by maps $\mathbb T^2 \to G$ of the form $[\alpha] * [\beta] := [\alpha * \beta]$, where $\alpha, \beta \colon \mathbb{T} \to G$ are piecewise smooth loops. For $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$ and the corresponding closed $\mathfrak{z}$-valued 2-form $\omega_G = \Psi(\omega)$ the flux homomorphism $F_\omega \colon \pi_1(G) \to H^1_c(\mathfrak{g}, \mathfrak{a}) \cong H^1_{\rm dR}(G, \mathfrak{z})$ satisfies

$$F_\omega([\alpha])([\beta]) = -C^{\mathfrak{a}}_\omega([\beta], [\alpha]) = C^{\mathfrak{a}}_\omega([\alpha], [\beta]) = \int_{\beta * \alpha} \omega^{\rm eq} = \int_{\beta * \alpha} \omega_G. \tag{7.5}$$

In view of

$$\begin{aligned} \mathrm{Hom}(\pi_1(G), H^1_c(\mathfrak{g}, \mathfrak{a})) &\cong \mathrm{Hom}(\pi_1(G), H^1_{\rm dR}(G, \mathfrak{z})) \cong \mathrm{Hom}(\pi_1(G), \mathrm{Hom}(\pi_1(G), \mathfrak{z})) \\ &\cong \mathrm{Hom}(\pi_1(G) \otimes \pi_1(G), \mathfrak{z}), \end{aligned}$$

this shows that the map

$$P_2 \colon H^2_c(\mathfrak{g}, \mathfrak{a}) \cong H^2_{\rm dR}(G, \mathfrak{z}) \cong \mathrm{Hom}(H_2(G), \mathfrak{z}) \to \mathrm{Hom}(\pi_1(G), H^1_c(\mathfrak{g}, \mathfrak{a})), \quad [\omega] \mapsto F_\omega$$

is injective because the alternating biadditive map

$$\pi_1(G) \otimes \pi_1(G) \to H_2(G), \quad [\alpha] \otimes [\beta] \mapsto [\alpha * \beta]$$

is surjective. This in turn implies that $D_2(H^2_s(G, A_0)) = \{0\}$, so that the sequence

$$H^1_s(\widetilde{G}, A_0) \to \mathrm{Hom}(\pi_1(G), A^G) = \mathrm{Hom}(\pi_1(G), Z) \to H^2_s(G, A_0) \to \mathbf{0}$$

is exact (Theorem 7.2). From

$$H^1_s(\widetilde{G}, A_0) \cong H^1_c(\mathfrak{g}, \mathfrak{a}) \cong H^1_{\rm dR}(G, \mathfrak{z}) \cong \mathrm{Hom}(\pi_1(G), \mathfrak{z})$$

we thus get

$$H^2_s(G, A_0) \cong \mathrm{Hom}(\pi_1(G), Z)/q_Z \circ \mathrm{Hom}(\pi_1(G), \mathfrak{z}) \cong \mathrm{Ext}_{\rm ab}(\pi_1(G), \Gamma_Z)$$

because the divisibility of $\mathfrak{z}$ implies $\mathrm{Ext}_{\rm ab}(\pi_1(G), \mathfrak{z}) = \mathbf{0}$.

To calculate $H^2_s(G, A)$, we use the exact sequence from Appendix E:

(7.6)

$$H^1_s(G, \pi_0(A)) \to H^2_s(G, A_0) \to H^2_s(G, A) \to H^2_s(G, \pi_0(A)) \to H^3_s(G, A_0) \to \ldots.$$

Since G is connected, $Z^1_s(G, \pi_0(A))$ is trivial and thus $H^1_s(G, \pi_0(A))$ vanishes, so that we have an injection $H^2_s(G, A_0) \hookrightarrow H^2_s(G, A)$. From our description of $H^2_s(G, A_0)$, it follows that the image of this injection coincides with the image of the connecting map $H^1(\pi_1(G), A)^{[G]} = \mathrm{Hom}(\pi_1(G), Z) \to H^2_s(G, A)$. We likewise have $H^1_s(\widetilde{G}, \pi_0(A)) = \mathbf{0}$, and Theorem 7.2 implies that

$$\begin{aligned} H^2_s(G, \pi_0(A)) &\cong \mathrm{Hom}(\pi_1(G), \pi_0(A)) \cong \mathrm{Hom}(\pi_1(G), \mathrm{Hom}(\pi_1(G), \Gamma_Z)) \\ &\cong \mathrm{Hom}(\pi_1(G) \otimes \pi_1(G), \Gamma_Z). \end{aligned}$$

If $q\colon \widehat{G} \to G$ is an A-extension of G, then $\widehat{G}/A_0$ is the corresponding extension of G by $\pi_0(A)$, which is a covering, hence given by a homomorphism $\pi_1(G) \to \pi_0(A)$, which coincides with the corresponding connecting homomorphism in the long exact homotopy sequence of the A-principal bundle $\widehat{G}$. Therefore the map $H^2_s(G,A) \to H^2_s(G,\pi_0(A)) \cong \mathrm{Hom}(\pi_1(G),\pi_0(A))$ assigns to an A-extension the corresponding connecting map δ, which satisfies $F_\omega = -\overline{\theta}_A \circ \delta$ if $\omega \in Z^2_c(\mathfrak{g},\mathfrak{a})$ is the corresponding Lie algebra cocycle (Proposition 6.2(2)).

Next we use the Integrability Criterion from Theorem 6.7. Since the characteristic homomorphism $\overline{\theta}_A$ is injective, for $\omega \in Z^2_c(\mathfrak{g},\mathfrak{a}) \cong H^2_{\mathrm{dR}}(G,\mathfrak{z})$ there exists a homomorphism $\gamma\colon \pi_1(G) \to \pi_0(A)$ with $F_\omega = \overline{\theta}_A \circ \gamma$ if and only if $\mathrm{im}(F_\omega) \subseteq \mathrm{im}(\overline{\theta}_A) = H^1_{\mathrm{dR}}(G,\Gamma_Z)$, and in this case $-F_\omega$, considered as a map $\pi_1(G) \to \pi_0(A)$, is the connecting map of the corresponding extension (Proposition 6.2(2)). In view of (7.5), this means that the Lie algebra cocycle ω is integrable if and only if $\omega_G \in H^2_{\mathrm{dR}}(G,\Gamma_Z)$ in the sense that all periods of the 2-form ω_G are contained in Γ_Z. Identifying $\mathrm{Hom}(\pi_1(G),\pi_0(A))$ with $\mathrm{Hom}(\pi_1(G)\otimes\pi_1(G),\Gamma_Z)$, the corresponding connecting map corresponds to the commutator map $C^{\mathfrak{a}}_\omega$, which is alternating. We conclude that a homomorphism $\pi_1(G)\otimes\pi_1(G) \to \Gamma_Z$ is a connecting map of an A-extension of G if and only if it factors through $*\colon \pi_1(G)\otimes\pi_1(G) \to H_2(G)$ to a homomorphism $H_2(G) \to \Gamma_Z$ given by integrating against a closed 2-form with periods in Γ_Z. Combining all this, we get an exact sequence

$$\mathbf{0} \to \mathrm{Ext}_{\mathrm{ab}}(\pi_1(G),\Gamma_Z) \to H^2_s(G,A) \to \mathrm{Hom}(H_2(G),\Gamma_Z) \to \mathbf{0}.$$

From [Ne02, Rem. 9.5(e)] we know that discrete subgroups of separable locally convex spaces are free. Therefore Γ_Z is free if $\mathfrak{z}$ is separable. If this is the case, then the fact that $H_2(G)$ is finitely generated implies that $\mathrm{Hom}(H_2(G),\Gamma_Z) \cong \Gamma_Z^{b_2(G)}$ is also free, so that the above sequence splits, and we get from the Universal Coefficient Theorem

$$H^2_s(G,A) \cong \mathrm{Ext}_{\mathrm{ab}}(\pi_1(G),\Gamma_Z) \oplus \mathrm{Hom}(H_2(G),\Gamma_Z) \cong H^2_{\mathrm{sing}}(G,\Gamma_Z). \tag{7.7}$$

Finally we observe that the homomorphism

$$\mathrm{Hom}(H_2(G),\Gamma_Z) \to \mathrm{Hom}(\pi_1(G)\otimes\pi_1(G),\Gamma_Z) \cong H^2_s(G,\pi_0(A))$$

is not surjective, which is due to the fact that the map $\pi_1(G)\otimes\pi_1(G) \to H_2(G)$ is alternating. This implies that the map

$$\mathrm{Hom}(\pi_1(G)\otimes\pi_1(G),\Gamma_Z) \to H^3_s(G,A_0)$$

obtained from (7.6) is non-zero.

To make this more explicit, let $K \subseteq G$ be a maximal compact subgroup. Then there exists a torus T with $K \cong (K,K) \rtimes T$ and $d := \dim T = \dim Z(K)$.

Since the inclusion $K \hookrightarrow G$ is a homotopy equivalence and all homology groups of T are free, we derive from the Künneth Theorem

$$H_2(G) \cong H_2(K) \cong H_2(T) \oplus H_1(T) \otimes H_1((K,K)) \oplus H_2((K,K)),$$

where the latter two summands are finite groups. Likewise $\pi_1(G) \cong \pi_1(T) \oplus \pi_1((K,K)) \cong \pi_1(T) \oplus H_1((K,K))$. Therefore $H_1(T) \cong \mathbb{Z}^d$ and $H_2(T) \cong \mathbb{Z}^{\binom{d}{2}}$ lead to

$$\mathrm{Hom}(H_2(G), \Gamma_Z) = \mathrm{Hom}(H_2(T), \Gamma_Z) \cong \Gamma_Z^{\binom{d}{2}}$$

and

$$\mathrm{Hom}(\pi_1(G) \otimes \pi_1(G), \Gamma_Z) = \mathrm{Hom}(\pi_1(T) \otimes \pi_1(T), \Gamma_Z) \cong \Gamma_Z^{d^2}.$$

We therefore obtain an injection

$$\Gamma_Z^{\binom{d+1}{2}} \hookrightarrow H^3_s(G, A_0).$$

It would be interesting to calculate the higher cohomology groups $H^n_s(G,A)$ and $H^n_s(G,A_0)$ explicitly, but for that one needs different tools. With van Est's Theorem, in the version of [HocMo62], one gets

$$H^n_{gs}(G, \mathfrak{a}) \cong H^n_c(\mathfrak{g}, \mathfrak{k}, \mathfrak{a}) \cong H^n_{\mathrm{dR}}(G/K, \mathfrak{z}) = \mathbf{0}$$

for $n > 0$, where $H^n_{gs}(G, \mathfrak{a})$ denotes the cohomology defined by the globally smooth cochains, but this provides not enough information on the groups $H^n_s(G, \mathfrak{a})$ defined by the locally smooth cochains. ■

8. Abelian extensions with smooth global sections

In this subsection we discuss the existence of a smooth cross section for an abelian Lie group extension $A \hookrightarrow \widehat{G} \twoheadrightarrow G$ which is equivalent to the existence of a smooth global cocycle $f \colon G \times G \to A$ with $\widehat{G} \cong G \times_f A$. Moreover, we will show that for simply connected groups, it is equivalent to the exactness of the equivariant 2-form ω^{eq} on G, where $\omega = Df$.

The following lemma will be helpful in the proof of Proposition 8.2.

Lemma 8.1. *Let G be a connected Lie group, A a smooth G-module and $f \in Z^2_s(G,A)$ such that all functions $f_g \colon G \to A, x \mapsto f(g,x)$ are smooth. Then $f \colon G \times G \to A$ is a smooth function.*

Proof. We write the cocycle condition as

$$f(xy,z) = f(x,yz) + \rho_A(x).f(y,z) - f(x,y), \quad x,y,z \in G.$$

For x fixed, this function is smooth as a function of the pair (y,z) in a neighborhood of $(\mathbf{1},\mathbf{1})$. This implies that f is smooth on a neighborhood of the points

$(x, \mathbf{1})$, $x \in G$. Fixing x and z shows that there exists a $\mathbf{1}$-neighborhood $V \subseteq G$ (independent of x) such that the functions $f(\cdot, z)$, $z \in V$, are smooth in a neighborhood of x. Since $x \in G$ was arbitrary, we conclude that the functions $f(\cdot, z)$, $z \in V$, are smooth. Now

$$f(\cdot, yz) = f(\cdot y, z) - \rho_A(\cdot).f(y, z) + f(\cdot, y)$$

shows that the same holds for the functions $f(\cdot, u)$, $u \in V^2$. Iterating this process, using $G = \bigcup_{n\in\mathbb{N}} V^n$, we derive that all functions $f(\cdot, x)$, $x \in G$, are smooth. Finally we see that the function

$$(x, y) \mapsto f(x, yz) = f(xy, z) - \rho_A(x).f(y, z) + f(x, y)$$

is smooth in a neighborhood of each point $(x_0, \mathbf{1})$, hence that f is smooth in each point (x_0, z_0), and this proves that f is smooth on $G \times G$. ■

Proposition 8.2. *Let G be a connected Lie group, $\mathfrak{a}$ a Mackey complete locally convex smooth G-module, $\omega \in Z^2_c(\mathfrak{g}, \mathfrak{a})$ a continuous 2-cocycle, and $\omega^{\rm eq} \in \Omega^2(G, \mathfrak{a})$ the corresponding equivariant 2-form on G with $\omega^{\rm eq}_{\mathbf{1}} = \omega$. We assume that*

(1) *$\omega^{\rm eq} = d\theta$ for some $\theta \in \Omega^1(G, \mathfrak{a})$ and*

(2) *for each $g \in G$ the closed 1-form $\lambda_g^*\theta - \rho_{\mathfrak{a}}(g) \circ \theta$ is exact.*

Then the product manifold $\widehat{G} := \mathfrak{a} \times G$ carries a Lie group structure which is given by a smooth 2-cocycle $f \in Z^2_s(G, \mathfrak{a})$ with $D[f] = [\omega]$ via

$$(a, g)(a', g') := (a + g.a' + f(g, g'), gg').$$

Proof. For each $g \in G$ the relation $\rho_{\mathfrak{a}}(g) \circ \omega^{\rm eq} = \lambda_g^*\omega^{\rm eq}$ implies

$$d\big(\rho_{\mathfrak{a}}(g) \circ \theta - \lambda_g^*\theta\big) = \rho_{\mathfrak{a}}(g) \circ \omega^{\rm eq} - \lambda_g^*\omega^{\rm eq} = 0.$$

In view of (2), for each $g \in G$ there exists a smooth function $f_g \colon G \to \mathfrak{a}$ with $f_g(\mathbf{1}) = 0$ and

$$df_g = \lambda_g^*\theta - \rho_{\mathfrak{a}}(g) \circ \theta.$$

Observe that $f_{\mathbf{1}} = 0$. For $g, h \in G$ this leads to

$$\begin{aligned} df_{gh} &= \lambda_{gh}^*\theta - \rho_{\mathfrak{a}}(gh) \circ \theta = \lambda_h^*(\lambda_g^*\theta - \rho_{\mathfrak{a}}(g) \circ \theta) + \lambda_h^*(\rho_{\mathfrak{a}}(g) \circ \theta) - \rho_{\mathfrak{a}}(gh) \circ \theta \\ &= \lambda_h^* df_g + \rho_{\mathfrak{a}}(g)(\lambda_h^*\theta - \rho_{\mathfrak{a}}(h) \circ \theta) = \lambda_h^* df_g + \rho_{\mathfrak{a}}(g) \circ df_h \\ &= d(f_g \circ \lambda_h + \rho_{\mathfrak{a}}(g) \circ f_h). \end{aligned}$$

Comparing values of both functions in $\mathbf{1}$, we get

$$f_{gh} = f_g \circ \lambda_h + \rho_{\mathfrak{a}}(g) \circ f_h - f_g(h). \tag{8.1}$$

Now we define $f \colon G \times G \to \mathfrak{a}$ by $f(x, y) := f_x(y)$. Then (8.1) means that

$$f(gh, u) = f(g, hu) + \rho_{\mathfrak{a}}(g).f(h, u) - f(g, h), \quad g, h, u \in G,$$

i.e., f is a group cocycle.

Moreover, the concrete local formula for f_x in the Poincaré Lemma ([Ne02, Lemma 3.3]) and the smooth dependence of the integral on x imply that f is smooth on a neighborhood of $(\mathbf{1},\mathbf{1})$, so that Lemma 8.1 implies that $f\colon G\times G \to \mathfrak{a}$ is a smooth function. We therefore obtain on the space $\widehat{G} := \mathfrak{a} \times G$ a Lie group structure with the multiplication given by

$$(a,g)(a',g') := (a + g.a' + f(g,g'), gg')$$

(Lemma 2.1), and Lemma 2.7 implies that the corresponding Lie bracket is given by

$$[(a,x),(a',x')] = \big(x.a' - x'.a + d^2f(\mathbf{1},\mathbf{1})(x,x') - d^2f(\mathbf{1},\mathbf{1})(x',x), [x,x']\big).$$

Now we relate this formula to the Lie algebra cocycle ω. The relation $df_g = \lambda_g^*\theta - \rho_{\mathfrak{a}}(g)\circ\theta$ leads to

$$df(g,\mathbf{1})(0,y) = df_g(\mathbf{1})y = (\lambda_g^*\theta - \rho_{\mathfrak{a}}(g)\circ\theta)_{\mathbf{1}}(y) = \langle\theta, y_l\rangle(g) - \rho_{\mathfrak{a}}(g).\theta_{\mathbf{1}}(y),$$

where y_l denotes the left invariant vector field with $y_l(\mathbf{1}) = y$. Taking second derivatives, we further obtain for $x \in \mathfrak{g}$:

$$\begin{aligned} d^2f(\mathbf{1},\mathbf{1})(x,y) &= x_l(\langle\theta,y_l\rangle)(\mathbf{1}) - x.\theta_{\mathbf{1}}(y) = (d\theta)(x_l,y_l)(\mathbf{1}) + y_l(\langle\theta,x_l\rangle)(\mathbf{1}) \\ &\qquad\qquad + \theta([x_l,y_l])(\mathbf{1}) - x.\theta_{\mathbf{1}}(y) \\ &= \omega(x,y) + y_l(\langle\theta,x_l\rangle)(\mathbf{1}) + \theta_{\mathbf{1}}([x,y]) - x.\theta_{\mathbf{1}}(y), \end{aligned}$$

Subtracting $d^2f(\mathbf{1},\mathbf{1})(y,x) = y_l(\langle\theta,x_l\rangle)(\mathbf{1}) - y.\theta_{\mathbf{1}}(x)$, leads to

$$(Df)(x,y) = \omega(x,y) + \theta_{\mathbf{1}}([x,y]) - x.\theta_{\mathbf{1}}(y) + y.\theta_{\mathbf{1}}(x) = \omega(x,y) - (d_{\mathfrak{g}}\theta_{\mathbf{1}})(x,y).$$

Since this cocycle is equivalent to ω, the assertion follows. ∎

Using the methods developed in [NV02], it is not hard to show that condition (2) in Proposition 8.2 is equivalent to:

(2') for each $x \in \mathfrak{g}$ the closed 1-form $\mathcal{L}_{x_r}\theta - \dot\rho_{\mathfrak{a}}(x)\circ\theta$ is exact.

In view of $\mathcal{L}_{x_r}\theta = di_{x_r}\theta + i_{x_r}\omega^{\mathrm{eq}}$, this means that $[i_{x_r}\omega^{\mathrm{eq}}] = [\dot\rho_{\mathfrak{a}}(x)\circ\theta]$ in $\widehat{H}^1_{\mathrm{dR}}(G,\mathfrak{a})$.

Corollary 8.3. *If G is simply connected and ω^{eq} is exact, then there exists a smooth cocycle $f\colon G\times G\to\mathfrak{a}$ with $D[f]=[\omega]$, so that $\widehat{G} := \mathfrak{a}\times_f G$ is a Lie group with Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{a}\oplus_\omega\mathfrak{g}$.*

Proof. Since $\pi_1(G)$ is trivial, condition (2) in Proposition 8.2 is automatically satisfied. ∎

For central extensions of finite-dimensional groups, the construction described in Proposition 8.2 is due to E. Cartan, who used it to construct a central extension of a simply connected finite-dimensional Lie group G by the group $\mathfrak{a}$. Since in this case

$$H^2_{\rm dR}(G,\mathfrak{a}) \cong \mathrm{Hom}(\pi_2(G),\mathfrak{a}) = \mathbf{0} \quad \text{and} \quad H^1_{\rm dR}(G,\mathfrak{a}) \cong \mathrm{Hom}(\pi_1(G),\mathfrak{a}) = \mathbf{0},$$

(cf. [God71]), the requirements of the construction are satisfied for every Lie algebra cocycle $\omega \in Z^2_c(\mathfrak{g},\mathfrak{a})$.

Proposition 8.4. *If G is a connected Lie group which is smoothly paracompact, then the conclusion of* Proposition 8.2 *remains valid under the assumptions:*
(1) $\omega^{\rm eq}$ *is an exact 2-form, and*
(2) $F_\omega = 0$.

Proof. In view of (1), we can apply Proposition 8.2 to the universal covering group $q_G \colon \widetilde{G} \to G$ of G, which leads to an $\mathfrak{a}$-extension

$$q^\sharp \colon G^\sharp := \mathfrak{a} \times_f \widetilde{G} \to \widetilde{G}, \quad (a,g) \mapsto g,$$

where $f \in Z^2_s(\widetilde{G},\mathfrak{a})$ is a smooth cocycle with $D[f] = [\omega]$. In view of Corollary 6.5, the vanishing of F_ω implies the existence of a homomorphism $\gamma \colon \pi_1(G) \to Z(G^\sharp)$ with $q^\sharp \circ \gamma = \mathrm{id}_{\pi_1(G)}$. Then $\mathrm{im}(\gamma)$ is a discrete central subgroup of $G^\sharp$, so that $\widehat{G} := G^\sharp/\,\mathrm{im}(\gamma)$ is a Lie group, and we obtain an $\mathfrak{a}$-extension of G by

$$q \colon \widehat{G} \to G, \quad g\,\mathrm{im}(\gamma) \mapsto q_G \circ q^\sharp(g).$$

As $\widehat{G}$ is a principal $\mathfrak{a}$-bundle over G, its fibers are affine spaces whose translation group is $\mathfrak{a}$. If G is smoothly paracompact, we can therefore use a smooth partition of unity subordinated to a trivializing open cover of the $\mathfrak{a}$-bundle $\widehat{G} \to G$ to patch smooth local sections together to a global smooth section $\sigma \colon G \to \widehat{G}$. Then the map

$$\mathfrak{a} \times_{f_G} G \to \widehat{G}, \quad (a,g) \mapsto a\sigma(g)$$

is an isomorphism of Lie groups, where $f_G \in Z^2_s(G,\mathfrak{a}), (g,g') \mapsto \sigma(g)\sigma(g')\sigma(gg')^{-1}$ is a globally smooth cocycle. ■

Remark 8.5. Let G be a connected Lie group and A a smooth G-module of the form $\mathfrak{a}/\Gamma_A$. Let $Z^2_{gs}(G,A)$ denote the group of smooth 2-cocycles $G \times G \to A$ and $B^2_{gs}(G,A) \subseteq Z^2_{gs}(G,A)$ the cocycles of the form $d_G h$, where $h \in C^\infty(G,A)$ is a smooth function with $h(\mathbf{1}) = 0$. Then one can show that we have an injection

$$H^2_{gs}(G,A) := Z^2_{gs}(G,A)/B^2_{gs}(G,A) \hookrightarrow H^2_s(G,A),$$

the space $H^2_{gs}(G,A)$ classifies those A-extensions of G with a smooth global section, and we have an exact sequence

$$\mathrm{Hom}(\pi_1(G),\mathfrak{a}^G) \xrightarrow{\delta} H^2_{gs}(G,A) \xrightarrow{D} H^2_c(\mathfrak{g},\mathfrak{a}) \xrightarrow{P} H^2_{\rm dR}(G,\mathfrak{a}) \times \mathrm{Hom}\big(\pi_1(G), H^1_c(\mathfrak{g},\mathfrak{a})\big),$$

where $P([\omega]) = ([\omega^{\rm eq}], F_\omega)$. The proof is an easy adaptation from the corresponding arguments for central extensions in Section 8 of [Ne02]. ■

9. Applications to diffeomorphism groups

In the present section we apply the general results of this paper to the diffeomorphism group G of a compact manifold M. In this case the Lie algebra $\mathfrak g$ is the Fréchet–Lie algebra $\mathcal{V}(M)$ of smooth vector fields on M and we obtain interesting Lie algebra 2-cocycles with values in the space $C^\infty(M,V)$ of smooth V-valued functions from closed V-valued 2-forms on M. In this case the period map and the flux cocycle can be made more concrete in geometric terms which makes it possible to evaluate the obstructions to the existence of abelian extensions in many concrete examples, even if $\pi_1(\mathrm{Diff}(M))$ and $\pi_2(\mathrm{Diff}(M))$ are not known.

The diffeomorphism group as a Lie group

Definition 9.1. Let M be a compact manifold.

(a) We write $\mathrm{Diff}(M)$ for the group of all diffeomorphisms of M and $\mathcal{V}(M)$ for the Lie algebra of smooth vector fields on M, i.e., the set of all smooth maps $X\colon M \to TM$ with $\pi_{TM}\circ X = \mathrm{id}_M$, where $\pi_{TM}\colon TM \to M$ is the bundle projection of the tangent bundle. We define the Lie algebra structure on $\mathcal{V}(M)$ in such a way that $[X,Y].f = X.(Y.f) - Y.(X.f)$ holds for $X,Y \in \mathcal{V}(M)$ and $f \in C^\infty(M,\mathbb{R})$.

Then $\mathrm{Diff}(M)$ is a Lie group whose Lie algebra is $\mathcal{V}(M)^{\rm op}$ (the same space with the apposite bracket $(X,Y)\mapsto -[X,Y]$) and we have a smooth exponential function

$$\exp\colon \mathcal{V}(M) \to \mathrm{Diff}(M)$$

given by $\exp(X) = \Phi^1_X$, where $\Phi^t_X \in \mathrm{Diff}(M)$ is the flow of the vector field X at time t ([KM97]).

The tangent bundle of $\mathrm{Diff}(M)$ can be identified with the set

$$T(\mathrm{Diff}(M)) := \{X \in C^\infty(M,TM)\colon \pi_{TM}\circ X \in \mathrm{Diff}(M)\},$$

where the map

$$\pi\colon T(\mathrm{Diff}(M)) \to \mathrm{Diff}(M), \quad X \mapsto \pi_{TM}\circ X$$

is the bundle projection. Then $T_\varphi(\mathrm{Diff}(M)) := \pi^{-1}(\varphi)$ is the fiber over the diffeomorphism φ.

In view of the natural action of $\mathrm{Diff}(M)$ on TM given by $\psi.v := T(\psi).v$, we obtain natural left and right actions of $\mathrm{Diff}(M)$ on $T(\mathrm{Diff}(M))$ by

$$(\varphi.X)(x) = \varphi(x).X(x), \quad X.\varphi := X\circ\varphi.$$

Then

$$\pi_{TM} \circ (\varphi.X) = \varphi \circ (\pi_{TM} \circ X) \quad \text{and} \quad \pi_{TM} \circ (X \circ \varphi) = (\pi_{TM} \circ X) \circ \varphi,$$

so that the left, resp., right action of $\mathrm{Diff}(M)$ on $T(\mathrm{Diff}(M))$ covers the left, resp., right multiplication action of the group $\mathrm{Diff}(M)$ on itself. In the following we shall mostly consider the opposite group $\mathrm{Diff}(M)^{\mathrm{op}}$ whose Lie algebra is $\mathcal{V}(M)$. The adjoint action of this group is given by

$$\mathrm{Ad}\colon \mathrm{Diff}(M)^{\mathrm{op}} \times \mathcal{V}(M) \to \mathcal{V}(M), \quad (\varphi, X) \mapsto \varphi^{-1}.(X \circ \varphi) = \varphi^{-1}.(X.\varphi).$$

(b) Let $J \subseteq \mathbb{R}$ be an interval and $\varphi\colon J \to \mathrm{Diff}(M)^{\mathrm{op}}$ be a smooth curve. Then for each $t \in J$ we obtain a vector field

$$\delta^r(\varphi)(t) := \varphi(t)^{-1}.\varphi'(t)$$

called the *right logarithmic derivative of* φ *in* t. We likewise define the *left logarithmic derivative* by

$$\delta^l(\varphi)(t) := \varphi'(t) \circ \varphi(t)^{-1}.$$ ■

Definition 9.2. Let M be a compact smooth manifold and $\mathfrak{g} := \mathcal{V}(M)$ the Lie algebra of smooth vector fields on M. If V is Fréchet space and $\mathfrak{a} := C^\infty(M, V)$ the space of smooth V-valued functions on M, then $(X.f)(p) := df(p)X(p)$ turns $\mathfrak{a}$ into a topological $\mathcal{V}(M)$-module. Note that $C^\infty(M, V)$ and $\mathcal{V}(M)$ are Fréchet modules of the Fréchet algebra $R := C^\infty(M, \mathbb{R})$.

In the Lie algebra complex $(C^p_c(\mathfrak{g}, \mathfrak{a}), d_\mathfrak{g})_{p \in \mathbb{N}_0}$ formed by the continuous alternating maps $\mathfrak{g}^p \to \mathfrak{a}$, we have the subcomplex given by the subspaces $C^p_R(\mathfrak{g}, \mathfrak{a}) \subseteq C^p_c(\mathfrak{g}, \mathfrak{a})$ consisting of R-multilinear maps $\mathfrak{g}^p \to \mathfrak{a}$. Using partitions of unity, it is easy to see that the elements of $C^p_R(\mathfrak{g}, \mathfrak{a})$ can be identified with smooth V-valued p-forms, so that $C^p_R(\mathfrak{g}, \mathfrak{a}) \cong \Omega^p(M, V)$ ([Hel78]), and the de Rham differential coincides with the Lie algebra differential $d_\mathfrak{g}$ to $C^p_R(\mathfrak{g}, \mathfrak{a})$.

We thus obtain natural maps $Z^p_{\mathrm{dR}}(M, V) \to Z^p_c(\mathfrak{g}, \mathfrak{a})$ and $j_p\colon H^p_{\mathrm{dR}}(M, V) \to H^p_c(\mathfrak{g}, \mathfrak{a})$. ■

Lemma 9.3. *If M is connected, then $V \cong C^\infty(M, V)^{\mathcal{V}(M)} = \mathfrak{a}^\mathfrak{g}$ consists of the constant functions $M \to V$.* ■

Lemma 9.4. *The map $j_1\colon H^1_{\mathrm{dR}}(M, V) \to H^1_c(\mathfrak{g}, \mathfrak{a})$ is injective.*

Proof. Let $\alpha \in \Omega^1(M, V)$ be a closed V-valued 1-form on M. If $j_1([\alpha]) = 0$, then there exists an element $f \in \mathfrak{a} = C^\infty(M, V)$ with $\alpha = d_\mathfrak{g} f$, which means that $\alpha = df$. Hence α is exact and therefore j_1 is injective. ■

Lemma 6.1 in [MN03] implies that we have a smooth action of the group $G := \mathrm{Diff}(M)_0^{\mathrm{op}}$ on $\mathfrak{a}$ by $\varphi.f := f \circ \varphi$. The derived action of $\mathcal{V}(M)$ on this space is given by

$$(X.f)(p) = \frac{d}{dt}\Big|_{t=0} (\exp(tX).f)(p) = \frac{d}{dt}\Big|_{t=0} f(\exp(tX).p) = df(p)X(p)$$

which is compatible with Definition 9.2. We view each smooth V-valued 2-form $\omega_M \in \Omega^2(M,V)$ as an element $\omega_{\mathfrak{g}} \in C^2_c(\mathfrak{g},\mathfrak{a})$. In the following we shall obtain some information on the period map and the flux homomorphism

$$\mathrm{per}_\omega\colon \pi_2(\mathrm{Diff}(M)) \to \mathfrak{a}^{\mathfrak{g}} \cong V \quad \text{ and } \quad F_\omega\colon \pi_1(\mathrm{Diff}(M)) \to H^1_c(\mathfrak{g},\mathfrak{a})$$

which makes it possible to verify the integrability criteria from Sections 6 and 7 in many special cases.

More on the period group

The following proposition is very helpful in verifying the discreteness of the image of the period map for the group $G := \mathrm{Diff}(M)_0^{\mathrm{op}}$. In the following we write $(m,g) \mapsto g(m)$ for the canonical right action of G on M.

Proposition 9.5. *Let $\omega_M \in Z^2_{\mathrm{dR}}(M,V)$ be a closed V-valued 2-form on M, $\sigma\colon \mathbb{S}^2 \to G = \mathrm{Diff}(M)_0^{\mathrm{op}}$ smooth and $m \in M$. Then*

$$\mathrm{per}_{\omega_{\mathfrak{g}}}([\sigma])(m) = \int_{\mathrm{ev}_m^D \circ \sigma} \omega_M \in V \cong C^\infty(M,V)^{\mathcal{V}(M)},$$

where $\mathrm{ev}_m^D\colon G \to M, g \mapsto g(m)$. In particular the period group $\Pi_{\omega_{\mathfrak{g}}} = \mathrm{im}(\mathrm{per}_{\omega_{\mathfrak{g}}})$ is contained in the group $\int_{\pi_2(M)} \omega_M$ of spherical periods of ω_M.

Proof. Since $\mathfrak{a}^{\mathfrak{g}}$ consists of constant functions $M \to V$, it suffices to calculate the value of $\mathrm{per}_{\omega_{\mathfrak{g}}}([\sigma]) \in C^\infty(M,V)$ in the point m.

We claim that

$$(\mathrm{ev}_m^D)^*\omega_M = \mathrm{ev}_m \circ \omega_{\mathfrak{g}}^{\mathrm{eq}}, \tag{9.1}$$

where $\mathrm{ev}_m\colon C^\infty(M,V) \to V$ is the evaluation in m. First we note that for $g \in G$ we have $\mathrm{ev}_m^D \circ \lambda_g = \mathrm{ev}_{g(m)}^D$. Further

$$d\,\mathrm{ev}_m^D(\mathbf{1})(X) = \frac{d}{dt}\Big|_{t=0} \exp(tX).m = X(m) \quad \text{ for } \quad X \in \mathcal{V}(M).$$

For $g \in G$ and vector fields $X, Y \in \mathfrak{g} = \mathcal{V}(M)$ this leads to

$$\begin{aligned}
&((\mathrm{ev}_m^D)^*\omega_M)(g.X, g.Y)\\
&= \omega_M(g(m))(d\,\mathrm{ev}_m^D(g)d\lambda_g(\mathbf{1}).X, d\,\mathrm{ev}_m^D(g)d\lambda_g(\mathbf{1}).Y)\\
&= \omega_M(g(m))(d(\mathrm{ev}_m^D \circ\lambda_g)(\mathbf{1}).X, d(\mathrm{ev}_m^D \circ\lambda_g)(\mathbf{1}).Y)\\
&= \omega_M(g(m))(d\,\mathrm{ev}_{g(m)}^D(\mathbf{1}).X, d\,\mathrm{ev}_{g(m)}^D(\mathbf{1}).Y)\\
&= \omega_M(g(m))(X(g(m)), Y(g(m))) = \Big(g.\big(\omega_{\mathfrak g}(X,Y)\big)\Big)(m) = (\mathrm{ev}_m \circ\omega_{\mathfrak g}^{\mathrm{eq}})(g.X, g.Y).
\end{aligned}$$

This proves (9.1). We now obtain

$$\mathrm{per}_{\omega_{\mathfrak g}}([\sigma])(m) = \mathrm{ev}_m \int_\sigma \omega_{\mathfrak g}^{\mathrm{eq}} = \int_\sigma \mathrm{ev}_m \circ\omega_{\mathfrak g}^{\mathrm{eq}} = \int_\sigma (\mathrm{ev}_m^D)^*\omega_M = \int_{\mathrm{ev}_m^D \circ\sigma} \omega_M.$$

■

We immediately derive the following sufficient criterion for the discreteness of $\mathrm{im}(\mathrm{per}_{\omega_{\mathfrak g}})$.

Corollary 9.6. *If the subgroup* $\int_{\pi_2(M)} \omega_M := \{\int_\sigma \omega_M : \sigma \in C^\infty(\mathbb{S}^2, M)\} \subseteq V$ *of spherical periods of* ω_M *is discrete, then the image of* $\mathrm{per}_{\omega_{\mathfrak g}}$ *is discrete.* ■

Example 9.7. (1) The preceding corollary applies in particular to all manifolds M for which $\pi_2(M)/\mathrm{tor}(\pi_2(M))$ is a cyclic group. In fact, for each torsion element $[\sigma] \in \pi_2(M)$ we have $\int_\sigma \omega_M = 0$, so that $\int_{\pi_2(M)} \omega_M$ is the image of the cyclic group $\pi_2(M)/\mathrm{tor}(\pi_2(M))$, hence cyclic and therefore discrete.

Examples of such manifolds are spheres and tori:

$$\pi_2(\mathbb{S}^d) \cong \begin{cases} \{0\} & \text{for } d \neq 2\\ \mathbb{Z} & \text{for } d = 2\end{cases} \qquad \text{and} \qquad \pi_2(\mathbb{T}^d) \cong \pi_2(\mathbb{R}^d) = \{0\}, \quad d \in \mathbb{N}.$$

The only compact connected manifolds M with $\dim M \leq 2$ and $\pi_2(M)$ non-trivial are the 2-sphere $\mathbb{S}^2$ and the real projective plane $\mathbb{P}_2(\mathbb{R})$. This follows from $\pi_2(M) \cong \pi_2(\widetilde{M})$ for the universal covering $\widetilde{M} \to M$ and the fact that a simply connected 2-dimensional manifold is diffeomorphic to $\mathbb{S}^2$ or $\mathbb{R}^2$. Further all orientable 3-manifolds which are irreducible in the sense of Kneser have trivial π_2. In particular the complement of a knot $K \subseteq \mathbb{S}^3$ has trivial π_2 (cf. [Mil03, p.1228]).

(2) For $M = \mathbb{S}^2$ we have

$$\pi_2(\mathrm{Diff}(M)) \cong \pi_2(\mathrm{SO}_3(\mathbb{R})) = \{\mathbf{1}\} \quad \text{and} \quad \pi_2(\mathbb{S}^2) \cong \mathbb{Z}.$$

If $\omega_M \in Z^2_{\mathrm{dR}}(M, \mathbb{R})$ is the closed 2-form with $\int_M \omega_M = 1$, we have $\int_{\pi_2(M)} \omega_M = \mathbb{Z}$ which is larger than $\Pi_{\omega_{\mathfrak g}} = \mathrm{im}(\mathrm{per}_{\omega_{\mathfrak g}}) = \{0\}$. ■

Problem 9. Find an example of a closed 2-form ω_M for which the group $\Pi_{\omega_{\mathfrak g}} = \mathrm{im}(\mathrm{per}_{\omega_{\mathfrak g}})$ is discrete and $\int_{\pi_2(M)} \omega_M$ is not. ■

The flux cocycle

We continue with the setting where M is a compact manifold and $G = \mathrm{Diff}(M)_0^{\mathrm{op}}$ is the identity component of its diffeomorphism group endowed with the opposite multiplication. For any Fréchet space V the space $\Omega^1(M,V)$ is a smooth G-module with respect to $(\varphi,\beta) \mapsto \varphi^*\beta$. To verify the smoothness of this action, we can think of $\Omega^1(M,V)$ as a closed subspace of $C^\infty(TM,V)$ and observe that $\mathrm{Diff}(M)$ acts smoothly on TM, so that Lemma 6.1 in [MN03] applies. The corresponding derived module of $\mathfrak{g} = \mathcal{V}(M)$ is given by $(X,\beta) \mapsto \mathcal{L}_X.\beta$, where $\mathcal{L}_X = d \circ i_X + i_X \circ d$ denotes the Lie derivative. The subspace $dC^\infty(M,V)$ of exact 1-forms is a closed subspace because

$$dC^\infty(M,V) = \Big\{\beta \in \Omega^1(M,V) \colon (\forall \gamma \in C^\infty(\mathbb{S}^1, M)) \int_\gamma \beta = 0\Big\} \tag{9.2}$$

and the linear maps $\Omega^1(M,V) \to V, \beta \mapsto \int_\gamma \beta$ are continuous. We can therefore form the quotient module

$$\widehat{H}^1_{\mathrm{dR}}(M,V) := \Omega^1(M,V)/dC^\infty(M,V)$$

containing $H^1_{\mathrm{dR}}(M,V) = Z^1_{\mathrm{dR}}(M,V)/dC^\infty(M,V)$ as a closed subspace.

Lemma 9.8. *For each closed V-valued 2-form $\omega \in \Omega^2(M,V)$ the continuous linear map*

$$f_\omega \colon \mathcal{V}(M) \to \widehat{H}^1_{\mathrm{dR}}(M,V), \quad X \mapsto [i_X\omega]$$

is a Lie algebra 1-cocycle.

Proof. For $X, Y \in \mathcal{V}(M)$ we use the formulas $i_{[X,Y]} = [\mathcal{L}_X, i_Y]$ and $\mathcal{L}_X = i_X \circ d + d \circ i_X$ to obtain

$$\begin{aligned} i_{[X,Y]}\omega &= \mathcal{L}_X i_Y\omega - i_Y\mathcal{L}_X\omega = d i_X i_Y \omega + i_X d(i_Y\omega) - i_Y(d i_X\omega + i_X d\omega) \\ &= d i_X i_Y \omega + i_X d(i_Y\omega) - i_Y(d i_X \omega). \end{aligned}$$

In view of $[\mathcal{L}_X i_Y\omega] = [d i_X i_Y\omega + i_X d i_Y\omega] = [i_X d i_Y \omega]$ in $\widehat{H}^1_{\mathrm{dR}}(M,V)$, this means that

$$f_\omega([X,Y]) = X.f_\omega(Y) - Y.f_\omega(X),$$

i.e., f_ω is a 1-cocycle. ∎

Definition 9.9. Let $q_G \colon \widetilde{G} \to G$ denote the universal covering morphism of $G = \mathrm{Diff}(M)_0^{\mathrm{op}}$ and define the $\widetilde{G}$-action on $C^\infty(M,V)$, $\Omega^1(M,V)$, $\widehat{H}^1_{\mathrm{dR}}(M,V)$ etc. by pulling it back by q_G to $\widetilde{G}$. Then Proposition 3.4 implies that there exists a smooth 1-cocycle

$$F_\omega \colon \widetilde{G} \to \widehat{H}^1_{\mathrm{dR}}(M,V) = \Omega^1(M,V)/dC^\infty(M,V) \quad \text{with} \quad dF_\omega(\mathbf{1}) = f_\omega.$$

This cocycle is called the *flux cocycle corresponding to* ω. Its differential dF_ω coincides with the equivariant 1-form f_ω^{eq}. ∎

Remark 9.10. (a) If $g \in \widetilde{G}$ and $\widetilde{\gamma}\colon [0,1] \to \widetilde{G}$ is a piecewise smooth curve with $\widetilde{\gamma}(0) = \mathbf{1}$ and $\widetilde{\gamma}(1) = g$, then $\widetilde{\gamma}$ is the unique lift of $\gamma := q_G \circ \widetilde{\gamma}\colon [0,1] \to G$. The value of the flux cocycle in g is determined by

$$\begin{aligned} F_\omega(g) &= \int_0^1 dF_\omega(\widetilde{\gamma}(t))(\widetilde{\gamma}'(t))\,dt = \int_0^1 (f_\omega^{\rm eq})(\widetilde{\gamma}(t))(\widetilde{\gamma}'(t))\,dt \\ &= \int_0^1 \gamma(t).f_\omega(\widetilde{\gamma}(t)^{-1}.\widetilde{\gamma}'(t))\,dt = \int_0^1 \gamma(t).f_\omega(\gamma(t)^{-1}.\gamma'(t))\,dt \\ &= \int_0^1 \gamma(t).f_\omega(\delta^l(\gamma)(t))\,dt = \int_0^1 [\gamma(t)^*.i_{\delta^l(\gamma)(t)}\omega]\,dt \\ &= \int_0^1 [i_{\delta^r(\gamma)(t)}(\gamma(t)^*\omega)]\,dt \in \widehat{H}^1_{\rm dR}(M,V). \end{aligned}$$

Here we have used the relation $\varphi^*(i_X\omega) = i_{\mathrm{Ad}(\varphi).X}(\varphi^*\omega)$ for $\varphi \in \mathrm{Diff}(M)$, $X \in \mathcal{V}(M)$ and $\omega \in \Omega^p(M,V)$.

(b) For the special case when the curve $\gamma\colon [0,1] \to \mathrm{Diff}(M)$ has values in the subgroup

$$\mathrm{Sp}(M,\omega) := \{\varphi \in \mathrm{Diff}(M)\colon \varphi^*\omega = \omega\},$$

all vector fields $\delta^l(\gamma)(t)$ are contained in the Lie algebra

$$\mathfrak{sp}(M,\omega) := \{X \in \mathcal{V}(M)\colon \mathcal{L}_X.\omega = 0\}$$

([NV03, Lemma 1.4]). For $\mathcal{L}_X\omega = 0$ we have $d(i_X\omega) = \mathcal{L}_X\omega = 0$, so that all 1-forms $i_X\omega$ are closed. This in turn implies that for each $\varphi \in \mathrm{Diff}(M)_0$ the 1-form $\varphi^*i_X\omega - i_X\omega$ is exact ([NV03, Lemma 1.3]). For the flux cocycle this leads to the simpler formula

$$F_\omega(g) = \int_0^1 [i_{\delta^l(\gamma)(t)}\omega]\,dt.$$

Hence $F_\omega(g)$ is the flux associated to the curve $\gamma\colon [0,1] \to \mathrm{Sp}(M,\omega)$ in the context of symplectic geometry [MDS98].

(c) If the closed form ω is exact, $\omega = d\theta$, then

$$f_\omega(X) = [i_X\omega] = [i_X d\theta] = [\mathcal{L}_X\theta] = X.[\theta]$$

in $\widehat{H}^1_{\rm dR}(M,V)$ implies that f_ω is a coboundary. Hence it integrates to a group cocycle given by

$$F_\omega\colon \mathrm{Diff}(M)^{\rm op} \to \widehat{H}^1_{\rm dR}(M,V), \quad \varphi \mapsto [\varphi^*\theta - \theta]. \qquad \blacksquare$$

On the space $\widehat{H}^1_{\rm dR}(M,V)$ the integration maps $\widehat{H}^1_{\rm dR}(M,V) \to V, [\beta] \mapsto \int_\alpha \beta$ for $\alpha \in C^\infty(\mathbb{S}^1,M)$ separate points (cf. (9.2)), so that the element $F_\omega(g) \in \widehat{H}^1_{\rm dR}(M,V)$ is determined by the integrals $\int_\alpha F_\omega(g)$ which are evaluated in the proposition below.

Proposition 9.11. *For $\alpha \in C^\infty(\mathbb{S}^1, M)$ and a smooth curve $\gamma\colon [0,1] \to G = \mathrm{Diff}(M)_0^{\mathrm{op}}$ with $\gamma(0) = \mathrm{id}_M$ we consider the smooth map*

$$H\colon [0,1] \times \mathbb{S}^1 \to M, \quad (t,s) \mapsto \gamma(t)(\alpha(s)).$$

Let $\widetilde{\gamma}\colon [0,1] \to \widetilde{G}$ be the smooth lift with $\widetilde{\gamma}(0) = \mathbf{1}$. Then the value of the flux cocycle in $\widetilde{\gamma}(1)$ is determined by the integrals

$$\int_\alpha F_\omega(\widetilde{\gamma}(1)) = \int_H \omega.$$

Proof. First we note that

$$\frac{\partial H}{\partial t}(t,s) = \gamma'(t)(\alpha(s)) = \gamma'(t) \circ \gamma(t)^{-1} \circ \gamma(t)(\alpha(s)) = \delta^l(\gamma)(t)(H(t,s))$$

and $\frac{\partial H}{\partial s}(t,s) = \gamma(t).\alpha'(s)$. Identifying $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$, we therefore obtain with Remark 9.10(a) the formula

$$\begin{aligned}
\int_\alpha F_\omega(\widetilde{\gamma}(1)) &= \int_\alpha \int_0^1 [\gamma(t)^*.i_{\delta^l(\gamma)(t)}\omega]\, dt \\
&= \int_0^1 \int_0^1 \omega_{\gamma(t).\alpha(s)}(\delta^l(\gamma)(t)(\gamma(t).\alpha(s)), \gamma(t).\alpha'(s))\, dt\, ds \\
&= \int_0^1 \int_0^1 \omega_{H(t,s)}\Big(\frac{\partial H(t,s)}{\partial t}(t,s), \frac{\partial H(t,s)}{\partial s}(t,s)\Big)\, dt\, ds \\
&= \int_{[0,1]^2} H^*\omega = \int_H \omega.
\end{aligned}$$

∎

The preceding proposition justifies the term ‘flux cocycle’ because it says that $\int_\alpha F_\omega(\widetilde{\gamma}(1))$ measures the ‘ω-surface area’ of the surface obtained by moving the loop α by the curve γ in $\mathrm{Diff}(M)$.

Corollary 9.12. *If $\gamma(1) = \gamma(0) = \mathrm{id}_M$, then $F_\omega(\widetilde{\gamma}(1)) \in H^1_{\mathrm{dR}}(M,V)$, and we obtain a homomorphism*

$$F_\omega\,|_{\pi_1(\mathrm{Diff}(M))}\colon \pi_1(\mathrm{Diff}(M)) \to H^1_{\mathrm{dR}}(M,V).$$

Proof. We keep the notation from Proposition 9.11. If the curve γ in $\mathrm{Diff}(M)$ is closed and $\widetilde{\gamma}$ is the corresponding map $\mathbb{S}^1 \to \mathrm{Diff}(M)$, then H induces a continuous map $\widetilde{H}\colon \mathbb{T}^2 \to M, (t,s) \mapsto \widetilde{\gamma}(t).\alpha(s)$ and

$$\int_\alpha F_\omega(\widetilde{\gamma}(1)) = \int_H \omega = \int_{\widetilde{H}} \omega = \widetilde{H}^*[\omega] \in H^2(\mathbb{T}^2, V) \cong V.$$

As homotopic curves α_1 and α_2 lead to homotopic maps $\widetilde{H}_1, \widetilde{H}_2\colon \mathbb{T}^2 \to M$, we obtain

$$\int_{\alpha_1} F_\omega(\widetilde{\gamma}(1)) = \int_{\alpha_2} F_\omega(\widetilde{\gamma}(1))$$

whenever α_1 and α_2 are homotopic, and this implies that $F_\omega(\widetilde{\gamma}(1)) \in H^1_{\mathrm{dR}}(M,V)$.

That the restriction of F_ω to $\pi_1(\mathrm{Diff}(M))$ is a homomorphism follows from the cocycle property of F_ω and the fact that $\pi_1(\mathrm{Diff}(M)) = \ker q_G$ acts trivially on $\widehat{H}^1_{\mathrm{dR}}(M,V)$. ■

Let $\omega_M \in \Omega^2(M,V)$ be a closed 2-form and identify it with a Lie algebra 2-cocycle $\omega_{\mathfrak{g}} \in Z^2_c(\mathfrak{g},\mathfrak{a})$ for $\mathfrak{g} = \mathcal{V}(M)$ and $\mathfrak{a} = C^\infty(M,V)$. Next we show that the flux cocycle

$$F_{\omega_{\mathfrak{g}}} : \widetilde{G} \to \widehat{H}^1_c(\mathfrak{g},\mathfrak{a})$$

coincides with flux cocycle F_{ω_M} from Definition 9.9. For that we recall from Lemma 9.4 that we can view $H^1_{\mathrm{dR}}(M,V)$ as a subspace of $H^1_c(\mathfrak{g},\mathfrak{a})$ because $B^1_c(\mathfrak{g},\mathfrak{a}) = dC^\infty(M,V)$, leads to an embedding

$$\widehat{H}^1_{\mathrm{dR}}(M,V) \hookrightarrow \widehat{H}^1_c(\mathfrak{g},\mathfrak{a}) := C^1_c(\mathfrak{g},\mathfrak{a})/B^1_c(\mathfrak{g},\mathfrak{a}).$$

Proposition 9.13. *For a closed 2-form $\omega_M \in \Omega^2(M,V)$ we have*

$$F_{\omega_{\mathfrak{g}}} = F_{\omega_M} : \widetilde{G} \to \widehat{H}^1_{\mathrm{dR}}(M,V) \subseteq \widehat{H}^1_c(\mathfrak{g},\mathfrak{a}).$$

Proof. We parametrize $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ by the unit interval $[0,1]$. Then we have for any smooth curve $\gamma\colon [0,1] \to G = \mathrm{Diff}(M)^{\mathrm{op}}_0$ starting in $\mathbf{1}$ and $X \in \mathfrak{g} = \mathcal{V}(M)$:

$$\begin{aligned}
I_\gamma(X) &:= \int_\gamma i_{X_r}\omega^{\mathrm{eq}}_{\mathfrak{g}} = \int_0^1 \omega^{\mathrm{eq}}_{\mathfrak{g}}(X\gamma(t), \gamma'(t))\,dt \\
&= \int_0^1 \gamma(t).\omega_{\mathfrak{g}}(\mathrm{Ad}(\gamma(t))^{-1}.X, \gamma(t)^{-1}\gamma'(t))\,dt \\
&= \int_0^1 \gamma(t).\omega_M(\gamma(t).(X \circ \gamma(t)^{-1}), \delta^l(\gamma)(t))\,dt \\
&= \int_0^1 \omega_M\big(\gamma(t).(X \circ \gamma(t)^{-1}), \delta^l(\gamma)(t)\big) \circ \gamma(t)\,dt.
\end{aligned}$$

From this formula it is easy to see that $I_\gamma \in \mathrm{Lin}(\mathfrak{g},\mathfrak{a})$ defines a 1-form on M whose value in $v \in T_p(M)$ is given by

$$I_\gamma(v) = \int_0^1 (\omega_M)_{\gamma(t).p}\big(\gamma(t).v, \delta^l(\gamma)(t)(\gamma(t).p)\big)\,dt.$$

This means that

$$I_\gamma = -\int_0^1 \gamma(t)^*\big(i_{\delta^l(\gamma)(t)}\omega_M\big)\,dt,$$

which, in view of Remark 9.10, implies that

$$F_{\omega_{\mathfrak{g}}}(\widetilde{\gamma}(1)) = [-I_\gamma] = F_{\omega_M}(\widetilde{\gamma}(1)) \in \widehat{H}^1_{\mathrm{dR}}(M,V) \subseteq \widehat{H}^1_c(\mathfrak{g},\mathfrak{a}).$$

The remaining assertions now follow from Corollary 9.12. ■

Corollary 9.14. *$F_\omega(\pi_1(G))$ vanishes if and only if for each smooth loop $\alpha\colon \mathbb{S}^1 \to M$ and each smooth loop $\gamma\colon \mathbb{S}^1 \to \mathrm{Diff}(M)$ we have $\int_H \omega = 0$ for the map $H\colon \mathbb{T}^2 \to M, H(t,s) = \gamma(t).\alpha(s)$.* ■

The condition in the preceding corollary is in particular satisfied if the set of homotopy classes of based maps $\mathbb{T}^2 \to M$ or at least the corresponding homology classes in $H_2(M)$ are trivial.

Remark 9.15. It is interesting to observe that the discreteness of the period map for $\omega \in \Omega^2(M,V)$ leads to a condition on the group of spherical cycles, i.e., the image of $\pi_2(M)$ in $H_2(M)$, and the vanishing of $F_\omega(\pi_1(G))$ leads to a condition on the larger subgroup of $H_2(M)$ generated by the cycles coming from maps $\mathbb{T}^2 \to M$. That the latter group contains the former follows from the existence of a map $\mathbb{T}^2 \to \mathbb{S}^2$, inducing an isomorphism $H_2(\mathbb{T}^2) \to H_2(\mathbb{S}^2)$. If M is a Lie group, then Proposition 6.11 implies that $H_2(M)$ is generated by the homology classes coming from continuous maps $\mathbb{T}^2 \to M$. ■

Examples

Example 9.16. Let $\mathfrak{z}$ be a Fréchet space, $\Gamma_Z \subseteq \mathfrak{z}$ a discrete subgroup, $Z := \mathfrak{z}/\Gamma_Z$ and $q_Z\colon \mathfrak{z} \to Z$ the quotient map, which can also be considered as the exponential map of the Lie group Z.

Further let $q\colon P \to M$ be a smooth Z-principal bundle over the compact manifold M, $\theta \in \Omega^1(P,\mathfrak{z})$ a principal connection 1-form and $\omega \in \Omega^2(M,\mathfrak{z})$ the corresponding curvature, i.e., $q^*\omega = -d\theta$. We call a vector field $X \in \mathcal{V}(P)$ *horizontal* if $\theta(X) = 0$. Write $\mathcal{V}(P)^Z$ for the Lie algebra of Z-invariant vector fields on P. Then we have a linear bijection

$$\sigma\colon \mathcal{V}(M) \to \mathcal{V}(P)^Z_{\mathrm{hor}} := \{X \in \mathcal{V}(P)^Z \colon \theta(X) = 0\}$$

which is uniquely determined by $q_*\sigma(X) = X$ for $X \in \mathcal{V}(M)$. For two horizontal vector fields $\widetilde{X}, \widetilde{Y}$ on P we then have

$$(q^*\omega)(\widetilde{X},\widetilde{Y}) = -d\theta(\widetilde{X},\widetilde{Y}) = \widetilde{Y}.\theta(\widetilde{X}) - \widetilde{X}.\theta(\widetilde{Y}) - \theta([\widetilde{Y},\widetilde{X}]) = \theta([\widetilde{X},\widetilde{Y}]).$$

This means that

$$\omega(X,Y) = (q^*\omega)(\sigma(X),\sigma(Y)) = \theta([\sigma(X),\sigma(Y)]) = \theta([\sigma(X),\sigma(Y)] - \sigma([X,Y])) \tag{9.3}$$

can be viewed as the cocycle of the abelian extension

$$\mathfrak{a} := \mathfrak{gau}(P) \cong C^\infty(M,\mathfrak{z}) \hookrightarrow \widehat{\mathfrak{g}} := \mathcal{V}(P)^Z \twoheadrightarrow \mathfrak{g} = \mathcal{V}(M)$$

with respect to the section $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$.

On the group level we find that the inverse image $\widehat{G}$ of $G = \mathrm{Diff}(M)_0^{\mathrm{op}}$ in $\mathrm{Aut}(P)^{\mathrm{op}}$ is an extension of G by the abelian gauge group $A := \mathrm{Gau}(P) \cong C^\infty(M, Z)$ and we have already seen above that its Lie algebra is $\widehat{\mathfrak{g}} \cong \mathfrak{a} \oplus_\omega \mathfrak{g}$.

The exponential function of the abelian Lie group A is given by

$$\exp_A : \mathfrak{a} = C^\infty(M, \mathfrak{z}) \to C^\infty(M, Z), \quad \xi \mapsto q_Z \circ \xi.$$

Its image is the identity component A_0 of A. The characteristic map

$$\overline{\theta}_A : \pi_0(A) \cong [M, Z] \to H^1_c(\mathfrak{g}, \mathfrak{a}), \quad [f] \mapsto [D(d_G f)]$$

considered in Proposition 6.4 can be made more explicit by observing that

$$(d_G f)(g) = g.f - f = f \circ g - f,$$

so that

$$D(d_G f)(X) = X.f = \langle df, X \rangle$$

(cf. Definition A.2). This means that $D(d_G f)$ can be identified with the 1-form $df \in H^1_{\mathrm{dR}}(M, \mathfrak{z}) \subseteq H^1_c(\mathfrak{g}, \mathfrak{a})$. Therefore the homomorphism $\overline{\theta}_A : \pi_0(A) \to H^1_c(\mathfrak{g}, \mathfrak{a})$ from Proposition 6.4 is obtained by factorization of the map

$$A = C^\infty(M, Z) \to H^1_{\mathrm{dR}}(M, \mathfrak{z}), \quad f \mapsto [df]$$

whose kernel is the identity component $A_0 = q_Z \circ C^\infty(M, \mathfrak{z})$ of A, to the injective homomorphism

$$\pi_0(A) \cong C^\infty(M, Z)/q_Z \circ C^\infty(M, \mathfrak{z}) \cong [M, Z] \to H^1_{\mathrm{dR}}(M, \mathfrak{z}), \quad [f] \mapsto [df].$$

According to [Ne02, Prop. 3.9], its image consists of the subspace

$$H^1_{\mathrm{dR}}(M, \Gamma_Z) := \Big\{ [\alpha] \in H^1_{\mathrm{dR}}(M, \mathfrak{z}) : (\forall \gamma \in C^\infty(\mathbb{S}^1, M)) \int_\gamma \alpha \in \Gamma_Z \Big\},$$

so that

$$\overline{\theta}_A : \pi_0(A) \to H^1_{\mathrm{dR}}(M, \Gamma_Z), \quad [f] \mapsto [df]$$

is an isomorphism.

In view of Proposition 6.3, the flux homomorphism satisfies $F_\omega = -\overline{\theta}_A \circ \delta$, where $\delta : \pi_1(G) \to \pi_0(A)$ is the connecting homomorphism corresponding to the long exact homotopy sequence of the A-bundle $\widehat{G} \to G$. As $\overline{\theta}_A$ is an isomorphism, F_ω is essentially the same as δ, and we can view it as a homomorphism

$$F_\omega : \pi_1(G) \to H^1_{\mathrm{dR}}(M, \Gamma_Z) \subseteq H^1_{\mathrm{dR}}(M, \mathfrak{z}).$$

Note that we cannot expect $F_\omega(\pi_1(G))$ to vanish because the abelian extension $A \hookrightarrow \widehat{G} \to G$ is not an extension by a connected group. ■

Example 9.17. (a) We consider the special case where the manifold M is a torus: $M = T = \mathfrak{t}/\Gamma_T$, where $\mathfrak{t}$ is a finite-dimensional vector space and $\Gamma_T \subseteq \mathfrak{t}$ is a discrete subgroup for which $\mathfrak{t}/\Gamma_T$ is compact.

Then the group T acts by multiplication maps on itself, and we obtain a homomorphism $T \hookrightarrow G = \mathrm{Diff}(M)_0^{\mathrm{op}}$ which induces a homomorphism $\eta_T \colon \pi_1(T) \to \pi_1(G)$.

Let $\omega_T \in \Omega^2(T, \mathfrak{z})$ be an invariant $\mathfrak{z}$-valued 2-form on T and $\omega = \Omega_{\mathbf{1}} \in Z^2_c(\mathfrak{t}, \mathfrak{z})$. Then ω_T is closed because T is abelian. If $e_1, \ldots, e_n$ is an integral basis of Γ_T, then the maps

$$\mathbb{T}^2 \to T, \quad (t,s) \mapsto te_i + se_j + \Gamma_T, \quad i < j$$

lead to an integral basis of $H_2(T) \cong \mathbb{Z}^{\binom{\dim T}{2}}$, so that the period group of ω_T is

$$\Gamma_\omega := \mathrm{span}_{\mathbb{Z}}\, \omega(e_i, e_j) = \mathrm{span}_{\mathbb{Z}}\, \omega(\Gamma_T, \Gamma_T) \subseteq \mathfrak{z}.$$

We assume that $\Gamma_Z \subseteq \mathfrak{z}$ is a discrete subgroup with

$$\omega(\Gamma_T, \Gamma_T) \subseteq \Gamma_Z$$

and put $Z := \mathfrak{z}/\Gamma_Z$.

In view of $\pi_2(T) = \{0\}$, we have $\mathrm{per}_\omega = 0$ by Proposition 9.5. Next we are making the map

$$F_\omega \circ \eta_T \colon \pi_1(T) = \Gamma_T \to H^1_{\mathrm{dR}}(T, \Gamma_Z) \cong \mathrm{Hom}(\Gamma_T, \Gamma_Z)$$

more explicit. For $x, y \in \Gamma_T$ and the corresponding loops $\gamma_x(t) = tx + \Gamma_T$ and $\gamma_y(t) = ty + \Gamma_T$ in T we have for

$$H \colon \mathbb{T}^2 \to T, (t,s) \mapsto \gamma_x(t) + \gamma_y(s) = [tx + sy]$$

the formula

$$\int_{\gamma_y} F_\omega([\gamma_x]) = \int_H \omega = \omega(x,y)$$

(Propositions 9.11 and 9.13). This means that $F_\omega \circ \eta_T \colon \pi_1(T) \to \mathrm{Hom}(\pi_1(T), \Gamma_Z)$ can be identified with the map $x \mapsto i_x\omega$.

On the other hand, the existence of a Z-bundle over T with curvature ω implies the existence of an abelian extension

$$A := C^\infty(T, Z) \hookrightarrow \widehat{T} \twoheadrightarrow T,$$

where T acts on A by $(t.f)(x) = f(x+t)$ (cf. Example 9.16). The corresponding Lie algebra cocycle $\omega \in Z^2_c(\mathfrak{t}, C^\infty(T, \mathfrak{z}))$ is given by $(x,y) \mapsto \omega(x,y) \in \mathfrak{z}$ whose values lie in $\mathfrak{z} \cong \mathfrak{a}^T$.

(b) We now explain how essentially everything said about bundles over tori can be generalized to bundles over their natural infinite-dimensional generalizations.

Let $\mathfrak{t}$ be a locally convex space, $\Gamma_T \subseteq \mathfrak{t}$ a discrete subgroup and consider the connected abelian Lie group $T := \mathfrak{t}/\Gamma_T$. Let further $\mathfrak{z}$ be a Mackey complete locally convex space, $\Gamma_Z \subseteq \mathfrak{z}$ be a discrete subgroup and $Z := \mathfrak{z}/\Gamma_Z$, considered as a trivial T-module. We fix a continuous bilinear map $f_{\mathfrak{z}} \colon \mathfrak{t} \times \mathfrak{t} \to \mathfrak{z}$ and define $f_Z \in Z^2_s(\mathfrak{t}, Z)$ by $f_Z := q_Z \circ f_{\mathfrak{z}}$, where $q_Z \colon \mathfrak{z} \to Z$ is the quotient map.

Let $H := Z \times_{f_Z} \mathfrak{t}$ denote the corresponding central extension of $\mathfrak{t}$ by Z. Then $Z^\sharp := Z \times_{f_Z} \Gamma_T$ is a normal subgroup of H because all commutators lie in Z. Since $H/Z^\sharp \cong \mathfrak{t}/\Gamma_T = T$, we can think of H as an extension

$$Z^\sharp \hookrightarrow H \twoheadrightarrow T.$$

Since Z is divisible and Γ_T discrete, the central extension $Z \hookrightarrow Z^\sharp \twoheadrightarrow \Gamma_T$ is trivial if and only if it is an abelian group, which means that its commutator map $\Gamma_T \times \Gamma_T \to Z$ vanishes. The commutator map is given by

$$\begin{aligned}(z,t)(z',t')(z,t)^{-1}(z',t')^{-1} &= (f_Z(t,t'), t+t')(f_Z(t',t), t+t')^{-1} \\ &= (f_Z(t,t') - f_Z(t',t), 0) \\ &= (q_Z(f_{\mathfrak{z}}(t,t') - f_{\mathfrak{z}}(t',t)), 0) = (q_Z(\omega(t,t')), 0)\end{aligned}$$

for $\omega(t,t') := f_{\mathfrak{z}}(t,t') - f_{\mathfrak{z}}(t',t)$. Therefore $Z^\sharp$ is a trivial extension of Γ_T if and only if

$$\omega(\Gamma_T, \Gamma_T) \subseteq \Gamma_Z. \tag{9.4}$$

The condition for the extistence of a Z-bundle $P \to T$ with curvature ω_T is also given by (9.4). The necessity of this condition in the infinite-dimensional case can be seen by restricting to two-dimensional subtori. If (9.4) is satisfied, then we can view Γ_T as a subgroup of $Z^\sharp$ because there exists a homomorphism $\sigma \colon \Gamma_T \to Z^\sharp$ splitting the extension $Z^\sharp \twoheadrightarrow \Gamma_T$. Now we form the homogeneous space $P := H/\sigma(\Gamma_T)$ which defines a Z-bundle

$$Z \hookrightarrow P = H/\sigma(\Gamma_T) \twoheadrightarrow T \cong H/Z^\sharp.$$

As Z is central in H, the left action of H on P induces a homomorphism

$$H \to \mathrm{Aut}(P) = \mathrm{Diff}(P)^Z$$

restricting to a homomorphism

$$j_Z \colon Z^\sharp \cong Z \times_{f_Z} \Gamma_T \to \mathrm{Gau}(P) \cong C^\infty(T, Z),$$

where the elements of Z correspond to constant functions. The group Γ_T acts on P by

$$x.(q_Z(z), y) = (q_Z(z + f_{\mathfrak{z}}(x,y)), y) = (q_Z(z), y).f_Z(x,y),$$

so that

$$j_Z(z,x)(y + \Gamma_T) = z + f_Z(x,y).$$

Identifying
$$H^1_{\rm dR}(T,\Gamma_Z) \cong dC^\infty(T,Z)/dC^\infty(T,\mathfrak{z}) \subseteq H^1_{\rm dR}(T,\mathfrak{z})$$
with a subspace of $H^1_c(\mathfrak{t},\mathfrak{a})$ (cf. Lemma 9.4), we can view F_ω as a map
$$\pi_1(T) \to H^1_{\rm dR}(T,\Gamma_Z) \hookrightarrow \mathrm{Hom}(\pi_1(T),\Gamma_Z). \qquad \blacksquare$$

10. The diffeomorphism group of the circle and its universal covering

In this section we apply the general results from Sections 6 and 7 to the group $\mathrm{Diff}(\mathbb{S}^1)_0$ of orientation preserving diffeomorphisms of the circle $\mathbb{S}^1$ and the modules $\mathcal{F}_\lambda$ of λ-densities on $\mathbb{S}^1$ whose cohomology for the group $\mathrm{Diff}(\mathbb{S}^1)_0$ has been determined in [OR98]. We shall also point out how the picture changes if $\mathrm{Diff}(\mathbb{S}^1)_0$ is replaced by its universal covering group.

The diffeomorphism group of the circle

Let $G := \mathrm{Diff}(\mathbb{S}^1)_0^{\rm op}$ be the group of orientation preserving diffeomorphisms of the circle $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$. Then its universal covering group $\widetilde{G}$ can be identified with the group
$$\widetilde{G} := \{f \in \mathrm{Diff}(\mathbb{R})^{\rm op} \colon (\forall x \in \mathbb{R})\, f(x+1) = f(x)+1\},$$
and the covering homomorphism $q_G \colon \widetilde{G} \to G$ is given by $q(f)([x]) = [f(x)]$, where $[x] = x + \mathbb{Z} \in \mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$. The kernel of q_G consists of all translations τ_a, $a \in \mathbb{Z}$, and since $\widetilde{G}$ is an open convex subset of a closed subspace of $C^\infty(\mathbb{R},\mathbb{R})$, it is a contractible manifold. In particular, we obtain
$$\pi_1(G) \cong \mathbb{Z} \quad \text{and} \quad \pi_k(G) = \{\mathbf{1}\} \quad \text{for} \quad k \neq 1.$$

The group G has an import series of representation $\mathcal{F}_\lambda$, $\lambda \in \mathbb{R}$, where $\mathcal{F}_\lambda$ is the space of λ-densities on the circle $\mathbb{S}^1$. As the tangent bundle $T\mathbb{S}^1$ is trivial, we may identify the space $\mathcal{F}_\lambda$ with the space $C^\infty(\mathbb{S}^1,\mathbb{R})$ of 1-periodic functions on $\mathbb{R}$ with the representation
$$\rho_\lambda(\varphi).\xi = (\varphi')^\lambda \cdot (\xi \circ \varphi)$$
which corresponds symbolically to $\varphi^*(\xi(dx)^\lambda) = (\xi\circ\varphi)\cdot(\varphi')^\lambda \cdot (dx)^\lambda$. Note that $\mathcal{F}_0 = C^\infty(\mathbb{S}^1,\mathbb{R})$ is a Fréchet algebra and that, as G-modules,
$$\mathcal{F}_1 \cong \Omega^1(\mathbb{S}^1,\mathbb{R}) \quad \text{and} \quad \mathcal{F}_{-1} \cong \mathcal{V}(\mathbb{S}^1) = \mathfrak{g}.$$

For the Lie algebra $\mathfrak{g} = \mathcal{V}(\mathbb{S}^1)$ of G the derived representation of the vector field $X = \xi\frac{d}{dx}$ is given by

(10.1) $$\rho_\lambda(\xi).f = \xi f' + \lambda f \xi'.$$

This follows directly from $\rho_\lambda(g).f = (g')^\lambda \cdot (f \circ g)$ and the product rule. In the following we shall identify $\mathfrak{g}$ with $C^\infty(\mathbb{S}^1,\mathbb{R})$ via $\xi\frac{d}{dt} \mapsto \xi$.

Lemma 10.1. *On the Fréchet–Lie group* $A := C^\infty(\mathbb{S}^1, \mathbb{R}^\times) = \mathcal{F}_0^\times$ *we have a smooth* G*-action by* $g.f := f \circ g$ *and the derivative* $\eta\colon G \to A, f \mapsto f'$ *is a smooth* 1*-cocycle.*

Proof. For $g, h \in G$ we have $\eta(gh) = (gh)' = (h \circ g)' = (h' \circ g) \cdot g' = (g.\eta(h)) \cdot \eta(g)$. ∎

Remark 10.2. The representation on $\mathcal{F}_\lambda$ has the form $\rho_\lambda(g).f = \eta(g)^\lambda \cdot (f \circ g)$ and the fact that $\eta^\lambda\colon G \to A$ is a cocycle implies that $\rho_\lambda\colon G \to \mathrm{GL}(\mathcal{F}_\lambda)$ is a group homomorphism. ∎

The cohomology on the Lie algebra level

Proposition 10.3. *The cohomology in degrees* 0, 1, 2 *of the* $\mathfrak{g}$*-module* $\mathcal{F}_\lambda$ *has the following structure:*

$$H^0_c(\mathfrak{g}, \mathcal{F}_\lambda) = \mathcal{F}_\lambda^{\mathfrak{g}} = \begin{cases} \{0\} & \text{for } \lambda \neq 0 \\ \mathbb{R}1 & \text{for } \lambda = 0. \end{cases}$$

For $n \in \mathbb{N}_0$ *let* $\alpha_n(\xi) = \xi^{(n)}$ *denote the* n*-fold derivative. Then*

$$H^1_c(\mathfrak{g}, \mathcal{F}_0) = \operatorname{span}\{[\alpha_0], [\alpha_1]\}, \quad H^1_c(\mathfrak{g}, \mathcal{F}_1) = \mathbb{R}[\alpha_2], \quad H^1_c(\mathfrak{g}, \mathcal{F}_2) = \mathbb{R}[\alpha_3]$$

and $H^1_c(\mathfrak{g}, \mathcal{F}_\lambda)$ *vanishes for* $\lambda \neq 0, 1, 2$. *In degree* 2 *we have*

$$H^2_c(\mathfrak{g}, \mathcal{F}_\lambda) \cong \begin{cases} \mathbb{R}^2 & \text{for } \lambda = 0, 1, 2 \\ \mathbb{R} & \text{for } \lambda = 5, 7 \\ \{0\} & \text{otherwise.} \end{cases}$$

For $\lambda = 0, 1, 2$ *the cohomology classes of the following elements form a basis of* $H^2_c(\mathfrak{g}, \mathcal{F}_\lambda)$*:*

$$\overline{\omega}_0(\xi, \eta) := \begin{vmatrix} \xi & \eta \\ \xi' & \eta' \end{vmatrix}, \quad \omega_0(\xi, \eta) := \int_0^1 \begin{vmatrix} \xi' & \eta' \\ \xi'' & \eta'' \end{vmatrix} \quad \text{for} \quad \lambda = 0,$$

$$\overline{\omega}_1(\xi, \eta) := \begin{vmatrix} \xi & \eta \\ \xi'' & \eta'' \end{vmatrix}, \quad \omega_1(\xi, \eta) := \begin{vmatrix} \xi' & \eta' \\ \xi'' & \eta'' \end{vmatrix} \quad \text{for} \quad \lambda = 1,$$

and

$$\overline{\omega}_2(\xi, \eta) := \begin{vmatrix} \xi & \eta \\ \xi''' & \eta''' \end{vmatrix}, \quad \omega_2(\xi, \eta) := \begin{vmatrix} \xi' & \eta' \\ \xi''' & \eta''' \end{vmatrix} \quad \text{for } \lambda = 2.$$

Proof. (cf. [OR98]) We have

$$\mathcal{F}_\lambda^{\mathfrak{g}} \cong \{f \in C^\infty(\mathbb{S}^1, \mathbb{R})\colon (\forall \xi \in C^\infty(\mathbb{S}^1, \mathbb{R}))\ \xi f' + \lambda \xi' f = 0\}.$$

For constant functions ξ the differential equation from above reduces to $f'\xi = 0$, so that f is constant, and now $\lambda\xi' f = 0$ for each ξ implies $\lambda f = 0$. This proves the assertion about $H^0_c(\mathfrak{g}, \mathcal{F}_\lambda)$.

According to [Fu86, p.176], we have

$$H^q_c(\mathfrak{g}, \mathcal{F}_\lambda) = 0 \quad \text{for} \quad \lambda \not\in \Big\{\frac{3r^2 \pm r}{2} : r \in \mathbb{N}_0\Big\} = \{0, 1, 2, 5, 7, 12, 15, \ldots\}.$$

If $r \in \mathbb{N}_0$ and $\lambda = \frac{3r^2 \pm r}{2}$, then

$$H^q_c(\mathfrak{g}, \mathcal{F}_\lambda) \cong \begin{cases} H^{q-r}_{\rm sing}(Y(\mathbb{S}^1), \mathbb{R}) & \text{for } q \geq r \\ \{0\} & \text{for } q < r, \end{cases}$$

where $Y(\mathbb{S}^1) = \mathbb{T}^2 \times \Omega\mathbb{S}^3$ and $\Omega\mathbb{S}^3$ is the loop space of $\mathbb{S}^3$. The cohomology algebra

$$H^\bullet_c(\mathfrak{g}, \mathcal{F}_0) \cong H^\bullet_{\rm sing}(Y(\mathbb{S}^1), \mathbb{R}) \cong H^\bullet_{\rm sing}(\mathbb{S}^1, \mathbb{R}) \otimes H^\bullet_{\rm sing}(\mathbb{S}^1, \mathbb{R}) \otimes H^\bullet_{\rm sing}(\Omega\mathbb{S}^3, \mathbb{R})$$

is a free anti-commutative real algebra with generators a, b, c satisfying

$$\deg(a) = \deg(b) = 1, \quad \deg(c) = 2, \quad a^2 = b^2 = 0.$$

It follows in particular that

$$H^0_c(\mathfrak{g}, \mathcal{F}_0) = \mathbb{R}, \quad H^1_c(\mathfrak{g}, \mathcal{F}_0) = \mathbb{R}a + \mathbb{R}b \cong \mathbb{R}^2, \quad H^2_c(\mathfrak{g}, \mathcal{F}_0) = \mathbb{R}c + \mathbb{R}ab \cong \mathbb{R}^2.$$

The structure of $H^\bullet_c(\mathfrak{g}, \mathcal{F}_\lambda)$ is now determined by the fact that it is a free module of the algebra $H^\bullet(Y(\mathbb{S}^1), \mathbb{R}) \cong H^\bullet_c(\mathfrak{g}, \mathcal{F}_0)$ with one generator in degree r. Here the algebra structure on $H^\bullet_c(\mathfrak{g}, \mathcal{F}_0)$ is obtained from the multiplication on $\mathcal{F}_0$ as in Appendix F, and the multiplication $\mathcal{F}_0 \times \mathcal{F}_\lambda \to \mathcal{F}_\lambda$ yields the $H^\bullet_c(\mathfrak{g}, \mathcal{F}_0)$-module structure $([\alpha], [\beta]) \mapsto [\alpha \wedge \beta]$ on $H^\bullet_c(\mathfrak{g}, \mathcal{F}_\lambda)$.

From [Fu86, Th. 2.4.12] we see that generators of $H^\bullet_c(\mathfrak{g}, \mathcal{F}_0)$ are given by the classes of α_0, α_1 and ω_0. Therefore a second basis element of $H^2_c(\mathfrak{g}, \mathcal{F}_0)$ is represented by

$$(\alpha_0 \wedge \alpha_1)(\xi, \eta) = \alpha_0(\xi)\alpha_1(\eta) - \alpha_0(\eta)\alpha_1(\xi) = \xi\eta' - \xi'\eta = \overline{\omega}_0(\xi, \eta).$$

The space $H^1_c(\mathfrak{g}, \mathcal{F}_\lambda)$ is non-zero for $r = 0, 1$ which corresponds to $\lambda \in \{0, 1, 2\}$. For $r = 0$ it is two-dimensional and for $r = 1$ it is one-dimensional. For $\lambda = 1$ a generator is given by $[\alpha_2]$ ([Fu86, Th. 2.4.12]; there is a misprint in the formula!). From the $H^\bullet_c(\mathfrak{g}, \mathcal{F}_0)$-module structure of $H^\bullet_c(\mathfrak{g}, \mathcal{F}_1)$ we obtain the generators of $H^2_c(\mathfrak{g}, \mathcal{F}_1)$:

$$(\alpha_0 \wedge \alpha_2)(\xi, \eta) = \xi\eta'' - \eta\xi'' = \overline{\omega}_1, \quad (\alpha_1 \wedge \alpha_2)(\xi, \eta) = \xi'\eta'' - \eta'\xi'' = \omega_1.$$

Averaging over the rotation group, we see that every cocycle is equivalent to a rotation invariant one. From that it is easy to verify that for $\lambda = 2$ a generator of $H^1_c(\mathfrak{g}, \mathcal{F}_2)$ is given by $[\alpha_3]$, and we obtain for the basis elements of $H^2_c(\mathfrak{g}, \mathcal{F}_2)$:

$$(\alpha_0 \wedge \alpha_3)(\xi, \eta) = \xi\eta''' - \eta\xi''' = \overline{\omega}_2, \quad (\alpha_1 \wedge \alpha_3)(\xi, \eta) = \xi'\eta''' - \eta'\xi''' = \omega_2. \quad \blacksquare$$

For an explicit description of a basis of $H^2_c(\mathfrak{g}, \mathcal{F}_\lambda)$ for $\lambda = 5, 7$ we refer to [OR98].

Integrating Lie algebra cocycles to group cocycles

Now we translate the information on the Lie algebra cohomology $H^p_c(\mathfrak{g}, \mathcal{F}_\lambda)$ for $p = 0, 1, 2$ (Proposition 10.3) to the group G. Since the group G is connected, we have

$$H^0_s(G, \mathcal{F}_\lambda) = \mathcal{F}_\lambda^G = \mathcal{F}_\lambda^{\mathfrak{g}} = \begin{cases} \{0\} & \text{for } \lambda \neq 0 \\ \mathbb{R}1 & \text{for } \lambda = 0. \end{cases}$$

In degree 1, we can use Proposition 3.4 to see that we have an exact sequence

$$\mathbf{0} \to H^1_s(G, \mathcal{F}_\lambda) \xrightarrow{D} H^1_c(\mathfrak{g}, \mathcal{F}_\lambda) \xrightarrow{P} \mathcal{F}_\lambda^{\mathfrak{g}}.$$

For $\lambda \neq 0$ this implies that $D\colon H^1_s(G, \mathcal{F}_\lambda) \to H^1_c(\mathfrak{g}, \mathcal{F}_\lambda)$ is an isomorphism. For $\lambda = 0$ we have to calculate the period map P. Let $\mathfrak{t} := \mathbb{R}1 \cong \mathbb{R}\frac{d}{dx} \subseteq \mathfrak{g}$ be the one-dimensional subalgebra corresponding to the rigid rotations of the circle $\mathbb{S}^1$ and $T \cong \mathbb{T} \subseteq G$ the corresponding subgroup. Then the inclusion $T \hookrightarrow G$ induces an isomorphism $\pi_1(T) \to \pi_1(G)$, so that we can calculate P by restricting to T. Since $\mathfrak{t}$ corresponds to constant functions, the cocycle α_1 vanishes on $\mathfrak{t}$, and the cocycle α_0 is non-trivial on $\mathfrak{t}$. Hence

$$H^1_s(G, \mathcal{F}_0) \cong \ker P = \mathbb{R}[\alpha_1].$$

The group cocycle corresponding to $\alpha_1(\xi) = \xi'$ is $\theta(\varphi) = \log \varphi'$ (cf. Lemma 10.1) because for $\varphi = \mathrm{id}_\mathbb{R} + \xi$ we have

$$\theta(\mathrm{id} + \xi) = \log(1 + \xi') \sim \xi' + \ldots,$$

which implies $D\theta = \alpha_1$. Since the map $d\colon \mathcal{F}_0 \cong C^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathcal{F}_1 \cong \Omega^1(\mathbb{S}^1, \mathbb{R})$ is equivariant, we obtain a group cocycle

$$d \circ \theta \in Z^1_s(G, \mathcal{F}_1), \quad (d \circ \theta)(f) := \log(f')' = \frac{f''}{f'},$$

and for $\varphi = \mathrm{id} + \xi$ the relation $(d \circ \theta)(\mathrm{id} + \xi) = \frac{\xi''}{1 + \xi'}$ directly leads to $D(d \circ \theta) = \alpha_2$. The Schwarzian derivative

$$S \in Z^1_s(G, \mathcal{F}_2), \quad S(\varphi) := \Big(\frac{\varphi'''}{\varphi'} - \frac{3}{2}\Big(\frac{\varphi''}{\varphi'}\Big)^2\Big)$$

satisfies $DS = \alpha_3$. We thus have

$$H^1_s(G, \mathcal{F}_\lambda) = \begin{cases} \{0\} & \text{for } \lambda \neq 0, 1, 2 \\ \mathbb{R}[\theta] & \text{for } \lambda = 0 \\ \mathbb{R}[d \circ \theta] & \text{for } \lambda = 1 \\ \mathbb{R}[S] & \text{for } \lambda = 2. \end{cases}$$

On the simply connected covering group $q_G\colon \widetilde{G} \to G$ we have $H^1_s(\widetilde{G}, \mathcal{F}_\lambda) \cong H^1_c(\mathfrak{g}, \mathcal{F}_\lambda)$ (Proposition 3.4), so that we need an additional 1-cocycle for $\lambda = 0$, which is given by

$$L(\varphi) := \varphi - \mathrm{id}_\mathbb{R}.$$

In fact, $L(\psi\varphi) = L(\varphi\circ\psi) := \varphi\circ\psi-\psi+\psi-\mathrm{id}_{\mathbb{R}} = \psi^* L(\varphi)+L(\psi)$. Since $DL = \alpha_0$, we get

$$H^1_s(\widetilde{G}, \mathcal{F}_0) = \mathbb{R}[L] + \mathbb{R}[\theta],$$

where $\theta(\varphi) = \log\varphi'$.

Now we turn to the group cohomology in degree 2: In view of $\pi_1(G) \cong \mathbb{Z}$ and Theorem 7.2, we have a map

$$\delta\colon \operatorname{Hom}(\pi_1(G), \mathcal{F}_\lambda^G) \cong \mathcal{F}_\lambda^G \to H^2_s(G, \mathcal{F}_\lambda), \quad \delta(\gamma) = (\mathcal{F}_\lambda \rtimes \widetilde{G})/\Gamma(\gamma).$$

The kernel of this map coincides with the image of the restriction map

$$R\colon H^1_s(\widetilde{G}, \mathcal{F}_\lambda) \cong H^1_c(\mathfrak{g}, \mathcal{F}_\lambda) \to \operatorname{Hom}(\pi_1(G), \mathcal{F}_\lambda^G) \cong \mathcal{F}_\lambda^G$$

and the image of D coincides with the kernel of the map

$$P\colon H^2_c(\mathfrak{g}, \mathcal{F}_\lambda) \to \operatorname{Hom}(\pi_1(G), H^1_c(\mathfrak{g}, \mathcal{F}_\lambda)) \cong H^1_c(\mathfrak{g}, \mathcal{F}_\lambda).$$

The following proposition clarifies the relation between second Lie algebra and Lie group cohomology for the modules $\mathcal{F}_\lambda$. We refer to Appendix F for the definition of the $\cap$-product of Lie group cocycles.

Proposition 10.4. *For each $\lambda \in \mathbb{R}$ the map $D\colon H^2_s(G, \mathcal{F}_\lambda) \to H^2_c(\mathfrak{g}, \mathcal{F}_\lambda)$ is injective. It is bijective for $\lambda \notin \{0,1,2\}$. For $\lambda \in \{0,1,2\}$ we have*

$$H^2_s(G, \mathcal{F}_0) = \mathbb{R}[B_0], \quad H^2_s(G, \mathcal{F}_1) = \mathbb{R}[B_1], \quad H^2_s(G, \mathcal{F}_2) = \mathbb{R}[B_2]$$

for

$$B_0(\varphi, \psi) := -\int_0^1 \log((\psi\circ\varphi)')d(\log\varphi'), \quad B_1 := \theta \cap (d\circ\theta) \quad \textit{and} \quad B_2 := \theta \cap S.$$

Proof. (cf. [OR98]) First we show that D is injective for each λ. As above, let $T \cong \mathbb{T} \subseteq G$ be the subgroup corresponding to $\mathfrak{t} = \mathbb{R}1$ in $\mathfrak{g}$. Since the inclusion $T \hookrightarrow G$ induces an isomorphism $\pi_1(T) \to \pi_1(G)$, we can calculate R by using the factorization

$$H^1_s(\mathfrak{g}, \mathcal{F}_0) \to H^1_s(\mathfrak{t}, \mathcal{F}_0) \to \operatorname{Hom}(\pi_1(T), \mathcal{F}_0^G) \cong \mathcal{F}_0^G \cong \operatorname{Hom}(\pi_1(G), \mathcal{F}_0^G).$$

It is clear that the cocycle α_1 vanishes on $\mathfrak{t}$, but α_0 satisfies $\mathrm{per}_{\alpha_0}([\mathrm{id}_T]) = 1 \in \mathcal{F}_0^G$. Therefore the restriction map R is surjective for $\lambda = 0$, which implies $\delta = 0$. For all other values of λ the map δ vanishes because $\mathcal{F}_\lambda^G$ is trivial. Therefore D is injective for each λ.

For $\lambda \notin \{0,1,2\}$ the space $H^1_c(\mathfrak{g}, \mathcal{F}_\lambda)$ vanishes, so that $P = 0$ and $\mathrm{im}(D) = \ker(P)$ imply that D is surjective.

For $\lambda = 0, 1, 2$ the space $H^2_c(\mathfrak{g}, \mathcal{F}_\lambda)$ is two-dimensional (Proposition 10.3). To calculate P in these cases, let

$$\gamma\colon [0,1] \to T \subseteq G, \quad t \mapsto (x \mapsto x + t + \mathbb{Z})$$

be the generator of $\pi_1(G)$. We have

$$-\widetilde{F}_\omega(\gamma)(x) = \int_0^1 (i_{x_r}.\omega^{\mathrm{eq}})(\gamma'(t))\,dt = \int_0^1 \gamma(t).\omega\big(\operatorname{Ad}(\gamma(t))^{-1}.x, 1\big)\,dt.$$

This means that $\widetilde{F}_\omega(\gamma)$ is the T-equivariant part of the linear map $-i_1\omega\colon \mathfrak{g} \to \mathcal{F}_\lambda$.

For the cocycle $\overline{\omega}_\lambda(\xi,\eta) := \xi\eta^{(\lambda+1)} - \eta\xi^{(\lambda+1)}$ we have

$$\big(i_1\overline{\omega}_\lambda\big)(\eta) = \overline{\omega}_\lambda(1,\eta) = \eta^{(\lambda+1)}.$$

As 1 acts on each $\mathcal{F}_\lambda$ by $\xi \mapsto \xi'$, the linear map $\overline{\omega}_\lambda(1,\cdot)$ is T-equivariant, hence equal to $-\widetilde{F}_\omega(\gamma)$, and we obtain

$$F_{\overline{\omega}_\lambda}(1) = [\widetilde{F}_\omega(\gamma)], \quad \widetilde{F}_\omega(\gamma)(\eta) = \eta^{(\lambda+1)}, \quad \text{for} \quad \lambda = 0,1,2.$$

For $\omega_0(\xi,\eta) := \int_{\mathbb{S}^1} \xi'\eta'' - \xi''\eta'$ we have $\omega_0(1,\eta) = 0$, so that $F_{\omega_0} = 0$, and likewise $\omega_\lambda(1,\eta) = 0$ for $\lambda = 1,2$ leads to $F_{\omega_\lambda} = 0$ for $\lambda = 1,2$.

We conclude that for $\lambda = 0,1,2$ the kernel of P is one-dimensional, and that

$$\operatorname{im}(D) = \ker(P) = \mathbb{R}[\omega_\lambda].$$

For $\lambda = 0$ the Thurston–Bott cocycle (for $\operatorname{Diff}(\mathbb{S}^1)^{\mathrm{op}}$)

$$B_0 \in Z^2_s(G,\mathbb{R}) \subseteq Z^2_s(G,\mathcal{F}_0), \quad B_0(\varphi,\psi) = -\int_0^1 \log((\psi\circ\varphi)')d(\log\varphi')$$

satisfies $DB_0 = \omega_0$ (cf. [GF68]). For $\lambda = 1,2$, we recall that $\omega_\lambda = \alpha_1 \wedge \alpha_{\lambda+1}$, so that Lemma F.3 implies that the cocycles

$$B_1 := \theta \cap (d\circ\theta) \quad \text{and} \quad B_2 := \theta \cap S$$

satisfy $DB_\lambda = \omega_\lambda$. This completes the proof. ∎

Proposition 10.5. *For the simply connected covering group $\widetilde{G}$ of G we have*

$$H^2_s(\widetilde{G},\mathcal{F}_\lambda) = \mathbb{R}[B_\lambda] \oplus \mathbb{R}[\overline{B}_\lambda] \cong \mathbb{R}^2 \quad \textit{for} \quad \lambda = 0,1,2,$$

where

$$\overline{B}_0 := L \cap \theta, \quad \overline{B}_1 := L \cap (d\circ\theta), \quad \overline{B}_2 := L \cap S,$$

and B_λ is the pull-back of the corresponding cocycle on G.

Proof. Since the simply connected covering group $\widetilde{G}$ is contractible, the derivation map

$$D\colon H^2_s(\widetilde{G}, \mathcal{F}_\lambda) \to H^2_c(\mathfrak{g}, \mathcal{F}_\lambda)$$

is bijective, so that we obtain larger cohomology spaces of $\widetilde{G}$ than for G. For $\lambda = 0, 1, 2$ we have $\overline{\omega}_\lambda = \alpha_0 \wedge \alpha_{\lambda+1}$, so that the cocycles $\overline{B}_j$, $j = 0, 1, 2$, satisfy $D\overline{B}_\lambda = \overline{\omega}_\lambda$ (Lemma F.3). Combining this with the pull-backs of the cocycles B_λ from G, the assertion follows. ■

A non-trivial abelian extension of $\mathrm{SL}_2(\mathbb{R})$

We consider the right action of $\mathrm{SL}_2(\mathbb{R})$ on the projective line $\mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ by

$$x.\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.x := \frac{dx - b}{-cx + a}.$$

In particular the action of the rotation group $\mathrm{SO}_2(\mathbb{R})$ is given by

$$\begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix}.x = \frac{\cos \pi t \cdot x - \sin \pi t}{\sin \pi t \cdot x + \cos \pi t},$$

so that

$$\begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix}.0 = -\tan \pi t$$

and the map $t \mapsto \tan \pi t$ induces a diffeomorphism $\mathbb{R}/\mathbb{Z} \to \mathbb{P}_1(\mathbb{R})$. We use this diffeomorphism to identify $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with $\mathbb{P}_1(\mathbb{R})$ and to obtain a smooth right action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{S}^1$. Then $\mathfrak{sl}_2(\mathbb{R})$ is isomorphic to a 3-dimensional subalgebra of $\mathcal{V}(\mathbb{S}^1)$ and $\mathfrak{so}_2(\mathbb{R})$ corresponds to $\mathbb{R}1 = \mathfrak{t}$. We put

$$U := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and observe that this element corresponds to the constant function $\frac{1}{\pi}$. From the relation $\operatorname{ad} U((\operatorname{ad} U)^2 + 4) = 0$ on $\mathfrak{sl}_2(\mathbb{R})$ and the formula for commutators in $\mathcal{V}(\mathbb{S}^1)$ we therefore derive

$$\mathfrak{sl}_2(\mathbb{R}) = \operatorname{span}\{1, \cos(2\pi t), \sin(2\pi t)\}$$

as a subalgebra of $\mathcal{V}(\mathbb{S}^1) \cong C^\infty(\mathbb{S}^1)$. We may therefore pick $H, P \in \mathfrak{sl}_2(\mathbb{R})$ with $[U, H] = -2P$ and $[U, P] = 2H$ such that H corresponds to the function $\cos(2\pi t)$ and P to the function $\sin(2\pi t)$.

The corresponding group homomorphism

$$\sigma\colon\ \mathrm{SL}_2(\mathbb{R}) \to \mathrm{Diff}(\mathbb{S}^1)_0^{\mathrm{op}}$$

is homotopy equivalent to the twofold covering of $T \cong \mathbb{S}^1$, hence induces an injection

$$\pi_1(\sigma)\colon \pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{Z} \to \pi_1(\mathrm{Diff}(\mathbb{S}^1)) \cong \mathbb{Z}$$

onto a subgroup of index 2.

From the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{S}^1$, we obtain a smooth action on the Fréchet spaces

$$\mathcal{F}_\lambda := C^\infty(\mathbb{S}^1, \mathbb{R}), \quad (g.f)(x) := \big(\sigma(g)'\big)^\lambda f(x.g).$$

By restriction to the subalgebra $\mathfrak{sl}_2(\mathbb{R}) \subseteq \mathcal{V}(\mathbb{S}^1)$, we obtain the 2-cocycle $\omega(\xi, \eta) = \xi'\eta'' - \xi''\eta'$ in $Z^2_c(\mathfrak{sl}_2(\mathbb{R}), \mathcal{F}_1)$. Let $\gamma\colon I \to \mathrm{SL}_2(\mathbb{R}), t \mapsto \exp(2\pi t U)$ be the canonical generator of $\pi_1(\mathrm{SL}_2(\mathbb{R}))$. As in the proof of Proposition 10.4, it then follows that

$$F_\omega\colon \pi_1(\mathrm{SL}_2(\mathbb{R})) \to H^1_c(\mathfrak{sl}_2(\mathbb{R}), \mathcal{F}_1)$$

is given by $F_\omega([\gamma]) = [\widetilde{F}_\omega(\gamma)]$, where $\widetilde{F}_\omega(\gamma)$ is the $\mathfrak{t}$-invariant part of $2i_1\omega = 0$, hence $F_\omega = 0$.

Next we show that $[\omega] \neq 0$ in $H^2_c(\mathfrak{sl}_2(\mathbb{R}), \mathcal{F}_1)$. If this is not the case, then there exists a linear map $\alpha\colon \mathfrak{sl}_2(\mathbb{R}) \to \mathcal{F}_1$ with $\omega = d_{\mathfrak{sl}_2(\mathbb{R})}\alpha$. Since ω is T-equivariant, we may assume, after averaging over the compact group T, that α is also T-invariant, i.e.,

$$\alpha([U, x]) = U.\alpha(x), \quad x \in \mathfrak{sl}_2(\mathbb{R}).$$

Now

$$0 = i_U\omega = i_U d_{\mathfrak{sl}_2(\mathbb{R})}\alpha = \mathcal{L}_U\alpha - di_U\alpha = -di_U\alpha$$

implies

$$i_U\alpha = \alpha(U) \in Z^0(\mathfrak{sl}_2(\mathbb{R}), \mathcal{F}_1) = \mathcal{F}_1^{\mathfrak{sl}_2(\mathbb{R})} = \{0\}.$$

We now derive from $[H, P] \in \mathbb{R}U$:

$$\omega(H, P) = d_{\mathfrak{sl}_2(\mathbb{R})}\alpha(H, P) = H.\alpha(P) - P.\alpha(H) - \alpha([H, P]) = H.\alpha(P) - P.\alpha(H).$$

Further the equivariance of α implies the existence of $a, b \in \mathbb{R}$ with

$$\alpha(P) = a\cos(2\pi t) + b\sin(2\pi t) \quad \text{and} \quad \alpha(H) = \frac{1}{2}\alpha([U, P]) = \frac{1}{2}U.\alpha(P)$$
$$= -a\sin(2\pi t) + b\cos(2\pi t).$$

We further have

$$H.\alpha(P) = \cos(2\pi t).(a\cos(2\pi t) + b\sin(2\pi t)) = (a\cos^2(2\pi t) + b\sin(2\pi t)\cos(2\pi t))'$$

and

$$P.\alpha(H) = \sin(2\pi t).(-a\sin(2\pi t) + b\cos(2\pi t)) = (-a\sin^2(2\pi t) + b\sin(2\pi t)\cos(2\pi t))',$$

so that

$$\omega(H,P) = H.\alpha(P) - P.\alpha(H) = a(\cos^2(2\pi t) + \sin^2(2\pi t))' = a1' = 0,$$

contradicting

$$\omega(H,P) = \cos(2\pi t)' \sin(2\pi t)'' - \cos(2\pi t)'' \sin(2\pi t)' = 8\pi^3(\sin^3(2\pi t) + \cos^3(2\pi t)) \neq 0.$$

Therefore $[\omega] \neq 0$. Since F_ω and per_ω vanish, and

$$H^2_{\mathrm{dR}}(\mathrm{SL}_2(\mathbb{R}), \mathcal{F}_1) \cong H^2_{\mathrm{dR}}(\mathbb{S}^1, \mathcal{F}_1) = \{0\},$$

there exists a smooth 2-cocycle $f \in Z^2_s(\mathrm{SL}_2(\mathbb{R}), \mathcal{F}_1)$ with $Df = \omega$ (Proposition 8.4). Then the group

$$\mathcal{F}_1 \times_f \mathrm{SL}_2(\mathbb{R})$$

is a non-trivial abelian extension of $\mathrm{SL}_2(\mathbb{R})$.

If V is a trivial $\mathfrak{sl}_2(\mathbb{R})$-module, then the range of each 2-cocycle lies in a 3-dimensional subspace, hence is a coboundary, because the corresponding assertion holds for finite-dimensional modules. Therefore all central extensions of $\mathrm{SL}_2(\mathbb{R})$ by abelian Lie groups of the form $A = \mathfrak{a}/\Gamma_A$ are trivial (Theorem 7.2). The preceding example shows that $H^2_c(\mathfrak{sl}_2(\mathbb{R}), \mathcal{F}_1) \neq \{0\}$, which provides the non-trivial extension of $\mathrm{SL}_2(\mathbb{R})$ constructed above.

The choice of the cocycle ω above is most natural because one can show that the cohomology of the $\mathfrak{sl}_2(\mathbb{R})$-modules $\mathcal{F}_\lambda$ satisfies

$$\dim H^2_c(\mathfrak{sl}_2(\mathbb{R}), \mathcal{F}_\lambda) = \begin{cases} 0 & \text{for } \lambda \neq 0,1 \\ 1 & \text{for } \lambda = 0 \\ 2 & \text{for } \lambda = 1, \end{cases} \qquad \dim H^1_c(\mathfrak{sl}_2(\mathbb{R}), \mathcal{F}_\lambda) = \begin{cases} 0 & \text{for } \lambda \neq 0,1 \\ 2 & \text{for } \lambda = 0 \\ 1 & \text{for } \lambda = 1. \end{cases}$$

For $\lambda = 0$ the flux homomorphism yields an injective map

$$H^2_c(\mathfrak{sl}_2(\mathbb{R}), \mathcal{F}_\lambda) \to \mathrm{Hom}(\pi_1(\mathrm{SL}_2(\mathbb{R})), H^1_c(\mathfrak{sl}_2(\mathbb{R}), \mathcal{F}_\lambda) \cong H^1_c(\mathfrak{sl}_2(\mathbb{R}), \mathcal{F}_\lambda), \tag{10.2}$$

so that we only obtain non-trivial abelian extensions of the universal covering group $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. For $\lambda = 1$ the kernel of (10.2) is one-dimensional and spanned by $[\omega]$, so that $[\omega]$ is, up to scalar multiples, the only non-trivial 2-cohomology class associated to the modules $\mathcal{F}_\lambda$ which integrates to a group cocycle on $\mathrm{SL}_2(\mathbb{R})$.

11. Central extensions of groups of volume preserving diffeomorphisms

In the present section we discuss certain central extensions of the group $\mathrm{Diff}(M,\mu)$ of diffeomorphisms of a compact connected orientable manifold M preserving a volume form μ, resp., its identity component $D(M,\mu)$ of $\mathrm{Diff}(M,\mu)_0$. Each closed

$\mathfrak{z}$-valued 2-form ω on M defines a central extension of the corresponding Lie algebra $\mathcal{V}(M,\mu)$ of μ-divergence free vector fields because composing integration over M with respect to μ with the $C^\infty(M,\mathfrak{z})$-valued cocycle defined by the 2-form (cf. Section 9) leads to a $\mathfrak{z}$-valued 2-cocycle, the so-called *Lichnerowicz cocycle* (cf. [Vi02], [Li74]). We shall see that if $\pi_2(M)$ vanishes, then the only obstruction to the integrability of the corresponding central extension is given by the flux homomorphism $\pi_1(D(M,\mu)) \to H^1_{\rm dR}(M,\mathfrak{z})$. If M is a compact Lie group, we show that the flux becomes trivial on the covering group $\widetilde{D}(M,\mu)$ of $D(M,\mu)$ acting on the universal covering manifold $\widetilde{M}$ of M, which leads to central Lie group extensions of $\widetilde{D}(M,\mu)$.

Some facts on the flux homomorphism for volume forms

In this short subsection we collect some facts on the flux homomorphism of a volume form on a compact connected manifold. These results will be used to show that each closed 2-form on a compact Lie group G defines a central extension of the covering $\widetilde{D}(G,\mu)$ of the identity component $D(G,\mu)$ of the group of volume preserving diffeomorphisms of G which acts faithfully on the universal covering group $\widetilde{G}$.

Let M be a smooth compact manifold, $\mathfrak{z}$ a Mackey complete locally convex space and $\omega \in \Omega^p(M,\mathfrak{z})$ a closed $\mathfrak{z}$-valued p-form. For a piecewise smooth curve $\alpha\colon I \to \mathrm{Diff}(M)$ we define the *flux form*

$$\widetilde{F}_\omega(\alpha) := \int_0^1 \alpha(t)^*\big(i_{\delta^l(\alpha)(t)}\omega\big)\,dt = \int_0^1 i_{\alpha(t)^{-1}.\alpha'(t)}(\alpha(t)^*\omega)\,dt \in \Omega^{p-1}(M,\mathfrak{z}).$$

Let $\alpha\colon I \to \mathrm{Diff}(M)$ be a piecewise smooth path and $\sigma\colon \Delta_{p-1} \to M$ a smooth singular simplex. Further define

$$\alpha.\sigma\colon I \times \Delta_{p-1} \to M, \quad (t,x) \mapsto \alpha(t).\sigma(x).$$

Then

$$\begin{aligned}
&((\alpha.\sigma)^*\omega)(t,x)\Big(\frac{\partial}{\partial t}, v_1,\ldots,v_{p-1}\Big)\\
&= \omega(\alpha(t).\sigma(x))\big(\alpha'(t)(\sigma(x)), \alpha(t).d\sigma(x)v_1,\ldots,\alpha(t).d\sigma(x)v_{p-1}\big)\\
&= (\alpha(t)^*\omega)(\sigma(x))\big(\alpha(t)^{-1}.\alpha'(t)(\sigma(x)), d\sigma(x)v_1,\ldots,d\sigma(x)v_{p-1}\big)\\
&= \big(i_{\alpha(t)^{-1}.\alpha'(t)}\big(\alpha(t)^*\omega\big)\big)(\sigma(x))\big(d\sigma(x)v_1,\ldots,d\sigma(x)v_{p-1}\big)
\end{aligned}$$

(cf. [NV03, Lemma 1.7]) implies

$$\int_{\alpha.\sigma}\omega = \int_{I\times\Delta_{p-1}}(\alpha.\sigma)^*\omega = \int_\sigma \widetilde{F}_\omega(\alpha).$$

We thus obtain

$$\int_{\alpha.\Sigma}\omega = \int_\Sigma \widetilde{F}_\omega(\alpha)$$

for each singular chain Σ if we extend the map $\sigma \mapsto \alpha.\sigma$ additively to the group of piecewise smooth singular chains. If Σ is a boundary and α is closed, then $\alpha.\Sigma$ is a boundary, so that the integral vanishes by Stoke's Theorem, and therefore $\int_\Sigma \widetilde{F}_\omega(\alpha)$ vanishes. We conclude that $\widetilde{F}_\omega(\alpha)$ is a closed $(p-1)$-form, so that we obtain a group homomorphism

$$F_\omega \colon \pi_1(\mathrm{Diff}(M)) \to H^{p-1}_{\mathrm{dR}}(M, \mathfrak{z}), \quad [\alpha] \mapsto [\widetilde{F}_\omega(\alpha)].$$

Lemma 11.1. *If M is an oriented compact manifold of dimension n, $m_0 \in M$, and μ a volume form on M with $\int_M \mu = 1$, then the kernel of the corresponding flux homomorphism*

$$F_\mu \colon \pi_1(\mathrm{Diff}(M)) \to H^{n-1}_{\mathrm{dR}}(M, \mathbb{R}), \quad [\alpha] \mapsto [\widetilde{F}_\mu(\alpha)]$$

contains the kernel of

$$\pi_1(\mathrm{ev}^D_{m_0}) \colon \pi_1(\mathrm{Diff}(M)) \to \pi_1(M, m_0).$$

Proof. (We are grateful to Stephan Haller for communicating to us the idea of the following proof.) To each smooth loop $\alpha \colon \mathbb{S}^1 \to \mathrm{Diff}(M)$ with $\alpha(1) = \mathrm{id}_M$ we associate a locally trivial fiber bundle $q_\alpha \colon P_\alpha \to \mathbb{S}^2$ whose underlying topological space is obtained as follows. We think of $\mathbb{S}^2$ as a union of two closed discs B_1 and B_2 with $B_1 \cap B_2 = \mathbb{S}^1$. Then we put

$$P_\alpha := \Big((B_1 \times M) \dot\cup (B_2 \times M)\Big) / \sim,$$

where

$$(x,m) \sim (x',m') \colon \Leftrightarrow \begin{cases} x = x' \notin \partial B_1 \cup \partial B_2, \ m = m' \\ x = x' \in \partial B_1, m' = \alpha(x)(m). \end{cases}$$

Then $q_\alpha([x,m]) := x$ defines the structure of a locally trivial fiber bundle with fiber M over $\mathbb{S}^2$.

A section of P_α is a pair of two continuous maps $\widetilde{\sigma}_j \colon B_j \to M$, $j = 1,2$, such that the restrictions $\sigma_j := \widetilde{\sigma}_j \mid_{\partial B_j}$ satisfy $\sigma_2(x) = \alpha(x)(\sigma_1(x))$ for all $x \in \partial B_j$. This means that σ_1 and σ_2 are contractible loops in M with $\alpha.\sigma_1 = \sigma_2$. Conversely, every pair of contractible loops σ_1 and σ_2 in M satisfying $\alpha.\sigma_1 = \sigma_2$ can be extended to continuous maps $B_j \to M$ and thus to a section of P_α.

If σ_1 is a contractible loop based in m_0, then $\alpha.\sigma_1$ is a loop based in m_0 homotopic to the loop $x \mapsto \alpha(x)(m_0)$. Therefore the existence of a continuous section of P_α is equivalent to $[\alpha] \in \ker \pi_1(\mathrm{ev}^D_{m_0})$.

Suppose that $[\alpha] \in \ker \pi_1(\mathrm{ev}^D_{m_0})$ and that $\sigma \colon \mathbb{S}^2 \to P_\alpha$ is a corresponding section. It follows easily from the construction of P_α that the manifold P_α is orientable if M is orientable. Hence the 2-cycle $[\sigma]$ has a Poincaré dual $[\beta] \in H^n_{\mathrm{sing}}(P_\alpha, \mathbb{Z})$ whose restriction to a fiber M is the Poincaré dual of the intersection of $\mathrm{im}(\sigma)$ with a fiber, hence the fundamental class $[\mu] \in H^n_{\mathrm{sing}}(M, \mathbb{Z})$ ([Bre93, p.372]). Therefore the fundamental class of M extends to an n-dimensional cohomology class in P_α.

On the other hand, we obtain from [Sp66, p.455] the exact Wang cohomology sequence associated to P_α:

$$\ldots \to H^n_{\rm sing}(P_\alpha, \mathbb{Z}) \to H^n_{\rm sing}(M, \mathbb{Z}) \xrightarrow{\partial_\alpha} H^{n-1}_{\rm sing}(M, \mathbb{Z}) \to H^{n-1}_{\rm sing}(P_\alpha, \mathbb{Z}) \to \ldots,$$

where ∂_α satisfies

$$\langle \partial_\alpha[\beta], [\Sigma]\rangle = \langle [\beta], [\alpha.\Sigma]\rangle$$

for each $(n-1)$-cycle Σ in M, and the kernel of ∂_α consists of those cohomology classes extending to P_α. As this is the case for the fundamental class of M, it follows that $[\alpha.\Sigma] = 0$ holds for all $(n-1)$-cycles Σ on M. We conclude that $\widetilde{F}_\mu(\alpha)$ is an exact $(n-1)$-form if $[\alpha] \in \ker \pi_1(\mathrm{ev}^D_{m_0})$. ∎

Remark 11.2. Suppose that G is a compact Lie group of dimension d. Then G is orientable and we can identify G with the group $\lambda(G)$ of left translations in $\mathrm{Diff}(G)$. Then

$$\mathrm{Diff}(G) = \mathrm{Diff}(G)_{\mathbf{1}}\lambda(G) \cong \mathrm{Diff}(G)_{\mathbf{1}} \times G$$

as smooth manifolds, where $\mathrm{Diff}(G)_{\mathbf{1}}$ denotes the stabilizer of $\mathbf{1} \in G$ in $\mathrm{Diff}(G)$. In particular we have

$$\pi_1(\mathrm{Diff}(G)) \cong \pi_1(\mathrm{Diff}(G)_{\mathbf{1}}) \times \pi_1(G).$$

If μ is a normalized biinvariant volume form on G, then Lemma 9.1 implies that the corresponding flux homomorphism

$$F_\mu \colon \pi_1(\mathrm{Diff}(G)) \to H^{d-1}_{\rm dR}(G, \mathbb{R})$$

factors through a homomorphism

$$F^\sharp_\mu \colon \pi_1(G) \to H^{d-1}_{\rm dR}(G, \mathbb{R}).$$

Let $q_G \colon \widetilde{G} \to G$ denote the universal covering homomorphism and

$$\widetilde{\mathrm{Diff}}(G) := \{\widetilde{\varphi} \in \mathrm{Diff}(\widetilde{G}) \colon (\exists \varphi \in \mathrm{Diff}(G))\ \varphi \circ q_G = q_G \circ \widetilde{\varphi}\}.$$

Then we have a canonical homomorphism

$$Q_G \colon \widetilde{\mathrm{Diff}}(G) \to \mathrm{Diff}(G), \quad \widetilde{\varphi} \mapsto \varphi$$

whose kernel coincides with the group of deck transformations that is isomorphic to $\pi_1(G)$. We endow $\widetilde{\mathrm{Diff}}(G)$ with the Lie group structure turning Q_G into a covering map. We then have

$$\widetilde{\mathrm{Diff}}(G) = \widetilde{\mathrm{Diff}}(G)_{\mathbf{1}}\widetilde{G} \cong \widetilde{\mathrm{Diff}}(G)_{\mathbf{1}} \rtimes \widetilde{G} \cong \mathrm{Diff}(G)_{\mathbf{1}} \rtimes \widetilde{G}$$

as smooth manifolds, so that

$$\pi_1(\widetilde{\mathrm{Diff}}(G)) \cong \pi_1(\mathrm{Diff}(G)_{\mathbf{1}}).$$

The identity component $\widetilde{\mathrm{Diff}}(G)_0$ is a covering of $\mathrm{Diff}(G)_0$, and since the flux homomorphism vanishes on its fundamental group (Lemma 9.1), the flux cocycle

$$f_\mu \colon \mathcal{V}(G) \to \widehat{H}^{d-1}_{\mathrm{dR}}(G, \mathbb{R}), \quad X \mapsto [i_X \mu]$$

integrates to a group cocycle

$$F_\mu \colon \widetilde{\mathrm{Diff}}(G)_0^{\mathrm{op}} \to \widehat{H}^{d-1}_{\mathrm{dR}}(G, \mathbb{R}) = \Omega^{d-1}(G, \mathbb{R})/d\Omega^{d-2}(G, \mathbb{R})$$

with $DF_\mu = f_\mu$.

Application to central extensions

In this subsection we apply the tools developed in the present paper to central extensions of groups of volume preserving diffeomorphisms of compact manifolds.

Let M denote an orientable connected compact manifold and μ a volume form on M, normalized by $\int_M \mu = 1$. We write

$$D(M, \mu) := \{\varphi \in \mathrm{Diff}(M)^{\mathrm{op}} \colon \varphi^*\mu = \mu\}_0$$

for the identity component of the *group of volume preserving diffeomorphisms of* (M, μ) and

$$\mathfrak{g}_\mu := \mathcal{V}(M, \mu) := \{X \in \mathcal{V}(M) \colon \mathcal{L}_X \mu = 0\}$$

for its Lie algebra. Further let $\widetilde{D}(M, \mu) \subseteq \mathrm{Diff}(\widetilde{M})$ denote the identity component of the inverse image of $D(M, \mu)$ in

$$\widetilde{\mathrm{Diff}}(M) := \{\widetilde{\varphi} \in \mathrm{Diff}(\widetilde{M}) \colon (\exists \varphi \in \mathrm{Diff}(M))\ \varphi \circ q_M = q_M \circ \widetilde{\varphi}\},$$

where $q_M \colon \widetilde{M} \to M$ denotes a universal covering. Then we have a covering map $\widetilde{D}(M, \mu) \to D(M, \mu)$ which need not be universal. We write $\widetilde{D(M, \mu)}$ for the universal covering group of $D(M, \mu)$ which also is a covering group of $\widetilde{D}(M, \mu)$.

Let $\mathfrak{z}$ be a Fréchet space. On the space $C^\infty(M, \mathfrak{z})$ of smooth $\mathfrak{z}$-valued functions on M we then have the integration map

$$I \colon C^\infty(M, \mathfrak{z}) \to \mathfrak{z}, \quad f \mapsto \int_M f\mu.$$

Then I is equivariant for the natural action of $D(M, \mu)$ on $C^\infty(M, \mathfrak{z})$, where we consider $\mathfrak{z}$ as a trivial module. On the infinitesimal level this means that

$$\int_M (X.f)\, \mu = 0 \quad \text{for} \quad f \in C^\infty(M, \mathbb{R}), \quad X \in \mathcal{V}(M, \mu).$$

Each closed $\mathfrak{z}$-valued p-form $\omega \in \Omega^p(M, \mathfrak{z})$ defines a $C^\infty(M, \mathfrak{z})$-valued p-cochain for the action of the Lie algebra $\mathfrak{g}_\mu$ on $C^\infty(M, \mathfrak{z})$ and since I is $\mathfrak{g}_\mu$-equivariant, we obtain continuous linear maps

$$\Phi \colon \Omega^p(M, \mathfrak{z}) \to C^p_c(\mathfrak{g}_\mu, \mathfrak{z}), \quad \Phi(\omega)(X_1, \ldots, X_p) := I(\omega(X_1, \ldots, X_p)) = \int_M \omega(X_1, \ldots, X_p)\, \mu.$$

The equivariance of I implies that $\Phi(d\omega) = d_{\mathfrak{g}_\mu} \Phi(\omega)$, so that Φ induces maps

$$\Phi \colon H^p_{\mathrm{dR}}(M, \mathfrak{z}) \to H^p_c(\mathfrak{g}_\mu, \mathfrak{z}).$$

Remark 11.3. If $\pi_2(M) = \{0\}$ and $\widetilde{D(M, \mu)}$ denotes the simply connected covering group of $D(M, \mu)$, then for each closed 2-form $\omega \in Z^2_{\mathrm{dR}}(M, \mathfrak{z})$ the period map of the corresponding Lie algebra cocycle vanishes (Proposition 9.5), so that, in view of Theorem 7.2, Φ induces a map

$$\Phi \colon H^2_{\mathrm{dR}}(M, \mathfrak{z}) \to H^2_s(\widetilde{D(M, \mu)}, \mathfrak{z}).$$

If, more generally, $\Gamma_Z \subseteq \mathfrak{z}$ is a discrete subgroup with $\int_{\pi_2(M)} \omega \subseteq \Gamma_Z$ and $Z := \mathfrak{z}/\Gamma_Z$, then Theorem 7.2 implies that the Lie algebra cocycle ω integrates to a central extension

$$Z \hookrightarrow \widehat{D}(M, \mu) \twoheadrightarrow \widetilde{D(M, \mu)}.$$

Let

$$\mathcal{V}(M, \mu)_{\mathrm{ex}} := \{X \in \mathcal{V}(M, \mu) \colon i_X \mu \in d\Omega^{p-2}(M, \mathbb{R})\}$$

denote the Lie algebra of exact divergence free vector fields. It can be shown that this is the commutator algebra of $\mathcal{V}(M, \mu)$ and even a perfect Lie algebra (cf. [Li74]). It follows in particular that

$$H^1_c(\mathcal{V}(M, \mu)_{\mathrm{ex}}, \mathfrak{z}) = \mathrm{Hom}_{\mathrm{Lie\ alg}}(\mathcal{V}(M, \mu)_{\mathrm{ex}}, \mathfrak{z}) = \{0\}$$

for each trivial module $\mathfrak{z}$. Therefore, restricing the cocycles from above to $\mathcal{V}(M, \mu)_{\mathrm{ex}}$, resp. the corresponding connected subgroup $D(M, \mu)_{\mathrm{ex}}$ of exact volume preserving diffeomorphisms leads to a trivial flux homomorphism. Hence $\int_{\pi_2(M)} \omega \subseteq \Gamma_Z$ implies the existence of a central Z-extension of $D(M, \mu)_{\mathrm{ex}}$. We refer to Ismagilov ([Is96]) and Haller-Vizman ([HV04]) for geometric constructions of these central extensions (for $\mathfrak{z} = \mathbb{R}, Z = \mathbb{T} = \mathbb{R}/\mathbb{Z}$). ■

Proposition 11.4. *Let G be a compact connected Lie group and μ an invariant normalized volume form on G. Then the flux cocycle restricts to a surjective Lie algebra homomorphism*

$$f_\mu \colon \mathcal{V}(G, \mu) \to H^{d-1}_{\mathrm{dR}}(G, \mathbb{R})$$

whose kernel is the commutator algebra and whose restriction to $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g} \subseteq \mathcal{V}(G,\mu)$ *is bijective. This Lie algebra homomorphism integrates to a homomorphism of connected Lie groups*

$$F_\mu^G \colon \widetilde{D}(G,\mu) \to H^{d-1}_{\rm dR}(G,\mathbb{R})$$

whose restriction to $Z(\widetilde{G})_0 \subseteq \widetilde{G} \subseteq \widetilde{D}(G,\mu)$ *is an isomorphism. Moreover, each Lie algebra homomorphism* $\varphi_{\mathfrak{g}} \colon \mathcal{V}(G,\mu) \to \mathfrak{a}$ *to an abelian Lie algebra integrates to a group homomorphism* $\varphi_G \colon \widetilde{D}(G,\mu) \to \mathfrak{a}$ *which factors through* F_μ^G *.*

Proof. Since f_μ defines a Lie algebra homomorphism $\mathcal{V}(G,\mu) \to H^{d-1}_{\rm dR}(G,\mathbb{R})$, the restriction of the flux cocycle $F_\mu \colon \widetilde{\rm Diff}(G)_0 \to \widehat{H}^{d-1}_{\rm dR}(G,\mathbb{R})$ to the subgroup $\widetilde{D}(G,\mu)$ is a group homomorphism

$$F_\mu \colon \widetilde{D}(G,\mu) \to H^{d-1}_{\rm dR}(G,\mathbb{R}) \cong H^{d-1}(\mathfrak{g},\mathbb{R})$$

which on the subgroup $\widetilde{G}$ of $\widetilde{D}(G,\mu)$ is the Lie group homomorphism obtained by integrating the Lie algebra quotient homomorphism

$$\mathfrak{g} \to H^{d-1}(\mathfrak{g},\mathbb{R}), \quad x \mapsto [i_x\mu_{\mathfrak{g}}],$$

where $\mu_{\mathfrak{g}} := \mu(\mathbf{1}) \in C^d(\mathfrak{g},\mathbb{R})$. Note that Poincaré Duality implies that

$$H^{d-1}_{\rm dR}(G,\mathbb{R}) \cong H^1_{\rm dR}(G,\mathbb{R}) \cong \operatorname{Hom}(\mathfrak{g},\mathbb{R}) \cong \mathfrak{z}(\mathfrak{g})^*$$

so that $H^{d-1}_{\rm dR}(G,\mathbb{R}) \cong Z(\widetilde{G})_0 \cong \mathfrak{z}(\mathfrak{g})$ and we can think of the flux homomorphism as a group homomorphism

$$F_\mu^G \colon \widetilde{D}(G,\mu) \to \mathfrak{z}(\mathfrak{g}).$$

On the Lie algebra level we have $\mathfrak{g} \subseteq \mathcal{V}(G,\mu)$, $[\mathcal{V}(G,\mu),\mathcal{V}(G,\mu)] \subseteq \ker f_\mu$, and f_μ maps $\mathfrak{z}(\mathfrak{g})$ isomorphically onto $H^{d-1}_{\rm dR}(G,\mathbb{R})$. This leads to

$$\mathcal{V}(G,\mu) = [\mathcal{V}(G,\mu),\mathcal{V}(G,\mu)] \rtimes \mathfrak{z}(\mathfrak{g})$$

with $H_1(\mathcal{V}(G,\mu)) \cong \mathfrak{z}(\mathfrak{g})$, and we conclude that the flux homomorphism $F_\mu^G \colon \widetilde{D}(G,\mu) \to \mathfrak{z}(\mathfrak{g})$ is universal in the sense that each Lie algebra homomorphism $\mathcal{V}(G,\mu) \to \mathfrak{a}$, where $\mathfrak{a}$ is an abelian Lie algebra, integrates to a Lie group homomorphism $\widetilde{D}(G,\mu) \to \mathfrak{a}$. ■

Theorem 11.5. *Let* G *be a connected compact Lie group,* μ *an invariant normalized volume form,* $\mathfrak{z}$ *a Mackey complete locally convex space and* $\omega \in \Omega^2(G,\mathfrak{z})$ *a closed* 2*-form. Then the Lichnerowicz cocycle on* $\mathcal{V}(G,\mu)$ *given by*

$$(X,Y) \mapsto \int_G \omega(X,Y)\cdot\mu$$

integrates to a central Lie group extension

$$\mathfrak{z} \to \widehat{D}(G,\mu) \to \widetilde{D}(G,\mu).$$

Proof. First we recall that $\pi_2(G) = \{0\}$ ([Car52]), so that Remark 11.3 implies that the period map of $\widetilde{D}(G,\mu)$ vanishes for each closed 2-form $\omega \in \Omega^2(G,\mathfrak{z})$ on G. Moreover, the flux cocycle is a Lie algebra homomorphism

$$f_\omega \colon \mathfrak{g}_\mu = \mathcal{V}(G,\mu) \to H^1_c(\mathfrak{g}_\mu,\mathfrak{z}) \cong \mathrm{Hom}(\mathfrak{g}_\mu,\mathfrak{z}) \cong \mathrm{Hom}(\mathfrak{z}(\mathfrak{g}),\mathfrak{z})$$

so that Proposition 11.4 implies that the corresponding flux homomorphism vanishes on the fundamental group $\pi_1(\widetilde{D}(G,\mu))$, and Theorem 7.2 implies that ω defines a Lie algebra cocycle in $Z^2_c(\mathcal{V}(G,\mu),\mathfrak{z})$ corresponding to a global central extension as required. ■

Remark 11.6. In view of

$$H^2_{\mathrm{dR}}(G,\mathfrak{z}) \cong H^2_c(\mathfrak{g},\mathfrak{z}) = H^2_c(\mathfrak{z}(\mathfrak{g}),\mathfrak{z}) = \mathrm{Alt}^2(\mathfrak{z}(\mathfrak{g}),\mathfrak{z}) = \mathrm{Lin}(\Lambda^2(\mathfrak{z}(\mathfrak{g})),\mathfrak{z}),$$

we obtain a universal Lichnerowicz cocycle with values in the space $\mathfrak{z} := \Lambda^2(\mathfrak{z}(\mathfrak{g}))$.■

Remark 11.7. The preceding remark applies in particular to the d-dimensional torus $G = \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. We write $x_1,\ldots,x_d$ for the canonical coordinate functions on $\mathbb{R}^d$ and observe that their differential dx_j can also be viewed as 1-forms on $\mathbb{T}^d$. In this sense we have

$$H^2_{\mathrm{dR}}(\mathbb{T}^d,\mathbb{R}) \cong \bigoplus_{i<j} \mathbb{R}[dx_i \wedge dx_j] \cong \mathbb{R}^{\binom{d}{2}}.$$

Therefore the central extensions of $\widetilde{D}(\mathbb{T}^d,\mu)$ described above correspond to the central extensions of the corresponding Cartan type algebras discussed in [Dz92]. We conclude in particular that these cocycle do not integrate to central extensions of $D(\mathbb{T}^d,\mu)$, but that they integrate to central extensions of the covering group $\widetilde{D}(\mathbb{T}^d,\mu)$ which we consider as a group of diffeomorphisms of $\mathbb{R}^d$. ■

Appendix A. Differential forms and Alexander–Spanier cohomology

In this appendix we discuss a smooth version of Alexander–Spanier cohomology for smooth manifolds and define a homomorphism of chain complexes from the smooth Alexander–Spanier complex $(C^n_{AS,s}(M,A), d_{AS}), n \geq 1$, with values in an abelian Lie group A with Lie algebra $\mathfrak{a}$ to the $\mathfrak{a}$-valued de Rham complex $(\Omega^\bullet(M,\mathfrak{a}), d)$. In Appendix B this map is used to relate Lie group cohomology to Lie algebra cohomology. The main point is Proposition A.6 which provides an explicit map from smooth Alexander–Spanier cohomology to de Rham cohomology.

Definition A.1. (1) Let M be a smooth manifold and A an abelian Lie group. For $n \in \mathbb{N}_0$ let $C^n_{AS,s}(M,A)$ denote the set of germs of smooth A-valued functions on the diagonal in M^{n+1}. For $n = 0$ this is the space $C^0_{AS,s}(M,A) \cong C^\infty(M,A)$ of smooth A-valued functions on M. An element $[F]$ of this space is represented by a smooth function $F \colon U \to A$, where U is an open neighborhood of the diagonal in M^{n+1}, and two functions $F_i \colon U_i \to A$, $i = 1,2$, define the same germ if and only of their difference vanishes on a neighborhood of the diagonal. The elements of the space $C^n_{AS,s}(M,A)$ are called *smooth A-valued Alexander–Spanier n-cochains on M*.

We have a differential

$$d_{AS} \colon C^n_{AS,s}(M,A) \to C^{n+1}_{AS,s}(M,A)$$

given by

$$(d_{AS}F)(m_0,\ldots,m_{n+1}) := \sum_{j=0}^{n+1} (-1)^j F(m_0,\ldots,\widehat{m_j},\ldots,m_{n+1}),$$

where $\widehat{m_j}$ indicates omission of the argument m_j. To see that $d_{AS}F$ defines a smooth function on an open neighborhood of the diagonal in M^{n+2}, consider for $i = 0,\ldots,n+1$ the projections $p_i \colon M^{n+2} \to M^{n+1}$ obtained by omitting the i-th component. Then for each open subset $U \subseteq M^{n+1}$ containing the diagonal the subset $\bigcap_{i=0}^{n+1} p_i^{-1}(U)$ is an open neighborhood of the diagonal in M^{n+2} on which $d_{AS}F$ is defined. It is easy to see that d_{AS} is well-defined on germs and that we thus obtain a differential complex $(C^\bullet_{AS,s}(M,A), d_{AS})$. Its cohomology groups are denoted $H^n_{AS,s}(M,A)$.

(2) If M is a smooth manifold, then an atlas for the tangent bundle TM is obtained directly from an atlas of M, but we do not consider the cotangent bundle as a manifold because this requires to choose a topology on the dual spaces, for which there are many possibilities. Nevertheless, there is a natural concept of a smooth p-form on M. If V is a locally convex space, then a *V-valued p-form ω* on M is a function ω which associates to each $x \in M$ a k-linear alternating map $T_x(M)^p \to V$ such that in local coordinates the map $(x, v_1,\ldots,v_p) \mapsto \omega(x)(v_1,\ldots,v_p)$ is smooth. We write $\Omega^p(M,V)$ for the space of smooth p-forms on M with values in V.

The *de Rham differential* $d \colon \Omega^p(M,V) \to \Omega^{p+1}(M,V)$ is defined by

$$\begin{aligned}(d\omega)(x)(v_0,\ldots,v_p) := &\sum_{i=0}^{p} (-1)^i \big(X_i.\omega(X_0,\ldots,\widehat{X}_i,\ldots,X_p)\big)(x)\\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i,X_j],X_0,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_p)(x)\end{aligned}$$

for $v_0,\ldots,v_p \in T_x(M)$, where $X_0,\ldots,X_p$ are smooth vector fields on a neighborhood of x with $X_i(x) = v_i$.

To see that d defines indeed a map $\Omega^p(M,V) \to \Omega^{p+1}(M,V)$ one has to verify that the right hand side of the above expression does not depend on the choice of the vector fields X_i with $X_i(x) = v_i$ and that it defines an element of $\Omega^{p+1}(M,V)$, i.e., in local coordinates the map

$$(x, v_0, \ldots, v_p) \mapsto (d\omega)(x)(v_0, \ldots, v_p)$$

is smooth, multilinear and alternating in $v_0, \ldots, v_p$. For the proof we refer to (cf. [KM97]).

Extending d to a linear map on $\Omega(M,V) := \bigoplus_{p \in \mathbb{N}_0} \Omega^p(M,V)$, we have the relation $d^2 = 0$. The space

$$Z^p_{\mathrm{dR}}(M,V) := \ker(d|_{\Omega^p(M,V)})$$

of *closed forms* therefore contains the space $B^p_{\mathrm{dR}}(M,V) := d(\Omega^{p-1}(M,V))$ of exact forms, and

$$H^p_{\mathrm{dR}}(M,V) := Z^p_{\mathrm{dR}}(M,V)/B^p_{\mathrm{dR}}(M,V)$$

is the *V-valued de Rham cohomology space of M*. ■

Definition A.2. If M is a smooth manifold, A an abelian Lie group, $\mathfrak{a}$ its Lie algebra, $f \colon M \to A$ a smooth function and $Tf \colon TM \to TA$ its tangent map, then we define the *logarithmic derivative of f* as the $\mathfrak{a}$-valued 1-form

$$df \colon TM \to \mathfrak{a}, \quad v \mapsto f(m)^{-1}.Tf(v), \quad \text{for} \quad v \in T_m(M).$$

In terms of the canonical trivialization $\theta \colon TA \to A \times \mathfrak{a}, v \mapsto a^{-1}.v$ (for $v \in T_a(A)$) of the tangent bundle of A, this means that

$$df = \mathrm{pr}_2 \circ \theta \circ Tf \colon TM \to \mathfrak{a}.$$ ■

Definition A.3. Let $M_1, \ldots, M_n$ be smooth manifolds, A an abelian Lie group, and

$$f \colon M_1 \times \ldots \times M_n \to A$$

be a smooth function. For $n \in \mathbb{N}$ we define a function

$$d^n f \colon TM_1 \times \ldots \times TM_n \to \mathfrak{a}$$

as follows. Let $q \colon TM \to M$ be the canonical projection. For $v_1, \ldots, v_n \in TM$ with $q(v_i) = m_i$ we consider smooth curves $\gamma_i \colon]-1,1[\to M$ with $\gamma_i(0) = m_i$ and $\gamma_i'(0) = v_i$ and define

$$(d^n f)(m_1, \ldots, m_n)(v_1, \ldots, v_n) := \frac{\partial^n}{\partial t_1 \cdots \partial t_n}\bigg|_{t_i=0} f(\gamma_1(t_1), \ldots, \gamma_n(t_n)),$$

where for $n \geq 2$ the iterated higher derivatives are derivatives of $\mathfrak{a}$-valued functions in the sense of Definition A.2. One readily verifies that the right hand side does not depend on the choice of the curves γ_i and that it defines for each tuple $(m_1, \ldots, m_n) \in M_1 \times \ldots \times M_n$ a continuous n-linear map

$$(d^n f)(m_1, \ldots, m_n) \colon T_{m_1}(M_1) \times \ldots \times T_{m_n}(M_n) \to \mathfrak{a}.$$

If X is a smooth vector field on M_i, then we also define a smooth function

$$\partial_i(X)f \colon M_1 \times \ldots \times M_n \to \mathfrak{a}, \quad (m_1, \ldots, m_n) \mapsto df(m_1, \ldots, m_n)(0, \ldots, 0, X(m_i), 0, \ldots, 0)$$

by the partial derivative of f in the direction of the vector field X. For vector fields X_i on M_i we then obtain by iteration of this process

$$\big(\partial_1(X_1) \cdots \partial_n(X_n) f\big)(m_1, \ldots, m_n) = (d^n f)(m_1, \ldots, m_n)(X_1(m_1), \ldots, X_n(m_n))$$

and

$$\partial_1(X_1) \cdots \partial_n(X_n) f \colon M_1 \times \ldots \times M_n \to \mathfrak{a}$$

is a smooth function. ■

Definition A.4. Let M be a smooth manifold and A an abelian Lie group. We write $\Delta_n \colon M \to M^{n+1}, m \mapsto (m, \ldots, m)$ for the diagonal map.

For $[F] \in C^n_{AS,s}(M, A)$, $p \in M$ and $v_1, \ldots, v_n \in T_p(M)$ we define

$$\tau(F)(p)(v_1, \ldots, v_n) := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot (d^n F)(p, \ldots, p)(0, v_{\sigma(1)}, \ldots, v_{\sigma(n)})$$

and observe that $\tau(F)$ defines a smooth $\mathfrak{a}$-valued n-form on M depending only on the germ $[F]$ of F. We thus obtain for $n \geq 1$ a group homomorphism

$$\tau \colon C^n_{AS,s}(M, A) \to \Omega^n(M, \mathfrak{a}).$$

If $A = \mathfrak{a}$, then we also define τ for $n = 0$ as the identical map

$$\tau \colon C^0_{AS,s}(M, A) \cong C^\infty(M, A) \to \Omega^0(M, \mathfrak{a}) \cong C^\infty(M, \mathfrak{a}).$$

If $X_1, \ldots, X_n$ are smooth vector fields on an open subset $V \subseteq M$, we have on V the relation

$$\tau(F)(X_1, \ldots, X_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot \big(\partial_1(X_{\sigma(1)}) \cdots \partial_n(X_{\sigma(n)}).F\big) \circ \Delta_n.$$

As the operators $\partial_i(X)$ and $\partial_j(Y)$ commute for $i \neq j$ and vector fields X and Y on M, this can also be written as

$$\tau(F)(X_1, \ldots, X_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot \big(\partial_{\sigma(1)}(X_1) \cdots \partial_{\sigma(n)}(X_n).F\big) \circ \Delta_n.$$

For small n we have in particular the formulas

$n = 0$: $\tau(F) = F$ (if $A = \mathfrak{a}$).

$n = 1$: $\tau(F)(X) = \partial_1(X).F$.

$n = 2$: $\tau(F)(X, Y) = \partial_1(X)\partial_2(Y).F - \partial_1(X)\partial_2(Y).F$. ■

The following proposition builds on a construction one finds in the appendix of [EK64]. First we need a combinatorial lemma.

Lemma A.5. *Let $\sigma \in S_{n+1}$ be a permutation with $k := \sigma(1) < \ell := \sigma(i+1)$ and such that the restriction of σ defines an increasing map $\{1,\ldots,n\} \setminus \{1, i+1\} \to \{1,\ldots,n\} \setminus \{k,\ell\}$. Then $\mathrm{sgn}(\sigma) = (-1)^{i+k+l}$.*

Proof. Replacing σ by $\sigma_1 := \sigma \circ \alpha$, where $\alpha = (i+1 \ \ i \ \ i-1 \ \ldots \ 3 \ 2)$ is a cycle of length i, we obtain a permutation σ_1 that restricts to an increasing map

$$\{3,4,\ldots,n\} \to \{1,\ldots,n\} \setminus \{k,\ell\}.$$

Next we put $\sigma_2 := \beta \circ \sigma_1$, where $\beta = (1 \ 2 \ 3 \ \ldots \ k-1 \ k)$ is a cycle of length k, to obtain an increasing map

$$\{3,4,\ldots,n\} \to \{2,\ldots,n\} \setminus \{\ell\}.$$

Eventually we put $\sigma_3 := \gamma \circ \sigma_2$, where $\gamma = (2 \ 3 \ \ldots \ \ell-1 \ \ell)$ is a cycle of length $\ell - 1$ to obtain an increasing map

$$\{3,4,\ldots,n\} \to \{3,4,\ldots,n\},$$

which implies that σ_3 fixes all these elements. Further

$$\sigma_3(1) = \gamma\beta\sigma\alpha(1) = \gamma\beta\sigma(1) = \gamma\beta(k) = \gamma(1) = 1$$

implies that $\sigma_3 = \mathrm{id}$. This implies that

$$\mathrm{sgn}(\sigma) = \mathrm{sgn}(\alpha)\,\mathrm{sgn}(\beta)\,\mathrm{sgn}(\gamma) = (-1)^{i-1}(-1)^{k-1}(-1)^{\ell} = (-1)^{i+k+\ell}. \qquad \blacksquare$$

The following proposition generalizes an observation of van Est and Korthagen in the Appendix of [EK64]:

Proposition A.6. (van Est–Korthagen) *If M is smooth manifold, then the map*

$$\tau \colon C^n_{AS,s}(M,A) \to \begin{cases} C^\infty(M,A) & \text{for } n = 0 \\ \Omega^n(M,\mathfrak{a}) & \text{for } n \geq 1 \end{cases}$$

intertwines the Alexander–Spanier differential with the de Rham differential, hence induces a map

$$\tau \colon H^n_{AS,s}(M,A) \to H^n_{\mathrm{dR}}(M,\mathfrak{a}).$$

Proof. We have to show that $\tau(d_{AS}F) = d\tau(F)$ holds for $F \in C^\infty(U,A)$, where U is an open neighborhood of the diagonal in M^{n+1}.

From the chain rule we obtain for a vector field Y on M the relation

$$Y.\big(\big(\partial_1(X_1)\cdots\partial_n(X_n).F\big)\circ\Delta_n\big) = \big(\partial_0(Y)\partial_1(X_1)\cdots\partial_n(X_n).F\big)\circ\Delta_n$$
$$+\sum_{i=1}^n \big(\partial_1(X_1)\cdots\partial_i(Y)\partial_i(X_i)\cdots\partial_n(X_n).F\big)\circ\Delta_n. \tag{A.1}$$

Now let

$$F_i(x_0,\ldots,x_{n+1}) := F(x_0,\ldots,\widehat{x_i},\ldots,x_{n+1}).$$

Then

$$F_i\circ\Delta_{n+1} = F\circ\Delta_n \tag{A.2}$$

and $d_{AS}F = \sum_{i=0}^{n+1}(-1)^i F_i$. Since the function F_i is independent of x_i, we have

$$\partial_1(X_1)\cdots\partial_{n+1}(X_{n+1}).F_i = 0, \quad i\geq 1. \tag{A.3}$$

Therefore

$$\partial_1(X_1)\cdots\partial_{n+1}(X_{n+1}).(d_{AS}F) = \partial_1(X_1)\cdots\partial_{n+1}(X_{n+1})(F_0)$$
$$= \big(\partial_0(X_1)\cdots\partial_n(X_{n+1})F\big)_0.$$

In view of (A.2) and (A.1), this leads to

$$\big(\partial_1(X_1)\cdots\partial_{n+1}(X_{n+1}).(d_{AS}F)\big)\circ\Delta_{n+1} = \big(\partial_0(X_1)\cdots\partial_n(X_{n+1}).F\big)\circ\Delta_n$$
$$= X_1.\Big(\big(\partial_1(X_2)\cdots\partial_n(X_{n+1}).F\big)\Big)\circ\Delta_n$$
$$-\sum_{i=1}^n \big(\partial_1(X_2)\cdots\partial_i(X_1)\partial_i(X_{i+1})\cdots\partial_n(X_{n+1}).F\big)\circ\Delta_n.$$

Alternating the first summand, we get an expression of the form

$$\sum_{\sigma\in S_{n+1}} \mathrm{sgn}(\sigma)X_{\sigma(1)}.\Big(\partial_1(X_{\sigma(2)})\cdots\partial_n(X_{\sigma(n+1)}).F\Big)\circ\Delta_n$$
$$=\sum_{i=1}^{n+1}\sum_{\sigma(1)=i}\mathrm{sgn}(\sigma)X_i.\Big(\partial_1(X_{\sigma(2)})\cdots\partial_n(X_{\sigma(n+1)}).F\Big)\circ\Delta_n.$$

We write any permutation $\sigma\in S_{n+1}$ with $\sigma(1)=i$ as $\sigma=\alpha_i\beta$, where $\beta(1)=1$ and $\alpha_i(1)=i$ and α_i is the cycle

$$\alpha_i = (i\ i-1\ i-2\ \ldots\ 2\ 1).$$

We further identify S_n with the stabilizer of 1 in S_{n+1}. Then the above sum turns into

$$=\sum_{i=1}^{n+1}\mathrm{sgn}(\alpha_i)\sum_{\beta\in S_n}\mathrm{sgn}(\beta)X_i.\Big(\partial_1(X_{\alpha_i\beta(2)})\cdots\partial_n(X_{\alpha_i\beta(n+1)}).F\Big)\circ\Delta_n$$

$$= \sum_{i=1}^{n+1} (-1)^{i-1} X_i.\tau(F)(X_{\alpha_i(2)}, \ldots, X_{\alpha_i(n+1)})$$
$$= \sum_{i=1}^{n+1} (-1)^{i-1} X_i.\tau(F)(X_1, \ldots, \widehat{X_i}, \ldots, X_{n+1}).$$

In view of

$$d(\tau(F))(X_1, \ldots, X_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} X_i.\tau(F)(X_1, \ldots, \widehat{X}_i, \ldots, X_{n+1}) + \sum_{k<\ell} (-1)^{k+\ell} \tau(F)([X_k, X_\ell], X_1, \ldots, \widehat{X}_k, \ldots, \widehat{X}_\ell, \ldots, X_{n+1}),$$

and

$$- \sum_{k<\ell} (-1)^{k+\ell} \tau(F)([X_k, X_\ell], X_1, \ldots, \widehat{X}_k, \ldots, \ldots, \widehat{X}_\ell, \ldots X_{n+1})$$
$$= \sum_{k<\ell} (-1)^{k+\ell+1} \sum_{\beta \in S_n} \mathrm{sgn}(\beta) \big(\partial_{\beta(1)}([X_k, X_\ell]) \partial_{\beta(2)}(X_1) \cdots \widehat{\partial}(X_k) \cdots \widehat{\partial}(X_\ell) \cdots \partial_{\beta(n)}(X_{n+1}).F\big) \circ \Delta_n,$$

it remains to show that, as operators on functions on M^{n+1}, alternation of

$$\sum_{i=1}^{n} \partial_1(X_2) \cdots \partial_i(X_1) \partial_i(X_{i+1}) \cdots \partial_n(X_{n+1}) \tag{A.3}$$

leads to

$$\sum_{k<\ell} (-1)^{k+\ell+1} \sum_{\beta \in S_n} \mathrm{sgn}(\beta) \partial_{\beta(1)}([X_k, X_\ell]) \partial_{\beta(2)}(X_1) \cdots \widehat{\partial}(X_k) \cdots \widehat{\partial}(X_\ell) \ldots \partial_{\beta(n)}(X_{n+1})$$
$$= \sum_{k<\ell} (-1)^{k+\ell+1} \langle \partial_1 \wedge \ldots \wedge \partial_n, [X_k, X_\ell] \wedge X_1 \wedge \cdots \wedge \widehat{X}_k \wedge \cdots \wedge \widehat{X}_\ell \wedge \cdots \wedge X_{n+1} \rangle$$
$$= \sum_{k<\ell} (-1)^{k+\ell+1} \sum_{i=1}^{n} (-1)^{i+1} \partial_i([X_k, X_\ell]) \circ \langle \partial_1 \wedge \ldots \wedge \widehat{\partial}_i \wedge \ldots \wedge \partial_n, X_1 \wedge \cdots \wedge \widehat{X}_k \wedge \cdots \wedge \widehat{X}_\ell \wedge \cdots \wedge X_{n+1} \rangle.$$

Alternating (A.3) leads to the expression

$$\sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma) \sum_{i=1}^{n} \partial_1(X_{\sigma(2)}) \cdots \partial_i(X_{\sigma(1)}) \partial_i(X_{\sigma(i+1)}) \cdots \partial_n(X_{\sigma(n+1)})$$
$$= \sum_{i=1}^{n} \sum_{\sigma(1)<\sigma(i+1)} \mathrm{sgn}(\sigma) \sum_{i=1}^{n} \partial_1(X_{\sigma(2)}) \cdots \partial_i([X_{\sigma(1)}, X_{\sigma(i+1)}]) \cdots \partial_n(X_{\sigma(n+1)})$$

$$= \sum_{i=1}^{n} \sum_{k<\ell} \sum_{\substack{\sigma(1)=k \\ \sigma(i+1)=\ell}} \operatorname{sgn}(\sigma) \sum_{i=1}^{n} \partial_1(X_{\sigma(2)}) \cdots \partial_i([X_k, X_\ell]) \cdots \partial_n(X_{\sigma(n+1)}).$$

We can write each permutation $\sigma \in S_{n+1}$ as $\sigma = \sigma_0\beta$, where β fixes 1 and $i+1$, so that we can identify it with an element of S_{n-1}, and

$$\sigma_0 \colon \{2, \ldots, n+1\} \setminus \{i+1\} \to \{1, \ldots, n+1\} \setminus \{k, \ell\}$$

is increasing. In view of Lemma A.5, we then have $\operatorname{sgn}(\sigma_0) = (-1)^{i+k+\ell}$ for $k = \sigma(1)$ and $\ell = \sigma(i+1)$. Therefore alternating (A.3) gives

$$= \sum_{i=1}^{n} \sum_{k<\ell} (-1)^{i+k+\ell} \sum_{\beta \in S_{n-1}} \operatorname{sgn}(\beta) \sum_{i=1}^{n} \partial_1(X_{\sigma_0\beta(2)}) \cdots \partial_i([X_k, X_\ell]) \cdots \partial_n(X_{\sigma_0\beta(n+1)})$$

$$= \sum_{i=1}^{n} \sum_{k<\ell} (-1)^{i+k+\ell} \partial_i([X_k, X_\ell]) \circ \langle \partial_1 \wedge \cdots \wedge \widehat{\partial_i} \wedge \cdots \wedge \partial_n, X_{\sigma_0(2)} \wedge \cdots \wedge X_{\sigma_0(n+1)} \rangle$$

$$= \sum_{k<\ell} \sum_{i=1}^{n} (-1)^{i+k+\ell} \partial_i([X_k, X_\ell]) \langle \partial_1 \wedge \cdots \widehat{\partial_i} \cdots \wedge \partial_n, X_2 \wedge \cdots \wedge \widehat{X_k} \wedge \cdots \wedge \widehat{X_\ell} \wedge \cdots \wedge X_{n+1} \rangle.$$

This completes the proof of Proposition A.6. ■

Appendix B. Cohomology of Lie groups and Lie algebras

In this appendix we show that for $n \geq 2$ there is a natural "derivation map"

$$D_n \colon H^n_s(G, A) \to H^n_c(\mathfrak{g}, \mathfrak{a})$$

from locally smooth Lie group cohomology to continuous Lie algebra cohomology. For $n = 1$ we have a map $D_1 \colon Z^1_s(G, A) \to Z^1_c(\mathfrak{g}, \mathfrak{a})$, and if, in addition, $A \cong \mathfrak{a}/\Gamma_A$ holds for a discrete subgroup Γ_A of $\mathfrak{a}$, then this map induces a map between the cohomology groups.

Definition B.1. Let V be a *topological module of the topological Lie algebra* $\mathfrak{g}$. For $p \in \mathbb{N}_0$, let $C^p_c(\mathfrak{g}, V)$ denote the space of continuous alternating maps $\mathfrak{g}^p \to V$, i.e., the *Lie algebra p-cochains with values in the module V*. Note that $C^1_c(\mathfrak{g}, V) = \operatorname{Lin}(\mathfrak{g}, V)$ is the space of continuous linear maps $\mathfrak{g} \to V$. We use the convention $C^0_c(\mathfrak{g}, V) = V$. We then obtain a chain complex with the differential

$$d_{\mathfrak{g}} \colon C^p_c(\mathfrak{g}, V) \to C^{p+1}_c(\mathfrak{g}, V)$$

given on $f \in C^p_c(\mathfrak{g}, V)$ by

$$(d_{\mathfrak{g}} f)(x_0, \ldots, x_p) := \sum_{j=0}^{p} (-1)^j x_j.f(x_0, \ldots, \widehat{x}_j, \ldots, x_p) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_p),$$

where $\widehat{x}_j$ indicates omission of x_j. Note that the continuity of the bracket on $\mathfrak{g}$ and the action on V imply that $d_{\mathfrak{g}} f$ is continuous.

We thus obtain a subcomplex of the algebraic Lie algebra complex associated to $\mathfrak{g}$ and V in [CE48]. Hence $d_{\mathfrak{g}}^2 = 0$, and the space $Z^p_c(\mathfrak{g}, V) := \ker(d_{\mathfrak{g}} |_{C^p_c(\mathfrak{g},V)})$ of *p-cocycles* contains the space $B^p_c(\mathfrak{g}, V) := d_{\mathfrak{g}}(C^{p-1}_c(\mathfrak{g}, V))$ of *p-coboundaries* (cf. [We95, Cor. 7.7.3]). The quotient

$$H^p_c(\mathfrak{g}, V) := Z^p_c(\mathfrak{g}, V)/B^p_c(\mathfrak{g}, V)$$

is the *p-th continuous cohomology space of $\mathfrak{g}$ with values in the $\mathfrak{g}$-module V*. We write $[f]$ for the cohomology class of the cocycle f in $H^p_c(\mathfrak{g}, V)$. ■

Definition B.2. Let G be a Lie group and A an abelian Lie group. We call A a *smooth G-module* if it is endowed with a G-module structure defined by a smooth action map $G \times A \to A$.

Let A be a smooth G-module. Then we define $\widetilde{C}^n_s(G, A)$ to be the space of all functions $F \colon G^{n+1} \to A$ which are smooth in a neighborhood of the diagonal, equivariant with respect to the action of G on G^{n+1} given by

$$g.(g_0, \ldots, g_n) := (gg_0, \ldots, gg_n),$$

and vanish on all tuples of the form $(g_0, \ldots, g, g, \ldots, g_n)$. As the G-action preserves the diagonal, it preserves the space $\widetilde{C}^n_s(G, A)$. Moreover, the Alexander–Spanier differential d_{AS} defines a group homomorphism

$$d_{AS} \colon \widetilde{C}^n_s(G, A) \to \widetilde{C}^{n+1}_s(G, A),$$

and we thus obtain a differential complex $(\widetilde{C}^\bullet_s(G, A), d_{AS})$.

Let $C^n_s(G, A)$ denote the space of all function $f \colon G^n \to A$ which are smooth in an identity neighborhood and normalized in the sense that $f(g_1, \ldots, g_n)$ vanishes if $g_j = \mathbf{1}$ holds for some j. We call these functions *normalized locally smooth group cochains.* Then the map

$$\Phi_n \colon C^n_s(G, A) \to \widetilde{C}^n_s(G, A), \quad \Phi_n(f)(g_0, \ldots, g_n) := g_0.f(g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_{n-1}^{-1} g_n)$$

is a linear bijection whose inverse is given by

$$\Phi_n^{-1}(F)(g_1, \ldots, g_n) := F(\mathbf{1}, g_1, g_1 g_2, \ldots, g_1 \cdots g_n).$$

By
$$d_G := \Phi_{n+1}^{-1} \circ d_{AS} \circ \Phi_n \colon C_s^n(G,A) \to C_s^{n+1}(G,A)$$
we obtain the differential $d_G \colon C_s^n(G,A) \to C_s^{n+1}(G,A)$ turning $(C_s^\bullet(G,A), d_G)$ into a differential complex. We write $Z_s^n(G,A)$ for the corresponding group of cocycles, $B_s^n(G,A)$ for the subgroup of coboundaries and
$$H_s^n(G,A) := Z_s^n(G,A)/B_s^n(G,A)$$
is called the *n-th Lie cohomology group with values in the smooth module A.* ■

Lemma B.3. *The group differential $d_G \colon C_s^n(G,A) \to C_s^{n+1}(G,A)$ is given by*
$$\begin{aligned}(d_G f)(g_0,\ldots,g_n) &= g_0.f(g_1,\ldots,g_n)\\ &\quad + \sum_{j=1}^n (-1)^j f(g_0,\ldots,g_{j-1}g_j,\ldots,g_n) + (-1)^{n+1} f(g_0,\ldots,g_{n-1}).\end{aligned}$$

Proof. In fact, $d_{AS}F = \sum_{i=0}^{n+1}(-1)^i F_i$ leads with $F = \Phi_n(f)$ to $d_G f = \sum_{i=0}^{n+1}(-1)^i \Phi_{n+1}^{-1}(F_i)$ and hence to
$$\begin{aligned}&(d_G f)(g_0,\ldots,g_n)\\ &= \sum_{i=0}^{n+1}(-1)^i F_i(\mathbf{1}, g_0, g_0g_1, \ldots, g_0\cdots g_n)\\ &= \sum_{i=0}^{n+1}(-1)^i F(\mathbf{1}, g_0, g_0g_1, \ldots, g_0\cdots g_{i-1}, g_0\cdots g_{i+1}, \ldots, g_0\cdots g_n)\\ &= g_0.f(g_1,\ldots,g_n) + \sum_{i=1}^n (-1)^i f(g_0, g_1, \ldots, g_ig_{i+1}, \ldots, g_n)\\ &\qquad\qquad + (-1)^{n+1} f(g_0,\ldots,g_{n-1}).\end{aligned}$$
■

For $n = 0$ we have in particular
$$(d_G f)(g_0) = g_0.f - f,$$
and for $n = 1$:
$$(d_G f)(g_0, g_1) = g_0.f(g_1) - f(g_0g_1) + f(g_0).$$

Definition B.4. Let G be a Lie group and $\mathfrak{a}$ a *smooth locally convex G-module*, i.e., $\mathfrak{a}$ is a locally convex space and the action map $\rho_\mathfrak{a} \colon G \times \mathfrak{a} \to \mathfrak{a}, (g,a) \mapsto g.a$ is smooth. We write $\rho_\mathfrak{a}(g) \colon \mathfrak{a} \to \mathfrak{a}, a \mapsto g.a$ for the corresponding continuous linear automorphisms of $\mathfrak{a}$.

We call a p-form $\Omega \in \Omega^p(G,\mathfrak{a})$ *equivariant* if we have for all $g \in G$ the relation

$$\lambda_g^*\Omega = \rho_{\mathfrak{a}}(g) \circ \Omega.$$

The complex of equivariant differential forms has been introduced in the finite-dimensional setting by Chevalley and Eilenberg in [CE48].

If $\mathfrak{a}$ is a trivial module, then an equivariant p-form is a left invariant $\mathfrak{a}$-valued p-form on G. An equivariant p-form is uniquely determined by the corresponding element $\Omega_{\mathbf{1}} \in C^p_c(\mathfrak{g},\mathfrak{a})$:

$(B.1)$

$$\Omega_g(g.x_1,\ldots,g.x_p) = \rho_{\mathfrak{a}}(g) \circ \Omega_{\mathbf{1}}(x_1,\ldots,x_p), \quad \text{for} \quad g \in G, x_i \in \mathfrak{g} \cong T_{\mathbf{1}}(G).$$

Here $G \times T(G) \to T(G), (g,x) \mapsto g.x$ denotes the natural action of G on its tangent bundle $T(G)$ obtained by restricting the tangent map of the group multiplication.

Conversely, (B.1) provides for each $\omega \in C^p_c(\mathfrak{g},\mathfrak{a})$ a unique equivariant p-form ω^{eq} on G with $\omega^{\mathrm{eq}}_{\mathbf{1}} = \omega$. ■

Lemma B.5. *For each $\omega \in C^p_c(\mathfrak{g},\mathfrak{a})$ we have $d(\omega^{\mathrm{eq}}) = (d_{\mathfrak{g}}\omega)^{\mathrm{eq}}$. In particular, the evaluation map*

$$\mathrm{ev}_{\mathbf{1}} \colon \Omega^p(G,\mathfrak{a})^{\mathrm{eq}} \to C^p_c(\mathfrak{g},\mathfrak{a}), \quad \omega \mapsto \omega_{\mathbf{1}}$$

defines an isomorphism from the chain complex of equivariant $\mathfrak{a}$-valued differential forms on G to the continuous $\mathfrak{a}$-valued Lie algebra cohomology.

Proof. (cf. [CE48, Th. 10.1]) For $g \in G$ we have

$$\lambda_g^* d\omega^{\mathrm{eq}} = d\lambda_g^*\omega^{\mathrm{eq}} = d(\rho_{\mathfrak{a}}(g) \circ \omega^{\mathrm{eq}}) = \rho_{\mathfrak{a}}(g) \circ (d\omega^{\mathrm{eq}}),$$

showing that $d\omega^{\mathrm{eq}}$ is equivariant.

For $x \in \mathfrak{g}$ we write $x_l(g) = g.x$ for the corresponding left invariant vector field on G. It suffices to calculate the value of $d\omega^{\mathrm{eq}}$ on $(p+1)$-tuples of left invariant vector fields in the identity element.

In view of

$$\omega^{\mathrm{eq}}(x_{1,l},\ldots,x_{p,l})(g) = \rho_{\mathfrak{a}}(g).\omega(x_1,\ldots,x_p),$$

we obtain

$$\big(x_{0,l}.\omega^{\mathrm{eq}}(x_{1,l},\ldots,x_{p,l})\big)(\mathbf{1}) = x_0.\omega(x_1,\ldots,x_p),$$

and therefore

$$\Big(d\omega^{\mathrm{eq}}(x_{0,l},\ldots,x_{p,l})\Big)(\mathbf{1}) = \sum_{i=0}^{p}(-1)^i x_{i,l}.\omega^{\mathrm{eq}}(x_{0,l},\ldots,\widehat{x_{i,l}},\ldots,x_{p,l})(\mathbf{1})$$
$$+ \sum_{i<j}(-1)^{i+j}\omega^{\mathrm{eq}}([x_{i,l},x_{j,l}],x_{0,l},\ldots,\widehat{x_{i,l}},\ldots,\widehat{x_{j,l}},\ldots,x_{p,l})(\mathbf{1})$$

$$= \sum_{i=0}^{p}(-1)^i x_i.\omega(x_0,\ldots,\widehat{x_i},\ldots,x_p)$$
$$+ \sum_{i<j}(-1)^{i+j}\omega([x_i,x_j],x_0,\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots,x_p)$$
$$= (d_{\mathfrak g}\omega)(x_0,\ldots,x_p).$$

This proves our assertion. ■

Theorem B.6. *The maps*

$$D_n\colon = \mathrm{ev}_{\mathbf 1}\circ\tau\circ\Phi_n\colon C^n_s(G,A)\to C^n_c(\mathfrak g,\mathfrak a),\quad n\geq 1,$$

induce a morphism of chain complexes

$$D\colon (C^n_s(G,A),d_G)_{n\geq 1}\to (C^n_c(\mathfrak g,\mathfrak a),d_{\mathfrak g})_{n\geq 1}$$

and in particular homomorphisms

$$D_n\colon H^n_s(G,A)\to H^n_c(\mathfrak g,\mathfrak a),\quad n\geq 2.$$

For $A=\mathfrak a$ these assertions hold for all $n\in\mathbb N_0$ and if $A\cong\mathfrak a/\Gamma_A$ for a discrete subgroup Γ_A of $\mathfrak a$, then D_1 also induces a homomorphism

$$D_1\colon H^1_s(G,A)\to H^1_c(\mathfrak g,\mathfrak a),\quad [f]\mapsto[df(\mathbf 1)].$$

Proof. In view of Proposition A.6 and the definition of the group differential d_G, the composition

$$\tau\circ\Phi_n\colon C^n_s(G,A)\to\widetilde C^n_s(G,A)\subseteq C^n_{AS,s}(G,A)\to\Omega^n(G,\mathfrak a),\quad n\geq 1,$$

defines a homomorphism of chain complexes. For $A=\mathfrak a$ this relation also holds for $n\geq 0$.

For $f\in C^n_s(G,A)$ the function $F:=\Phi_n(f)\colon G^{n+1}\to A$ is G-equivariant with respect to the diagonal action. For $g\in G$ let

$$\mu_g\colon G^{n+1}\to G^{n+1},\quad (g_0,\ldots,g_n)\mapsto(gg_0,\ldots,gg_n)$$

and write $\rho_A(g)(a):=g.a$ for $a\in A$. Then the equivariance of F means that $\mu_g^*F=F\circ\mu_g=\rho_A(g)\circ F$ which implies that

$$\rho_A(g)\circ\tau(F)=\tau(\rho_A(g)\circ F)=\tau(\mu_g^*F)=\lambda_g^*\tau(F).$$

This shows that the image of $\tau\circ\Phi_n$ consists of equivariant $\mathfrak a$-valued n-forms on G. According to Lemma B.5, evaluating an equivariant n-form in the identity intertwines the de Rham differential with the Lie algebra differential $d_{\mathfrak g}$. This implies

$$d_{\mathfrak g}\circ D_n=D_{n+1}\circ d_G$$

for each $n \in \mathbb{N}$, i.e., the D_n define a morphism of chain complexes (truncated to $n \geq 1$). For $A = \mathfrak{a}$ it also holds for $n \geq 0$.

If $A \cong \mathfrak{a}/\Gamma_A$ and $n = 1$, then $D_1(B^1_s(G,A)) = B^1_c(\mathfrak{g},\mathfrak{a})$ implies that D_1 induces a map $H^1_s(G,A) \to H^1_c(\mathfrak{g},\mathfrak{a})$. If A is not of this form, then we cannot conclude that D_1 maps $B^1_s(G,A)$ into $B^1_c(\mathfrak{g},\mathfrak{a})$. ∎

To make D_n, $n \geq 2$, better accessible to calculations, we need a more concrete formula for the Lie algebra cochain $D_n f$ for $f \in C^n_s(G,A)$. As f vanishes on all tuples of the form $(g_1, \ldots, \mathbf{1}, \ldots, g_n)$, its $(n-1)$-jet in $\mathbf{1}$ vanishes and the term of order n is the n-linear map

$$(d^n f)(\mathbf{1}, \ldots, \mathbf{1})\colon \mathfrak{g}^n = T_{\mathbf{1}}(G)^n \to \mathfrak{a}$$

(cf. Definition A.3). In fact, in local coordinates the n-th order term of the Taylor expansion of f in $(\mathbf{1}, \ldots, \mathbf{1})$ is given by a symmetric n-linear map

$$(d^{[n]} f)(\mathbf{1}, \ldots, \mathbf{1})\colon (\mathfrak{g}^n)^n \to \mathfrak{a}$$

as

$$\frac{1}{n!}(d^{[n]} f)(\mathbf{1}, \ldots, \mathbf{1})(x, \ldots, x), \quad x = (x_1, \ldots, x_n) \in \mathfrak{g}^n.$$

The normalization condition on f implies that $(d^{[n]} f)(\mathbf{1}, \ldots, \mathbf{1})$ vanishes on all elements $(x^1, \ldots, x^n)$, $x^i = (x^i_l) \in \mathfrak{g}^n$, for which the j-th component (in $\mathfrak{g}$) vanishes for some j, i.e., $x^i_j = 0$ for all i. This implies that $(d^{[n]} f)(\mathbf{1}, \ldots, \mathbf{1})(x, \ldots, x)$ is a sum of $n!$ terms of the form

$$(d^{[n]} f)(\mathbf{1}, \ldots, \mathbf{1})((0, \ldots, x_{\sigma(1)}, \ldots, 0), (0, \ldots, x_{\sigma(2)}, \ldots, 0), \ldots, (0, \ldots, x_{\sigma(n)}, \ldots, 0)).$$

Since all these terms are equal, we find

$$\begin{aligned}\frac{1}{n!}(d^{[n]} f)(\mathbf{1}, \ldots, \mathbf{1})(x, \ldots, x) &= (d^{[n]} f)(\mathbf{1}, \ldots, \mathbf{1})((x_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, x_n)) \\ &= (d^n f)(\mathbf{1}, \ldots, \mathbf{1})(x_1, \ldots, x_n).\end{aligned}$$

Lemma B.7. *For $f \in C^n_s(G,A)$ and $x_1, \ldots, x_n \in \mathfrak{g}$ we have*

$$(D_n f)(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(d^n f)(\mathbf{1}, \ldots, \mathbf{1})(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Proof. Recall that on an n-tuple $(x_1, \ldots, x_n) \in \mathfrak{g}^n$ the map $d^n f$ can be calculated by choosing smooth vector fields X_n on an open identity neighborhood of G with $X_i(\mathbf{1}) = x_i$ via

$$(d^n f)(\mathbf{1}, \ldots, \mathbf{1})(x_1, \ldots, x_n) := (\partial_1(X_1) \cdots \partial_n(X_n).f)(\mathbf{1}, \ldots, \mathbf{1}).$$

For $F = \Phi_n(f)$ we now get

$$\begin{aligned}(D_n f)(x_1, \ldots, x_n) &= \tau(F)(x_1, \ldots, x_n) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(d^n F)(\mathbf{1}, \ldots, \mathbf{1})(0, x_{\sigma(1)}, \ldots, x_{\sigma(n)}).\end{aligned}$$

In view of

$$F(\mathbf{1}, g_1, \ldots, g_n) = f(g_1, g_1^{-1}g_2, \ldots, g_{n-1}^{-1}g_n)$$

and $f(g_1, \mathbf{1}, \ldots) = 0$, we have

$$(\partial_1(X_1)F)(\mathbf{1}, \mathbf{1}, g_2, \ldots, g_n) = (\partial_1(X_1)f)(\mathbf{1}, g_2, g_2^{-1}g_3, \ldots, g_{n-1}^{-1}g_n),$$

and inductively we obtain

$$\begin{aligned}(\partial_1(X_1)\cdots\partial_n(X_n)F)(\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}) &= (\partial_1(X_1)\ldots\partial_n(X_n)f)(\mathbf{1}, \ldots, \mathbf{1}) \\ &= (d^n f)(\mathbf{1}, \ldots, \mathbf{1})(x_1, \ldots, x_n).\end{aligned}$$

This implies the assertion. ∎

For $n = 1$ we obtain $(D_1 f)(x) = df(\mathbf{1}).x$, and for $n = 2$ we have

$$(D_2 f)(x, y) = (d^2 f)(\mathbf{1}, \mathbf{1})(x, y) - (d^2 f)(\mathbf{1}, \mathbf{1})(y, x).$$

If $(d^{[n]} f)(\mathbf{1}, \mathbf{1})$ denotes the symmetric n-linear map $(\mathfrak{g}^n)^n \to \mathfrak{a}$ representing the n-jet of f, this expression equals

$$(d^{[2]} f)(\mathbf{1}, \mathbf{1})((x, 0)(0, y)) - (d^{[2]} f)(\mathbf{1}, \mathbf{1})((y, 0), (0, x)).$$

Appendix C. Split Lie subgroups

In this appendix we collect some general material on Lie group structures on groups, (normal) Lie subgroups and quotient groups. In particular Theorem C.2 provides a tool to construct Lie group structures on groups for which a subset containing the identity is an open 0-neighborhood of a locally convex space such that the group operations are locally smooth in these coordinates. We also give a condition on a normal subgroup $N \lhd G$ for the quotient group G/N being a manifold such that the quotient map $q\colon G \to G/N$ defines on G the structure of a smooth N-principal bundle.

Lemma C.1. *Let G be a group and $\mathcal{F}$ a filter basis of subsets with $\bigcap \mathcal{F} = \{\mathbf{1}\}$ satisfying:*
(U1) $(\forall U \in \mathcal{F})(\exists V \in \mathcal{F})\ VV \subseteq U.$
(U2) $(\forall U \in \mathcal{F})(\exists V \in \mathcal{F})\ V^{-1} \subseteq U.$
(U3) $(\forall U \in \mathcal{F})(\forall g \in G)(\exists V \in \mathcal{F})\ gVg^{-1} \subseteq U.$

Then there exists a unique group topology on G such that $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods in G. It is given by $\{U \subseteq G\colon (\forall g \in U)(\exists V \in \mathcal{F})\ gV \subseteq U\}$.

Proof. [Bou88, Ch. III, §1.2, Prop. 1] ∎

Theorem C.2. *Let G be a group and $U = U^{-1}$ a symmetric subset. We further assume that U is a smooth manifold such that*

(L1) *there exists an open $\mathbf{1}$-neighborhood $V \subseteq U$ with $V^2 = V \cdot V \subseteq U$ such that the group multiplication $\mu_V : V \times V \to U$ is smooth,*

(L2) *the inversion map $\eta_U : U \to U, u \mapsto u^{-1}$ is smooth, and*

(L3) *for each $g \in G$ there exists an open $\mathbf{1}$-neighborhood $U_g \subseteq U$ with $c_g(U_g) \subseteq U$ and such that the conjugation map*

$$c_g : U_g \to U, \quad x \mapsto gxg^{-1}$$

is smooth.

Then there exists a unique structure of a Lie group on G for which there exists an open $\mathbf{1}$-neighborhood $U_1 \subseteq U$ such that the inclusion map $U_1 \to G$ induces a diffeomorphism onto an open subset of G.

Proof. (cf. [Ch46, §14, Prop. 2] or [Ti83, p.14] for the finite-dimensional case) First we consider the filter basis

$$\mathcal{F} := \{W \subseteq G : W \in \mathcal{U}_U(\mathbf{1})\}$$

of all those subsets of U which are $\mathbf{1}$-neighborhoods in U. Then (L1) implies (U1), (L2) implies (U2), and (L3) implies (U3). Moreover, the assumption that U is Hausdorff implies that $\bigcap \mathcal{F} = \{\mathbf{1}\}$. Therefore Lemma C.1 implies that G carries a unique structure of a (Hausdorff) topological group for which $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods.

After shrinking V and U, we may assume that there exists a diffeomorphism $\varphi : U \to \varphi(U) \subseteq E$, where E is a topological $\mathbb{K}$-vector space, $\varphi(U)$ an open subset, that V satisfies $V = V^{-1}$, $V^4 \subseteq U$, and that $m : V^2 \times V^2 \to U$ is smooth. For $g \in G$ we consider the maps

$$\varphi_g : gV \to E, \quad \varphi_g(x) = \varphi(g^{-1}x)$$

which are homeomorphisms of gV onto $\varphi(V)$. We claim that $(\varphi_g, gV)_{g \in G}$ is an atlas of G.

Let $g_1, g_2 \in G$ and put $W := g_1V \cap g_2V$. If $W \neq \emptyset$, then $g_2^{-1}g_1 \in VV^{-1} = V^2$. The smoothness of the map

$$\psi := \varphi_{g_2} \circ \varphi_{g_1}^{-1}|_{\varphi_{g_1}(W)} : \varphi_{g_1}(W) \to \varphi_{g_2}(W)$$

given by

$$\psi(x) = \varphi_{g_2}(\varphi_{g_1}^{-1}(x)) = \varphi_{g_2}(g_1\varphi^{-1}(x)) = \varphi(g_2^{-1}g_1\varphi^{-1}(x))$$

follows from the smoothness of the multiplication $V^2 \times V^2 \to U$. This proves that $(\varphi_g, gU)_{g \in G}$ is an atlas of G. Moreover, the construction implies that all left translations of G are smooth maps.

The construction also shows that for each $g \in G$ the conjugation $c_g \colon G \to G$ is smooth in a neighborhood of $\mathbf{1}$. Since all left translations are smooth, and

$$c_g \circ \lambda_x = \lambda_{c_g(x)} \circ c_g,$$

the smoothness of c_g in a neighborhood of $x \in G$ follows. Therefore all conjugations and hence also all right multiplications are smooth. The smoothness of the inversion follows from its smoothness on V and the fact that left and right multiplications are smooth. Finally the smoothness of the multiplication follows from the smoothness in $\mathbf{1} \times \mathbf{1}$ because of

$$\mu_G(g_1 x, g_2 y) = g_1 x g_2 y = g_1 g_2 c_{g_2^{-1}}(x) y = g_1 g_2 \mu_G(c_{g_2^{-1}}(x), y).$$

The uniqueness of the Lie group structure is clear because each locally diffeomorphic bijective homomorphism between Lie groups is a diffeomorphism. ■

Remark C.3. Suppose that the group G in Theorem C.2 is generated by each $\mathbf{1}$-neighborhood V in U. Then condition (L3) can be omitted. Indeed, the construction of the Lie group structure shows that for each $g \in V$ the conjugation $c_g \colon G \to G$ is smooth in a neighborhood of $\mathbf{1}$. Since the set of all these g is a submonoid of G containing V, it contains V^n for each $n \in \mathbb{N}$, hence all of G because G is generated by V. Therefore all conjugations are smooth, and one can proceed as in the proof of Theorem C.2. ■

Definition C.4. (a) (Split Lie subgroups) Let G be a Lie group. A subgroup H is called a *split Lie subgroup* if it carries a Lie group structure for which the canonical right action of H on G defined by restricting the multiplication map of G to $G \times H \to G$ defines a smooth principal bundle, i.e., the coset space G/H is a smooth manifold and the quotient map $\pi \colon G \to G/H$ has smooth local sections.

(b) If G is a Banach–Lie group and $\exp \colon \mathfrak{g} \to G$ its exponential function, then a closed subgroup $H \subseteq G$ is called a *Lie subgroup* if there exists an open 0-neighborhood $U \subseteq \mathfrak{g}$ such that $\exp|_U \colon U \to \exp(U)$ is a diffeomorphism onto an open subset of G and the Lie algebra

$$\mathfrak{h} := \{x \in \mathfrak{g} \colon \exp(\mathbb{R}x) \subseteq H\}$$

of H satisfies

$$H \cap \exp(U) = \exp(U \cap \mathfrak{h}).$$ ■

Since the Lie algebra $\mathfrak{h}$ of a Lie subgroup H of a Banach Lie group G need not have a closed complement in $\mathfrak{g}$, not every Lie subgroup is split. A simple example is the subgroup $H := c_0(\mathbb{N}, \mathbb{R})$ in $G := \ell^\infty(\mathbb{N}, \mathbb{C})$ (cf. [We95, Satz IV.6.5]).

Lemma C.5. *If H is a split Lie subgroup of G or a Lie subgroup of the Banach–Lie group G, then for any smooth manifold X each smooth map $f\colon X \to G$ with $f(X) \subseteq H$ is also smooth as a map $X \to H$. If H is a normal split Lie subgroup, then the conjugation action of G on H is smooth.*

Proof. The condition that H is a split Lie subgroup implies that there exists an open subset U of some locally convex space V and a smooth map $\sigma\colon U \to G$ such that the map

$$U \times H \to G, \quad (x, h) \mapsto \sigma(x)h$$

is a diffeomorphism onto an open subset of G. Let $p\colon \sigma(U)H \to U$ denote the smooth map given by $p(\sigma(x)h) = x$. If X is a manifold and $f\colon X \to G$ is a smooth map with values in H, then f is smooth as a map to $\sigma(U)H \cong U \times H$, hence smooth as a map $X \to H$.

If H is a Lie subgroup of a Banach–Lie group and $f\colon X \to G$ is a smooth map with $f(X) \subseteq H$, then we have to see that f is smooth as a map $X \to H$. To verify smoothness in a neighborhood of some $x_0 \in X$, it suffices to consider the map $x \mapsto f(x)f(x_0)^{-1}$, so that we may w.l.o.g. assume that $f(x_0) = \mathbf{1}$. Then we can use the natural chart of H in $\mathbf{1}$ given by the exponential function to see that f is smooth in a neighborhood of x_0 because any smooth map $X \to \mathfrak{g}$ with values in $\mathfrak{h}$ is smooth as a map $X \to \mathfrak{h}$.

Now suppose that $H \trianglelefteq G$ is normal. Then the conjugation map $G \times H \to G, (g, h) \mapsto ghg^{-1}$, is smooth with values in H, hence smooth as a map $G \times H \to H$. ■

Theorem C.6. *Let G be a Lie group and $N \trianglelefteq G$ a split normal subgroup. Then the quotient group G/N has a natural Lie group structure such that the quotient map $q\colon G \to G/N$ defines on G the structure of a principal N-bundle.*

Proof. There exists an open subset U of a locally convex space V and a smooth map $\sigma\colon U \to G$ such that the map

$$U \times N \to G, \quad (u, n) \mapsto \sigma(u)n$$

is a diffeomorphism onto an open subset $W = \sigma(U)N$ of G. As N is in particular closed, the quotient group G/N has a natural (Hausdorff) group topology.

Let $q\colon G \to G/N$ denote the quotient map. Then $q(W) = q \circ \sigma(U)$ is an open subset of G/N and $q(W) \cong W/N \cong (U \times N)/N \cong U$. Therefore the map $\varphi := q \circ \sigma\colon U \to q(W)$ is a homeomorphism.

Let $K = K^{-1} \subseteq q(W)$ be a symmetric open subset, $U_K := \varphi^{-1}(K)$, and endow K with the manifold structure obtained from the homeomorphism $\varphi\colon U_K \to K$.

(L1): Let $V \subseteq K$ be an open $\mathbf{1}$-neighborhood with $V^2 \subseteq K$. We identify V with the corresponding open subset $U_V \subseteq U$. Then the group multiplication $\mu_V\colon V \times V \to K$ is given by

$$\varphi(x)\varphi(y) = \sigma(x)N \cdot \sigma(y)N = \sigma(x)\sigma(y)N = \varphi(\varphi^{-1}(\sigma(x)\sigma(y)N)),$$

and since the map $p\colon W \to U, \sigma(u)n \to u$ is smooth, the map

$$(x,y) \mapsto \varphi^{-1}(\sigma(x)\sigma(y)N) = p(\sigma(x)\sigma(y))$$

is smooth.

(L2): We likewise see that the inversion map $K \to K$ corresponds to the smooth map

$$x \mapsto \varphi^{-1}(\varphi(x)^{-1}) = \varphi^{-1}(\sigma(x)^{-1}N) = p(\sigma(x)^{-1}).$$

(L3): For each $g \in G$ we find an open $\mathbf{1}$-neighborhood $K_g \subseteq K$ with $c_g(K_g) \subseteq K$. Then the conjugation map

$$c_g\colon K_g \to K, \quad x \mapsto gxg^{-1}$$

is written in φ-coordinates as

$$\varphi(x) \mapsto \varphi(\varphi^{-1}(g\sigma(x)g^{-1}N)) = \varphi(p(g\sigma(x)g^{-1}))$$

and therefore smooth.

Now Theorem C.2 applies and shows that there exists a unique structure of a Lie group on G/N for which there exists an open 0-neighborhood in U such that the map $\varphi\colon U \to G/N$ induces a diffeomorphism onto an open subset of G/N. ■

Appendix D. The exact Inflation-Restriction Sequence

In this section G denotes a Lie group, $N \trianglelefteq G$ a split normal Lie subgroup (cf. Definition C.4) and A a smooth G-module. We write $q\colon G \to G/N$ for the quotient map.

Definition D.1. (a) (Inflation and restriction) Restriction of cochains leads for each $p \in \mathbb{N}_0$ to a map

$$\widetilde{R}\colon C^p_s(G,A) \to C^p_s(N,A),$$

and since $R \circ d_G = d_N \circ R$, it follows that $\widetilde{R}(B^p(G,A)) \subseteq B^p(N,A)$, $\widetilde{R}(Z^p_s(G,A)) \subseteq Z^p_s(N,A)$, so that $\widetilde{R}$ induces a homomorphism

$$R\colon H^p_s(G,A) \to H^p_s(N,A).$$

(b) Since N is a normal subgroup of G, the subgroup

$$A^N := \{a \in A\colon (\forall n \in N)\ n.a = a\}$$

is a G-submodule of A. If A^N is a split Lie subgroup of A, it inherits a natural structure of a smooth G/N-module (Lemma C.2) but we do not want to make this restrictive assumption. We therefore define the chain complex $(C_s^\bullet(G/N, A^N), d_{G/N})$ as the complex whose cochain space $C_s^p(G/N, A^N)$ consists of those functions $f\colon (G/N)^p \to A^N$ for which the pull-back

$$q^*f\colon G^p \to A^N, \quad (q^*f)(g_1, \ldots, g_p) := f(q(g_1), \ldots, q(g_p))$$

is an element of $C_s^p(G, A)$. With this definition we do not need a Lie group structure on the subgroup A^N of A. For a cochain $f \in C_s^p(G/N, A^N)$ we define

$$\widetilde{I} := q^*\colon C_s^p(G/N, A^N) \to C_s^p(G, A).$$

Then $(C_s^\bullet(G/N, A^N), d_{G/N})$ becomes a chain complex with the group differential from Lemma B.3. Moreover, $q^* \circ d_{G/N} = d_G \circ q^*$, so that $q^*(B_s^p(G/N, A^N)) \subseteq B_s^p(G, A)$, and $q^*(Z_s^p(G/N, A^N)) \subseteq Z_s^p(G, A)$, showing that q^* induces the so called *inflation map*

$$I\colon H_s^p(G/N, A^N) \to H_s^p(G, A), \quad [f] \mapsto [q^*f]. \qquad \blacksquare$$

The restriction and inflation maps

$$C_s^p(G/N, A^N) \xrightarrow{I} C_s^p(G, A) \xrightarrow{R} C_s^p(N, A)$$

clearly satisfy $R \circ I = 0$, which is inherited by the corresponding maps

$$H_s^p(G/N, A^N) \xrightarrow{I} H_s^p(G, A) \xrightarrow{R} H_s^p(N, A).$$

Lemma D.2. *The restriction maps $\widetilde{R}\colon C_s^p(G, A) \to C_s^p(N, A)$ are surjective.*

Proof. Since N is a split Lie subgroup of G, there exists an open 0-neighborhood U in a locally convex space V and a smooth map $\varphi\colon U \to G$ with $\varphi(0) = \mathbf{1}$ such that the map

$$\Phi\colon N \times U \to G, \quad (n, x) \mapsto n\varphi(x)$$

is a diffeomorphism onto an open subset $N\varphi(U)$ of G.

Let $f \in C_s^p(N, A)$. We extend f to a function $\widetilde{f}\colon (N\varphi(U))^p \to A$ by

$$\widetilde{f}(n_1\varphi(x_1), \ldots, n_p\varphi(x_p)) := f(n_1, \ldots, n_p).$$

Then $\widetilde{f}$ is smooth in an identity neighborhood and vanishes if one argument $n_i\varphi(x_i)$ is $\mathbf{1}$, because this implies $x_i = 0$ and $n_i = \mathbf{1}$. Now we extend $\widetilde{f}$ to a function on G^p vanishing in all tuples $(g_1, \ldots, \mathbf{1}, \ldots, g_p)$. Then $\widetilde{f} \in C_s^p(G, A)$ satisfies $\widetilde{R}(\widetilde{f}) = f$. $\blacksquare$

Although the the inflation map I is injective on cochains and R is surjective on cochains, in general there are many cochains with trivial restrictions on N which are not in the image of the inflation map. Therefore we do not have a short exact sequence of chain complexes, hence cannot expect a long exact sequence in cohomology. In this appendix we discuss what we still can say on the corresponding maps in low degree. It would be interesting to see if these results can also be obtained from a generalization of the Hochschild–Serre spectral sequence to Lie groups. As we shall see below, it is clear that the construction in [HS53] has to be modified substantially for the locally smooth infinite-dimensional setting.

Lemma D.3. (a) *Each cohomology class in* $H^p_s(G,A)$ *annihilated by* R *can be represented by a cocycle in* $\ker \widetilde{R}$.

(b) *We have* $B^p_s(N,A) \subseteq \mathrm{im}(\widetilde{R})$ *and therefore* $[f] \in \mathrm{im}(R)$ *is equivalent to* $f \in \mathrm{im}(\widetilde{R})$.

Proof. (a) We may w.l.o.g. assume that $p \geq 1$. If $R[f] = 0$, then $\widetilde{R}(f) = d_N\alpha$ for some $\alpha \in C^{p-1}_s(N,A)$. Let $\widetilde{\alpha} \in C^{p-1}_s(G,A)$ be an extension of α to G (Lemma D.2). Then $f' := f - d_G\widetilde{\alpha}$ restricts to $\widetilde{R}(f) - d_N\alpha = 0$ and $[f'] = [f]$.

(b) For $\alpha \in C^{p-1}_s(G,A)$ we have $\widetilde{R}(d_G\alpha) = d_N\widetilde{R}(\alpha)$, so that $C^{p-1}_s(N,A) \subseteq \mathrm{im}(\widetilde{R})$ implies that $\widetilde{R}(B^p_s(G,A)) = B^p_s(N,A)$.

For $f \in Z^p_s(N,A)$ it follows that $[f] \in \mathrm{im}(R)$ is equivalent to the existence of $\alpha \in B^{p-1}_s(N,A)$ with $f - d_N\alpha \in \mathrm{im}(\widetilde{R})$, which implies that $f \in \mathrm{im}(\widetilde{R})$. ∎

Lemma D.4. *The coboundary operator* d_N *is equivariant with respect to the action of* G *on* $C^p_s(N,A)$, $p \in \mathbb{N}_0$, *given by*

$$(g.f)(n_1,\ldots,n_p) := g.f(g^{-1}n_1g^{-1},\ldots,g^{-1}n_pg).$$

In particular, this action leaves the space of cochains invariant and induces actions on the cohomology groups $H^p_s(N,A)$. ∎

The preceding lemma applies in particular to the case $N = G$, showing that the coboundary operator d_G is equivariant for the natural action of G on the spaces $C^p_s(G,A)$.

Definition D.5. In the following we need a refined concept of invariance of cohomology classes in $H^p_s(N,A)$ under the action of the group G. We call $f \in Z^p_s(N,A)$ *smoothly cohomologically invariant* if there exists a map

$$\theta\colon G \to C^{p-1}_s(N,A) \quad \text{with} \quad d_N(\theta(g)) = g.f - f \quad \text{for all} \quad g \in G$$

for which the map

$$G \times N^{p-1} \to A, \quad (g,n_1,\ldots,n_{p-1}) \to \theta(g)(n_1,\ldots,n_{p-1})$$

is smooth in an identity neighborhood of $G \times N^{p-1}$.

We write $Z^p_s(N,A)^{[G]}$ for the set of smoothly cohomologically invariant cocycles in the group $Z^p_s(N,A)$. If $f = d_Nh$ for some $h \in C^{p-1}_s(N,A)$, then we may put $\theta(g) := g.h - h$ to find

$$d_N(\theta(g)) = d_N(g.h - h) = g.d_N(h) - d_N(h) = g.f - f,$$

and the map

$$\begin{aligned} G \times N^{p-1} &\to A, \\ (g, n_1, \ldots, n_{p-1}) &\mapsto (g.h - h)(n_1, \ldots, n_{p-1}) \\ &\quad = g.h(g^{-1}n_1 g, \ldots, g^{-1}n_{p-1}g) - h(n_1, \ldots, n_{p-1}) \end{aligned}$$

is smooth in an identity neighborhood. This shows that $B^p_s(N, A) \subseteq Z^p_s(N, A)^{[G]}$, and we define the space of *smoothly invariant cohomology classes* by

$$H^p_s(N, A)^{[G]} := Z^p_s(N, A)^{[G]} / B^p_s(N, A). \qquad \blacksquare$$

For a generalization of the following fact to general p for discrete groups and modules we refer to [HS53] or [Gui80, Chap. I, Prop. 7.1].

Proposition D.6. *Let $N \lhd G$ be a split normal Lie subgroup and $p \in \{0, 1, 2\}$. Then the restriction map R maps $H^p_s(G, A)$ into $H^p_s(N, A)^{[G]}$. In particular*

$$(D.1) \qquad H^p_s(G, A) = H^p_s(G, A)^{[G]} \quad \text{for} \quad p = 0, 1, 2.$$

Proof. In view of the G-equivariance of the restriction map $C^p_s(G, A) \to C^p_s(N, A)$, it suffices to prove the assertion in the case $N = G$.

For $p = 0$ we have $C^0_s(G, A) = A$, and $Z^0_s(G, A) = H^0_s(G, A) = A^G$ is the submodule of G-invariants. Clearly G acts trivially on this space, so that there is nothing to prove.

For $p = 1$ and a cocycle $f \in Z^1_s(G, A)$ we have for $g, x \in G$:

$$\begin{aligned} (g.f - f)(x) &= g.f(g^{-1}xg) - f(x) = g.(g^{-1}.f(xg) + f(g^{-1})) - f(x) \\ &= f(xg) + g.f(g^{-1}) - f(x) \\ &= x.f(g) + f(x) - f(g) - f(x) = d_G(f(g))(x). \end{aligned}$$

This shows that

$$(D.2) \qquad g.f - f = d_G(f(g)),$$

so that $f \in Z^2_s(G, A)^{[G]}$ follows from the local smoothness of f.

For $p = 2$ and $f \in Z^2_s(G, A)$ we have

$$\begin{aligned} &(g.f - f)(x, x') \\ &= g.f(g^{-1}xg, g^{-1}x'g) - f(x, x') \\ &= -f(g, g^{-1}xx'g) + f(g, g^{-1}xg) + f(xg, g^{-1}x'g) - f(x, x') \\ &= -f(g, g^{-1}xx'g) + f(g, g^{-1}xg) - f(x, g) + x.f(g, g^{-1}x'g) + f(x, x'g) - f(x, x') \\ &= -f(g, g^{-1}xx'g) + f(g, g^{-1}xg) - f(x, g) + x.f(g, g^{-1}x'g) - x.f(x', g) + f(xx', g) \end{aligned}$$

and the function

$$\theta(g) \colon G \to A, \quad \theta(g)(x) := f(g, g^{-1}xg) - f(x, g)$$

satisfies

$$\begin{aligned}(d_G\theta(g))(x,x') &= x.\theta(g)(x') + \theta(g)(x) - \theta(g)(xx') \\ &= x.f(g,g^{-1}x'g) - x.f(x',g) + f(g,g^{-1}xg) - f(x,g) \\ &\qquad\qquad - f(g,g^{-1}xx'g) + f(xx',g) \\ &= (g.f - f)(x,x').\end{aligned}$$

Since the function $G^2 \to A, (g,x) \mapsto \theta(g)(x)$ is smooth in an identity neighborhood of G^2, the assertion follows for $p = 2$. ∎

Lemma D.7. *For each $f \in Z^1_s(N,A)^{[G]}$ there exists $a \in C^1_s(G,A)$ with*

$$d_N(a(g)) = g.f - f, \quad a(gn) = a(g) + g.f(n), \quad g \in G, n \in N.$$

Then $d_G a \in B^2_s(G,A)$ is A^N-valued and constant on $(N \times N)$-cosets, hence factors to a cocycle $\overline{d_G a} \in Z^2_s(G/N, A^N)$. The cohomology class $[\overline{d_G a}]$ does not depend on the choice of f in $[f]$ and the function a, and we thus obtain a group homomorphism

$$\delta\colon H^1_s(N,A)^{[G]} \to H^2_s(G/N, A^N), \quad [f] \mapsto [\overline{d_G a}].$$

Proof. Since N is a split Lie subgroup, there exists an open 0-neighborhood of some locally convex space V and a smooth map $\varphi\colon U \to G$ with $\varphi(0) = \mathbf{1}$ such that the multiplication map

$$N \times U \to G, (x,n) \mapsto \varphi(x)n$$

is a diffeomorphism onto an open subset of G. Let $E \subseteq G$ be a set of representatives of the N-cosets containing $\varphi(U)$, so that the multiplication map $E \times N \to G$ is bijective.

The requirement $f \in Z^1_s(N,A)^{[G]}$ implies the existence of a function $\alpha \in C^1_s(G,A)$ with $d_N(\alpha(g)) = g.f - f$. We now define

$$a\colon G = EN \to A, \quad x \cdot n \mapsto \alpha(x) + x.f(n).$$

Then a is smooth on an identity neighborhood because E contains $\varphi(U)$. Since f is a 1-cocycle, we have for $x \in E$ and $n, n' \in N$ the relation

$$a(xnn') = a(x) + x.f(nn') = a(x) + x.f(n) + (xn).f(n') = a(xn) + (xn).f(n'),$$

which means that

$$a(gn) = a(g) + g.f(n), \qquad g \in G, n \in N.$$

In view of (D.2), we have for $n \in N$ the relation $n.f - f = d_N(f(n))$, so that

$$\begin{aligned}(xn).f - f = x.(n.f - f) + x.f - f = x.d_N(f(n)) + d_N(a(x)) &= d_N(x.f(n) + a(x)) \\ &= d_N(a(xn)),\end{aligned}$$

and hence $d_N(a(g)) = g.f - f$ for all $g \in G$.

That the values of the function $d_G a$ lie in A^N follows from

$$\begin{aligned} d_N(a(g_1 g_2)) &= (g_1 g_2).f - f = g_1.(g_2.f - f) + g_1.f - f \\ &= g_1.d_N(a(g_2)) + d_N(a(g_1)) = d_N(g_1.a(g_2) + a(g_1)) \end{aligned}$$

in $C^1_s(N, A)$. The coboundary $d_G a$ is a cocycle, hence an element of $Z^2_s(G, A^N)$. We show that $d_G a$ is constant on the cosets of N. We have

$$\begin{aligned} (d_G a)(g_1, g_2 n) &= g_1.a(g_2 n) + a(g_1) - a(g_1 g_2 n) \\ &= g_1.a(g_2) + g_1 g_2.f(n) + a(g_1) - a(g_1 g_2) - g_1 g_2.f(n) \\ &= (d_G a)(g_1, g_2) \end{aligned}$$

and

$$\begin{aligned} (d_G a)(g_1 n, g_2) &= g_1 n.a(g_2) + a(g_1 n) - a(g_1 n g_2) \\ &= g_1 n.a(g_2) + a(g_1) + g_1.f(n) - a(g_1 g_2 (g_2^{-1} n g_2)) \\ &= g_1 n.a(g_2) + a(g_1) + g_1.f(n) - a(g_1 g_2) - (g_1 g_2).f(g_2^{-1} n g_2) \\ &= g_1 n.a(g_2) + a(g_1) + g_1.f(n) - a(g_1 g_2) - g_1.((g_2.f)(n)) \\ &= (d_G a)(g_1, g_2) + g_1.(na(g_2) - a(g_2)) + g_1.f(n) - g_1.f(n) \\ &\qquad - g_1.(n.a(g_2) - a(g_2)) \\ &= (d_G a)(g_1, g_2) \end{aligned}$$

We now define

$$\overline{d_G a} \colon G/N \times G/N \to A^N, \quad (xN, yN) \mapsto (d_G a)(x, y).$$

Since $d_G a$ is a cocycle on G, the function $\overline{d_G a}$ is an element of $Z^2_s(G/N, A^N)$. It remains to show that the cohomology class of $\overline{d_G a}$ in $H^2_s(G/N, A^N)$ does not depend on the choices of a and f. If $a' \in C^1_s(G, A)$ is another function with

$$d_N(a'(g)) = g.f - f, \quad a'(gn) = a'(g) + g.f(n), \quad g \in G, n \in N,$$

then $d_N(a'(g) - a(g)) = 0$ implies that

$$\beta(g) := a'(g) - a(g) \in A^N, \quad g \in G.$$

Moreover,

$$\beta(gn) = a'(gn) - a(gn) = a'(g) + g.f(n) - a(g) - g.f(n) = a'(g) - a(g) = \beta(g),$$

so that β factors through a function $\gamma \colon G/N \to A^N$, and we have

$$(d_{G/N}\gamma)(xN, yN) = x.\beta(y) - \beta(xy) + \beta(x) = (d_G \beta)(x, y) = (d_G a - d_G a')(x, y).$$

Moreover, the fact that the quotient map $G \to G/N$ defines on G the structure of a smooth N-principal bundle implies that γ is smooth in an identity neighborhood of G/N. Hence the cocycle $\overline{d_G a'}$ is an element of $Z^2_s(G/N, A^N)$ and satisfies $\overline{d_G a'} = \overline{d_G a} - d_{G/N}\gamma$, so that $[\overline{d_G a}] = [\overline{d_G a'}]$.

Now suppose that $f' \in Z^1_s(N, A)$ satisfies $f' = f + d_N c$ for some $c \in A$. In view of the G-equivariance of the differential d_N, we have

$$g.(d_N c) - d_N c = d_N(g.c - c) \quad \text{and} \quad (d_G c)(gn) = (d_G c)(g) + g.((d_G c)(n)),$$

so that the function $a' := a + d_G c$ satisfies

$$d_N(a'(g)) = d_N(a(g) + g.c - c) = g.f - f + g.d_N(c) - d_N(c) = g.f' - f',$$

$$a'(gn) = a'(g) + g.f'(n).$$

As $d_G c$ is a cocycle, we have $d_G a' = d_G a$, so that we obtain in particular the same cocycles on G/N. ■

With the preceding lemma, we can prove the exactness of the Inflation-Restriction Sequence:

Proposition D.8. *Let A be a smooth G-module and $N \lhd G$ a split normal Lie subgroup. Then we have the following exact Inflation-Restriction Sequence:*

$$\mathbf{0} \to H^1_s(G/N, A^N) \xrightarrow{I} H^1_s(G, A) \xrightarrow{R} H^1_s(N, A)^{[G]} \xrightarrow{\delta} H^2_s(G/N, A^N) \xrightarrow{I} H^2_s(G, A).$$

Proof. (see [We95, 6.8.3] or [MacL63, pp.347–354] for the case of abstract groups)

Exactness in $H^1_s(G/N, A^N)$: Let $\alpha \in Z^1_s(G/N, A^N)$. We have $[q^*\alpha] = 0$ if and only if there exists an $a \in A$ with $\alpha(gN) = g.a - a$ for all $g \in G$. That this function is constant on N-cosets implies that $a \in A^N$, and hence that $\alpha = d_{G/N} a \in B^1_s(G/N, A^N)$. Therefore the inflation map I is injective on $H^1_s(G/N, A^N)$.

Exactness in $H^1_s(G, A)$: That the restriction map $\widetilde{R}$ maps into smoothly G-invariant cohomology classes follows from Proposition D.6 and the G-equivariance of R. The relation $R \circ I = 0$ is clear.

To see that $\ker R \subseteq \operatorname{im} I$, let $f \in Z^1_s(G, A)$ vanishing on N (Lemma D.3). Then f is constant on the N-cosets because

$$f(gn) = f(g) + g.f(n) = f(g) \quad \text{for} \quad g \in G, n \in N.$$

Moreover,

$$n.f(g) = f(ng) - f(n) = f(ng) = f(gg^{-1}ng) = f(g)$$

implies that $\operatorname{im}(f) \subseteq A^N$. Hence $[f]$ is contained in the image of the inflation map I.

Exactness in $H^1_s(N,A)^{[G]}$: If $f \in Z^1_s(N,A)$ is the restriction of a 1-cocycle $\alpha \in Z^1_s(G,A)$, then (D.2) implies

$$(g.f - f)(n) = (d_N(\alpha(g)))(n),$$

so that we may take α as the function a in the definition of δ. Then $d_G a = d_G\alpha = 0$ because α is a cocycle, and hence $\delta([f]) = 0$.

If, conversely, $\delta([f]) = 0$, then there exists $b \in C^1_s(G/N, A^N)$ with $\overline{d_G a} = d_{G/N} b$, where $\overline{d_G a}(xN, yN) = (d_G a)(x,y)$ is defined as in Lemma D.7. Then the function $a' := a - (b \circ q)$ satisfies

$$a'(gn) = a'(g) + g.f(n), \quad d_N(a'(g)) = g.f - f, \quad g \in G, n \in N,$$

and, in addition,

$$d_G a' = d_G a - d_G(q^* b) = d_G a - q^*(d_{G/N} b) = q^*(\overline{d_G a} - d_{G/N} b) = 0.$$

This means that $a' \in Z^1_s(G,A)$, so that $a'|_N = a|_N = f$ implies that $[f]$ is in the image of the restriction map R.

Exactness in $H^2_s(G/N, A^N)$: If $f \in Z^1_s(N,A)$ has a smoothly invariant cohomology class and $[\overline{d_G a}] = \delta([f])$ as in Lemma D.7, then the image of $[\overline{d_G a}]$ in $Z^2_s(G,A)$ under I is given by $d_G a = q^*\overline{d_G a}$, hence a coboundary.

Suppose, conversely, that for $\alpha \in Z^2_s(G/N, A^N)$ the cocycle $q^*\alpha$ on G is a coboundary and $\beta \in C^1_s(G,A)$ satisfies $q^*\alpha = d_G\beta$. Then $d_G\beta$ vanishes on N, so that $f := \beta|_N$ is a cocycle. We have

$$\alpha(xN, yN) = x.\beta(y) - \beta(xy) + \beta(x) \quad \text{for} \quad x, y \in G.$$

For $y \in N$ we obtain from $\alpha(xN, N) = \alpha(N, xN) = \{0\}$ the relations

$$\beta(gn) = \beta(g) + g.\beta(n) \quad \text{and} \quad \beta(ng) = \beta(n) + n.\beta(g).$$

For $g \in G$ and $n \in N$ we therefore have

$$\begin{aligned}(g.f - f)(n) &= g.\beta(g^{-1}ng) - \beta(n) = \beta(ng) - \beta(g) - \beta(n) \\ &= \beta(n) + n.\beta(g) - \beta(g) - \beta(n) = n.\beta(g) - \beta(g) = d_N(\beta(g))(n).\end{aligned}$$

This means that $[f]$ is smoothly G-invariant and that $\delta([f]) = [\alpha]$. ∎

Example D.9. The following example shows that the exact Inflation-Restriction sequence cannot be continued in an exact fashion by the restriction map $R\colon H^2_s(G,A) \to H^2_s(N,A)^{[G]}$.

For that we consider the group $G := \mathbb{R}^2$, $N := \mathbb{Z}^2$, $G/N = \mathbb{T}^2$ and the trivial module $A = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Then

$$H^2_s(G/N, A^N) = H^2_s(\mathbb{T}^2, \mathbb{T}) = \{0\}, \quad H^2_s(G,A) = H^2_s(\mathbb{R}^2, \mathbb{T}) \cong H^2_c(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R},$$

and $H^2_s(N,A)^{[G]} = H^2(\mathbb{Z}^2,\mathbb{T}) \cong \mathbb{T}$. Now the assertion follows from the fact that the natural map $R\colon H^2_s(\mathbb{R}^2,\mathbb{T}) \cong \mathbb{R} \to H^2_s(\mathbb{Z}^2,\mathbb{R}) \cong \mathbb{T}$ is not injective. It corresponds to restricting an alternating $\mathbb{T}$-valued bilinear form to the lattice $\mathbb{Z}^2$. If the form is integral on this lattice, the corresponding extension of $\mathbb{Z}^2$ is abelian, hence trivial (cf. Example 6.10). ■

Remark D.10. If A is a trivial G-module, then the connecting map has a simpler description. Then we have $H^1_s(N,A) = \mathrm{Hom}(N,A) = Z^1_s(N,A)$, and the condition that a homomorphism $f\colon N \to A$ is invariant under G means that it vanishes on the normal subgroup $[G,N] \lhd N$.

The only condition on the function $a\colon G \to A$ that we need to describe δ is

$$a(gn) = a(g) + f(n), \quad g \in G, n \in N.$$

Then the function $(d_G a)(x,y) = a(y) - a(xy) + a(x)$ is constant on $(N \times N)$-cosets and defines a 2-cocycle in $Z^2_s(G/N,A)$. ■

Example D.11. (a) If G is a Lie group, then its identity component G_0 is a split normal subgroup and the quotient group $\pi_0(G)$ is discrete. Therefore the Inflation-Restriction Sequence yields an exact sequence

$$\mathbf{0} \to H^1(\pi_0(G), A^{G_0}) \xrightarrow{I} H^1_s(G,A) \xrightarrow{R} H^1_s(G_0,A)^{[G]} \xrightarrow{\delta} H^2(\pi_0(G), A^{G_0}) \xrightarrow{I} H^2_s(G,A).$$

(b) Assume that $A \cong \mathfrak{a}/\Gamma_A$ for a discrete subgroup Γ_A of the Mackey complete locally convex space $\mathfrak{a}$. If G is a connected Lie group, $q_G\colon \widetilde{G} \to G$ its universal covering and $\pi_1(G)$ its kernel, then $\pi_1(G)$ is discrete, hence a split Lie subgroup, and we obtain for any smooth G-module A from Proposition D.8 the exact sequence

$$\mathbf{0} \to H^1_s(G,A) \xrightarrow{I} H^1_s(\widetilde{G},A) \xrightarrow{R} H^1_s(\pi_1(G),A)^{[G]} \xrightarrow{\delta} H^2_s(G,A) \xrightarrow{I} H^2_s(\widetilde{G},A).$$

As $\pi_1(G)$ acts trivially on A and $\pi_1(G)$ is central in $\widetilde{G}$, we have

$$H^1_s(\pi_1(G),A) = \mathrm{Hom}(\pi_1(G),A), \quad H^1_s(\pi_1(G),A)^{[G]} = H^1_s(\pi_1(G),A)^G = \mathrm{Hom}(\pi_1(G),A^G).$$

In view of Corollary 7.3, we may identify $H^2_s(\widetilde{G},A)$ with the subgroup

$$\{[\omega] \in H^2_c(\mathfrak{g},\mathfrak{a})\colon q_A \circ \mathrm{per}_\omega = 0\}.$$

On this subgroup the map $[\omega] \mapsto F_\omega$ given by the flux homomorphism defines a homomorphism

$$\widetilde{P}_2\colon H^2_s(\widetilde{G},A) \to \mathrm{Hom}(\pi_1(G), H^1_c(\mathfrak{g},\mathfrak{a})) \cong \mathrm{Hom}(\pi_1(G), H^1_s(\widetilde{G},A))$$

whose kernel coincides with the image of I (Theorem 7.2). In Remark 6.9 we have seen that the image of $[\omega] \in H^2_s(\widetilde{G}, A) \subseteq H^2_c(\mathfrak{g}, \mathfrak{a})$ in $H^2_s(\pi_1(G), A)$ is given by the commutator map

$$C^A_\omega([\gamma], [\eta]) = -P_1(F_\omega([\gamma]))([\eta])$$

of the corresponding central extension where P_1 is defined in Proposition 3.4. From Example D.9 we know that the vanishing of C does not imply the vanishing of F_ω. ■

Remark D.12. Let $f_N \in Z^1_s(N, A)^{[G]}$ and $f \in C^1_s(G, A)$ with

$$f(gn) = f(g) + g.f_N(n), \quad d_N(f(g)) = g.f_N - f_N, \quad g \in G, n \in N.$$

Then $\delta(f_N) = [\overline{d_G f}] \in Z^2_s(G/N, A^N)$ defines an abelian extension of G/N by A^N. We now describe this abelian extension directly in terms of f_N. Here we assume that A^N is a split Lie group (cf. Appendix C).

Using the smooth action of G on A, we can form the semi-direct product Lie group $A \rtimes G$. Then we consider the map

$$\sigma\colon G \to A \rtimes G, \quad g \mapsto (f(g), g).$$

In view of $f\mid_N = f_N \in Z^1_s(N, A)$, the restriction $\sigma\mid_N$ is a homomorphism. Moreover, for $g, g' \in G$ we have

$$\sigma(g)\sigma(g') = (f(g) + g.f(g'), gg') \quad \text{and} \quad \sigma(gg') = (f(gg'), gg'),$$

which implies that

$$\delta_\sigma(g, g') := \sigma(g)\sigma(g')\sigma(gg')^{-1} = ((d_G f)(g, g'), \mathbf{1}) \in A^N \times \{\mathbf{1}\}.$$

Therefore the induced map $\overline{\sigma}\colon G \to (A/A^N) \rtimes G$ is a group homomorphism, and the pull-back of the abelian extension

$$A^N \hookrightarrow A \rtimes G \twoheadrightarrow (A/A^N) \rtimes G$$

is isomorphic to the abelian extension $\widehat{G} := A^N \times_{d_G f} G$ defined by $d_G f \in Z^2_s(G, A^N)$. Since f vanishes on $N \times G$ and $G \times N$, the subset $\{0\} \times N$ is a normal subgroup of $\widehat{G}$, and $\widehat{G}/N \cong A^N \times_{\overline{d_G f}} G/N$. ■

Appendix E. A long exact sequence for Lie group cohomology

Let G be a Lie group and

$$\mathbf{0} \to A_1 \xrightarrow{q_1} A_2 \xrightarrow{q_2} A_3 \to \mathbf{0} \tag{E.1}$$

be an extension of abelian Lie groups which are smooth G-modules such that q_1 and q_2 are G-equivariant. We assume that (E.1) is an extension of Lie groups, i.e., that there exists a section $\sigma\colon A_3 \to A_2$ of q_2 which is smooth in an identity neighborhood. Then the map

$$A_1 \times A_3 \to A_2, \quad (a,b) \mapsto a + \sigma(b)$$

is a local diffeomorphism (not necessarily a group homomorphism). This assumption implies that the natural maps

$$C^p_s(G, A_1) \to C^p_s(G, A_2) \to C^p_s(G, A_3)$$

define a short exact sequence of chain complexes, hence induce a long exact sequence in cohomology

$$\mathbf{0} \to H^0_s(G, A_1) \to H^0_s(G, A_2) \to H^0_s(G, A_3) \to H^1_s(G, A_1) \to \ldots$$
$$\ldots \to H^{p-1}_s(G, A_3) \xrightarrow{\delta} H^p_s(G, A_1) \to H^p_s(G, A_2) \to H^p_s(G, A_3) \to \ldots$$

The connecting map $\delta\colon H^p_s(G, A_3) \to H^{p+1}_s(G, A_1)$ is constructed as follows. For $f \in Z^p_s(G, A_3)$ we first find $f_1 \in C^p_s(G, A_2)$ with $f = q_2 \circ f_1$. Then $0 = d_G f = q_2 \circ d_G f_1$ implies that $d_G f_1$ is A_1-valued, hence an element of $Z^{p+1}_s(G, A_1)$, and then $\delta([f]) = [d_G f_1]$.

For $p = 0$ we have $H^0_s(G, A) = A^G$, so that the exact sequence starts with

$$A_1^G \hookrightarrow A_2^G \to A_3^G \to H^1_s(G, A_1) \to H^1_s(G, A_2) \to \ldots.$$

Remark E.1. A particularly interesting case arises if A is a smooth G-module, A_0 its identity component and $\pi_0(A) := A/A_0$. Then $\pi_0(A)$ is discrete. Let us assume, in addition, that G is connected. Then G acts trivially on the discrete group $\pi_0(A)$. We therefore have an exact sequence

$$A_0^G \hookrightarrow A^G \to \pi_0(A) \xrightarrow{\overline{\theta}_A} H^1_s(G, A_0) \to H^1_s(G, A) \to H^1_s(G, \pi_0(A)) = \mathbf{0},$$

where we use $Z^1_s(G, \pi_0(A)) \subseteq C^\infty(G, \pi_0(A)) = \mathbf{0}$ (Lemma 3.1) to see that $H^1_s(G, \pi_0(A))$ is trivial. Note that $\overline{\theta}_A$ is the characteristic homomorphism of the smooth G-module A, considered as a map into $H^1_s(G, A_0)$, which we may consider as a subspace of $H^1_c(\mathfrak{g}, \mathfrak{a})$ (Definition 3.6). It follows in particular that the natural map $H^1_s(G, A_0) \to H^1_s(G, A)$ is surjective.

Moreover, we obtain an exact sequence

$$\mathbf{0} \to H^2_s(G, A_0) \to H^2_s(G, A) \to H^2_s(G, \pi_0(A)) \xrightarrow{\delta} H^3_s(G, A_0) \to \ldots$$

Since G is connected and $\pi_0(A)$ is a trivial module, the group $H^2_s(G, \pi_0(A))$ classifies the central extensions of G by $\pi_0(A)$, which is parametrized by the abelian group $\mathrm{Hom}(\pi_1(G), \pi_0(A))$ (Theorem 7.2). This leads to an exact sequence

$$(E.2) \qquad \mathbf{0} \to H^2_s(G, A_0) \to H^2_s(G, A) \xrightarrow{\gamma} \mathrm{Hom}(\pi_1(G), \pi_0(A)) \to H^3_s(G, A_0),$$

where γ assigns to an extension of G by A the corresponding connecting homomorphism $\pi_1(G) \to \pi_0(A)$ in the long exact homotopy sequence. For the universal covering group $q_G : \widetilde{G} \to G$ we thus obtain an isomorphism

$$H^2_s(\widetilde{G}, A_0) \to H^2_s(\widetilde{G}, A). \tag{E.3}$$

With the results of Section 7 we have determined $H^2_s(G, A_0)$ in terms of the topology of G and the Lie algebra cohomology space $H^2_c(\mathfrak{g}, \mathfrak{a})$. To determine $H^2_s(G, A)$ in terms of $H^2_s(G, A_0)$ and known data, one has to determine the image of $H^2_s(G, A)$ in $\mathrm{Hom}(\pi_1(G), \pi_0(A))$. Proposition 6.4 shows that for each $f \in Z^2_s(G, A)$ the flux homomorphism f_{Df} satisfies

$$F_{Df} = -\overline{\theta}_A \circ \gamma([f]).$$

If A is a trivial G-module, then the divisibility of A_0 implies that $A \cong A_0 \times \pi_0(A)$ as Lie groups, hence as G-modules, and we thus obtain

$$H^2_s(G, A) \cong H^2_s(G, A_0) \times H^2_s(G, \pi_0(A)) \cong H^2_s(G, A_0) \times \mathrm{Hom}(\pi_1(G), \pi_0(A)). \quad \blacksquare$$

We refer to Example 7.6 for the discussion of a situation, where the relation between $H^2_s(G, A_0)$ and $H^2_s(G, A)$ is more complicated.

Problem E. Calculate $H^p_s(G, A)$ for connected Lie groups G and discrete abelian groups A. In this case A is a trivial G-module and the cohomology groups are defined by cochains which are constant 0 in an identity neighborhood. Clearly $H^0_s(G, A) = A$, $H^1_s(G, A) = \mathbf{0}$ follows from Proposition 3.4, and $H^2_s(G, A) \cong \mathrm{Hom}(\pi_1(G), A) \cong H^1_{\mathrm{sing}}(G, A)$ from Theorem 7.2. What happens for $p \geq 3$? $\blacksquare$

Appendix F. Multiplication in Lie algebra and Lie group cohomology

In this appendix we collect some information concerning multiplication of Lie algebra and Lie group cocycles which is used in Section 9.

Multiplication of Lie algebra cochains

Let U, V, W be topological modules of the topological Lie algebra $\mathfrak{g}$ and $m : U \times V \to W, (u, v) \mapsto u \cdot v$ a $\mathfrak{g}$-equivariant continuous bilinear map, i.e., $x.m(u, v) = m(x.u, v) + m(u, x.v)$ for all $x \in \mathfrak{g}$, $u \in U$ and $v \in V$. Then we define a product

$$C^p_c(\mathfrak{g}, U) \times C^q_c(\mathfrak{g}, V) \to C^{p+q}_c(\mathfrak{g}, W), \quad (\alpha, \beta) \mapsto \alpha \wedge \beta$$

by

$$(\alpha\wedge\beta)(x_1,\ldots,x_{p+q}) := \frac{1}{p!q!}\sum_{\sigma\in S_{p+q}} \mathrm{sgn}(\sigma)\alpha(x_{\sigma(1)},\ldots,x_{\sigma(p)})\beta(x_{\sigma(p+1)},\ldots,x_{\sigma(p+q)}).$$

For $p = q = 1$ we have in particular

$$(\alpha\wedge\beta)(x,y) = \alpha(x)\cdot\beta(y) - \alpha(y)\cdot\beta(x).$$

In the following we write for a p-linear map $\alpha\colon \mathfrak{g}^p \to V$:

$$\mathrm{Alt}(\alpha)(x_1,\ldots,x_p) := \sum_{\sigma\in S_p} \mathrm{sgn}(\sigma)\alpha(x_{\sigma(1)},\ldots,x_{\sigma(p)}).$$

In this sense we have

$$\alpha\wedge\beta = \frac{1}{p!q!}\,\mathrm{Alt}(\alpha\cdot\beta),$$

where $(\alpha\cdot\beta)(x_1,\ldots,x_{p+q}) := \alpha(x_1,\ldots,x_p)\cdot\beta(x_{p+1},\ldots,x_{p+q})$.

Lemma F.1. *For $\alpha\in C^p_c(\mathfrak{g},U)$ and $\beta\in C^q_c(\mathfrak{g},V)$ we have*

$$d_{\mathfrak{g}}(\alpha\wedge\beta) = d_{\mathfrak{g}}\alpha\wedge\beta + (-1)^p\alpha\wedge d_{\mathfrak{g}}\beta. \tag{F.1}$$

Proof. First we verify that for $x\in\mathfrak{g}$ the insertion map i_x satisfies

$$i_x(\alpha\wedge\beta) = i_x\alpha\wedge\beta + (-1)^p\alpha\wedge i_x\beta. \tag{F.2}$$

For $p = 0$ or $q = 0$ this formula is a trivial consequence of the definitions. We may therefore assume $p, q \geq 1$. We calculate for $x_1,\ldots,x_{p+q}\in\mathfrak{g}$:

$$\begin{aligned}
&i_{x_1}(\alpha\wedge\beta)(x_2,\ldots,x_{p+q}) = (\alpha\wedge\beta)(x_1,x_2,\ldots,x_{p+q})\\
&= \frac{1}{p!q!}\sum_{\sigma\in S_{p+q}} \mathrm{sgn}(\sigma)\alpha(x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(p)})\beta(x_{\sigma^{-1}(p+1)},\ldots,x_{\sigma^{-1}(p+q)})\\
&= \frac{1}{p!q!}\sum_{\sigma(1)\leq p}\ldots + \frac{1}{p!q!}\sum_{\sigma(1)>p}\ldots.
\end{aligned}$$

For $\sigma(1)\leq p$ we get

$$\begin{aligned}
\alpha(x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(p)}) &= (-1)^{\sigma(1)+1}\alpha(x_1,x_{\sigma^{-1}(1)},\ldots,\widehat{x_1},\ldots,x_{\sigma^{-1}(p)})\\
&= (-1)^{\sigma(1)+1}(i_{x_1}\alpha)(x_{\sigma^{-1}(1)},\ldots,\widehat{x_1},\ldots,x_{\sigma^{-1}(p)}),
\end{aligned}$$

which leads to

$$\begin{aligned}
&\frac{1}{p!q!}\sum_{\sigma(1)\leq p}\cdots\\
&=\frac{1}{p!q!}\sum_{i=1}^{p}\sum_{\sigma(1)=i}\mathrm{sgn}(\sigma)(-1)^{i+1}(i_{x_1}\alpha)(x_{\sigma^{-1}(1)},\ldots,\widehat{x_1},\ldots,x_{\sigma^{-1}(p)})\\
&\qquad\qquad\qquad\qquad\beta(x_{\sigma^{-1}(p+1)},\ldots,x_{\sigma^{-1}(p+q)})\\
&=\frac{1}{p!q!}\sum_{i=1}^{p}\mathrm{Alt}(i_{x_1}\alpha\cdot\beta)(x_2,\ldots,x_{p+q})=\frac{1}{(p-1)!q!}\mathrm{Alt}(i_{x_1}\alpha\cdot\beta)(x_2,\ldots,x_{p+q})\\
&=(i_{x_1}\alpha\wedge\beta)(x_2,\ldots,x_{p+q}).
\end{aligned}$$

We likewise obtain

$$\frac{1}{p!q!}\sum_{\sigma(1)>p}\ldots=(-1)^p(\alpha\wedge(i_{x_1}\beta))(x_2,\ldots,x_{p+q}).$$

This proves (F.2).

We now prove (F.1) by induction on p and q. For $p=0$ we have

$$(\alpha\wedge\beta)(x_1,\ldots,x_q)=\alpha\cdot\beta(x_1,\ldots,x_q)$$

and

$$\begin{aligned}
d_{\mathfrak g}(\alpha\wedge\beta)(x_0,\ldots,x_q)&=\sum_{i=0}^{q}(-1)^i x_i.(\alpha\cdot\beta)(x_0,\ldots,\widehat{x_i},\ldots,x_q)\\
&\quad+\sum_{i<j}(-1)^{i+j}\alpha\cdot\beta([x_i,x_j],\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots,x_q)\\
&=\sum_{i=0}^{q}(-1)^i(x_i.\alpha)\cdot\beta(x_0,\ldots,\widehat{x_i},\ldots,x_q)+\alpha\cdot(d_{\mathfrak g}\beta)(x_0,\ldots,x_q)
\end{aligned}$$

and

$$\begin{aligned}
(d_{\mathfrak g}\alpha\wedge\beta)(x_0,\ldots,x_q)&=\frac{1}{q!}\sum_{\sigma\in S_{q+1}}\mathrm{sgn}(\sigma)(d_{\mathfrak g}\alpha)(x_{\sigma(0)})\cdot\beta(x_{\sigma(1)},\ldots,x_{\sigma(q)})\\
&=\frac{1}{q!}\sum_{i=0}^{q}\sum_{\sigma(0)=i}\mathrm{sgn}(\sigma)(x_i.\alpha)\cdot\beta(x_{\sigma(1)},\ldots,x_{\sigma(q)})\\
&=\frac{1}{q!}\sum_{i=0}^{q}(-1)^i(x_i.\alpha)\cdot\mathrm{Alt}(\beta)(x_0,\ldots,\widehat{x_i},\ldots,x_q)\\
&=\sum_{i=0}^{q}(-1)^i(x_i.\alpha)\cdot\beta(x_0,\ldots,\widehat{x_i},\ldots,x_q).
\end{aligned}$$

This proves (F.1) for $p = 0$. A similar argument works for $q = 0$. We now assume that $p, q \geq 1$ and that (F.1) hold for the pairs $(p-1, q)$ and $(p, q-1)$. Then we obtain with the Cartan formulas and (F.2) for $x \in \mathfrak{g}$:

$$\begin{aligned}
&i_x(d_{\mathfrak{g}}\alpha \wedge \beta + (-1)^p \alpha \wedge d_{\mathfrak{g}}\beta) \\
&= (i_x d_{\mathfrak{g}}\alpha) \wedge \beta + (-1)^{p+1} d_{\mathfrak{g}}\alpha \wedge i_x\beta + (-1)^p i_x\alpha \wedge d_{\mathfrak{g}}\beta + \alpha \wedge i_x d_{\mathfrak{g}}\beta \\
&= x.\alpha \wedge \beta - d_{\mathfrak{g}}(i_x\alpha) \wedge \beta + (-1)^{p+1} d_{\mathfrak{g}}\alpha \wedge i_x\beta + (-1)^p i_x\alpha \wedge d_{\mathfrak{g}}\beta \\
&\qquad\qquad + \alpha \wedge x.\beta - \alpha \wedge d_{\mathfrak{g}}(i_x\beta) \\
&= x.(\alpha \wedge \beta) - d_{\mathfrak{g}}(i_x\alpha \wedge \beta) + (-1)^{p+1} d_{\mathfrak{g}}(\alpha \wedge i_x\beta) \\
&= x.(\alpha \wedge \beta) - d_{\mathfrak{g}}(i_x(\alpha \wedge \beta)) = i_x(d_{\mathfrak{g}}(\alpha \wedge \beta)).
\end{aligned}$$

Since x was arbitrary, this proves (F.1). ∎

The preceding lemma implies that products of two cocycles are cocycles and that the product of a cocycle with a coboundary is a coboundary, so that we obtain bilinear maps

$$H_c^p(\mathfrak{g}, U) \times H_c^q(\mathfrak{g}, V) \to H_c^{p+q}(\mathfrak{g}, W), \quad ([\alpha], [\beta]) \mapsto [\alpha \wedge \beta]$$

which can be combined to a product $H_c^\bullet(\mathfrak{g}, U) \times H_c^\bullet(\mathfrak{g}, V) \to H_c^\bullet(\mathfrak{g}, W)$.

Multiplication of group cochains

Now let U, V, W be smooth modules of the Lie group G and $m\colon U \times V \to W, (u, v) \mapsto u \cdot v$ a G-equivariant biadditive smooth map. Then we define a product

$$C_s^p(G, U) \times C_s^q(G, V) \to C_s^{p+q}(G, W), \quad (\alpha, \beta) \mapsto \alpha \cup \beta,$$

where

$$(\alpha \cup \beta)(g_1, \ldots, g_{p+q}) := \alpha(g_1, \ldots, g_p) \cdot (g_1 \cdots g_p).\beta(g_{p+1}, \ldots, g_{p+q})$$

(cf. [Bro82, p.110] up to the different signs which are caused by different signs for the group differential).

Lemma F.2. *For $\alpha \in C_s^p(G, U)$ and $\beta \in C_s^q(G, V)$ we have*

$$d_G(\alpha \cup \beta) = d_G\alpha \cup \beta + (-1)^p \alpha \cup d_G\beta.$$

Proof. For $g_0, \ldots, g_{p+q} \in G$ we have

$$\begin{aligned}
&d_G(\alpha \cup \beta)(g_0, \ldots, g_{p+q}) \\
&= g_0.(\alpha \cup \beta)(g_1, \ldots, g_{p+q}) + \sum_{i=1}^{p+q} (-1)^i (\alpha \cup \beta)(g_0, \ldots, g_{i-1}g_i, \ldots, g_{p+q}) \\
&\quad + (-1)^{p+q+1} (\alpha \cup \beta)(g_0, \ldots, g_{p+q-1})
\end{aligned}$$

$$
\begin{aligned}
&= (g_0.\alpha(g_1,\ldots,g_p))\cdot(g_0\cdots g_p).\beta(g_{p+1},\ldots,g_{p+q})\\
&+\sum_{i=1}^{p}(-1)^i\alpha(g_0,\ldots,g_{i-1}g_i,\ldots,g_p)\cdot g_0\cdots g_p.\beta(g_{p+1},\ldots,g_{p+q})\\
&+\sum_{i=p+1}^{p+q}(-1)^i\alpha(g_0,\ldots,g_{p-1})\cdot g_0\cdots g_{p-1}.\beta(g_p,\ldots,g_{i-1}g_i,\ldots,g_{p+q})\\
&\quad+(-1)^{p+q+1}\alpha(g_0,\ldots,g_{p-1})\cdot(g_0\cdots g_{p-1}).\beta(g_p,\ldots,g_{p+q-1})\\
&=(d_G\alpha)(g_0,\ldots,g_p)\cdot(g_0\cdots g_p).\beta(g_{p+1},\ldots,g_{p+q})\\
&+(-1)^p\alpha(g_0,\ldots,g_{p-1})\cdot(g_0\cdots g_p).\beta(g_{p+1},\ldots,g_{p+q})\\
&+\alpha(g_0,\ldots,g_{p-1})\cdot g_0\cdots g_{p-1}.\Big(\sum_{i=p+1}^{p+q}(-1)^i\beta(g_p,\ldots,g_{i-1}g_i,\ldots,g_{p+q})\\
&\quad+(-1)^{p+q+1}\beta(g_p,\ldots,g_{p+q-1})\Big)\\
&=(d_G\alpha\cup\beta)(g_0,\ldots,g_{p+q})+(-1)^p(\alpha\cup d_G\beta)(g_0,\ldots,g_{p+q}).
\end{aligned}
$$

■

Lemma F.2 implies that products of two cocycles are cocycles and that the product of a cocycle with a coboundary is a coboundary, so that we obtain biadditive maps

$$H^p_s(G,U)\times H^q_s(G,V)\to H^{p+q}_s(G,W),\quad ([\alpha],[\beta])\mapsto[\alpha\cup\beta].$$

The following lemma shows that for Lie groups the multiplication of group and Lie algebra cochains is compatible with the differentiation map D.

Lemma F.3. *If G is a Lie group, U, V and W are smooth modules and $m\colon U\times V\to W$ is continuous bilinear and equivariant, then we have for $\alpha\in C^p(G,U)$ and $\beta\in C^q(G,V)$ we have in $C^{p+q}_c(\mathfrak{g},W)$:*

$$D(\alpha\cup\beta)=D\alpha\wedge D\beta.$$

Proof. In view of $D\alpha=\mathrm{Alt}(d^p\alpha(\mathbf{1},\ldots,\mathbf{1}))$, we get

$$
\begin{aligned}
D\alpha\wedge D_\beta &= \frac{1}{p!q!}\mathrm{Alt}(D\alpha\cdot D\beta)=\frac{1}{p!q!}\mathrm{Alt}(\mathrm{Alt}(d^p\alpha(\mathbf{1},\ldots,\mathbf{1}))\cdot\mathrm{Alt}(d^q\beta(\mathbf{1},\ldots,\mathbf{1})))\\
&=\mathrm{Alt}(d^p\alpha(\mathbf{1},\ldots,\mathbf{1})\cdot d^q\beta(\mathbf{1},\ldots,\mathbf{1})),
\end{aligned}
$$

so that it remains to see that

$$d^{p+q}(\alpha\cup\beta)(\mathbf{1},\ldots,\mathbf{1})=(d^p\alpha)(\mathbf{1},\ldots,\mathbf{1})\cdot(d^q\beta)(\mathbf{1},\ldots,\mathbf{1}),$$

but this follows immediately from the normalization of the cocycles and the chain rule for jets, applied to the multiplication map m. ■

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