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Variations on Prequantization

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Abstract

We extend known prequantization procedures for Poisson and presymplectic manifolds by defining the prequantization of a Dirac manifold P as a principal $U(1)$-bundle Q with a compatible Dirac-Jacobi structure. We study the action of Poisson algebras of admissible functions on P on various spaces of locally (with respect to P) defined functions on Q, via hamiltonian vector fields. Finally, guided by examples arising in complex analysis and contact geometry, we propose an extension of the notion of prequantization in which the action of $U(1)$ on Q is permitted to have some fixed points.

Dedicated to the memory of Professor Shiing-Shen Chern

1 Introduction

Prequantization in symplectic geometry attaches to a symplectic manifold P a hermitian line bundle K (or the corresponding principal $U(1)$-bundle Q), with a connection whose curvature form is the symplectic structure. The Poisson Lie algebra $C^\infty(P)$ then acts faithfully on the space $\Gamma(K)$ of sections of K (or antiequivariant functions on Q). Imposing a polarization Π cuts down $\Gamma(K)$ to a smaller, more "physically appropriate" space $\Gamma_\Pi(K)$ on which a subalgebra of $C^\infty(P)$ may still act. By polarizing and looking at the "ladder" of sections of tensor powers $K^{\otimes n}$ (or functions on Q transforming according to all the negative tensor powers of the standard representation of $U(1)$), one gets an "asymptotic representation" of the full algebra $C^\infty(P)$. All of this often goes under the name of geometric quantization, with the last step closely related to deformation quantization.

For systems with constraints or systems with symmetry, the phase space P may be a presymplectic or Poisson manifold. Prequantization, and sometimes the full procedure of geometric quantization, has been carried out in these settings by several authors; their work is cited below.

The principal aim of this paper is to suggest two extensions of the prequantization construction which originally arose in an example coming from contact geometry. The first, which unifies the presymplectic and Poisson cases and thus permits the simultaneous application of constraints and symmetry, is to allow P to be a Dirac manifold. The second is to allow the $U(1)$ action on Q to have fixed points when P has a boundary, so that the antiequivariant functions become sections of a sheaf rather than a line bundle over P. In the course of the paper, we also make some new observations concerning the Poisson and presymplectic cases.

1.1 Symplectic prequantization

On a symplectic manifold (P, ω), one defines the hamiltonian vector field X_f of the function f by $\omega(X_f, \cdot) = df$, and one has the Lie algebra bracket $\{f, g\} = \omega(X_f, X_g)$ on $C^\infty(P)$. A closed 2-form ω is called integral if its de Rham cohomology class $[\omega] \in H^2(M, \mathbb{R})$ is integral, i.e. if it is in the image of the homomorphism $i_* : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$ associated with the inclusion $i : \mathbb{Z} \to \mathbb{R}$ of coefficient groups.

When ω is integral, following Kostant [21], we prequantize (P, ω) by choosing a hermitian line bundle K bundle over P with first Chern class in $i_*^{-1}[\omega]$. Then there is a connection ∇ on K with curvature $2\pi i\omega$. Associating to each function f the operator $\hat{f}$ on $\Gamma(K)$ defined by[1] $\hat{f}(s) = -[\nabla_{X_f} s + 2\pi i f s]$, we obtain a faithful Lie algebra representation of $C^\infty(P)$ on $\Gamma(K)$.

The construction above is equivalent to the following, due to Souriau [27]: let Q be the principal $U(1)$-bundle associated to K. Denote by σ the connection form on Q corresponding to ∇ (so $d\sigma = \pi^*\omega$, where $\pi : Q \to P$), and by E the infinitesimal generator of the $U(1)$ action on Q. We can identify the sections of K with functions $\bar{s} : Q \to \mathbb{C}$ which are $U(1)$-antiequivariant (i.e. $\bar{s}(x \cdot t) = \bar{s}(x) \cdot t^{-1}$ for $x \in Q$, $t \in U(1)$, or equivalently $E(\bar{s}) = -2\pi i \bar{s}$), and then the operator $\hat{f}$ on $\Gamma(K)$ corresponds to the action of the vector field

$$-X_f^H + \pi^* f E, \tag{1.1}$$

where the superscript ^{H} denotes the horizontal lift to Q of a vector field on P. Notice that σ is a contact form on Q and that $X_f^H - \pi^* f E$ is just the hamiltonian vector field of $\pi^* f$ with respect to this contact form (viewed as a Jacobi structure; see Section 3).

1.2 Presymplectic prequantization

Prequantization of a presymplectic manifolds (P, ω) for which ω is integral and of constant rank[2] was introduced by Günther [15] (see also Gotay and Sniatycki [12] and Vaisman [33]). Günther represents the Lie algebra of functions constant

[1] Our convention for the Poisson bracket differs by a sign from that of [15] and [21]; consequently our formula for $\hat{f}$ and Equation (1.1) below differ by a sign too. Our sign has the property that the map from functions to their hamiltonian vector fields is an antihomomorphism from Poisson brackets to Lie brackets.

[2] Unlike many other authors (including some of those cited here), we will use the work "presymplectic" to describe any manifold endowed with a closed 2-form, even if the form does not have constant rank.

along the leaves of $\ker\omega$ by assigning to each such function f the equivalence class of vector fields on Q given by formula (1.1), where X_f now stands for the equivalence class of vector fields satisfying $\omega(X_f,\cdot) = df$.

1.3 Poisson prequantization

Prequantization of Poisson manifolds (P,Λ) was first investigated algebraically by Huebschmann [16], in terms of line bundles by Vaisman [31], and then in terms of circle bundles by Chinea, Marrero, and de Leon [5]. When the Poisson cohomology class $[\Lambda] \in H^2_\Lambda(P)$ is the image of an integral de Rham class $[\Omega]$ under the map given by contraction with Λ, a $U(1)$-bundle Q with first Chern class in $i_*^{-1}[\Omega]$ may be given a Jacobi structure for which the map that assigns to $f \in C^\infty(P)$ the hamiltonian vector field (with respect to the Jacobi structure) of $-\pi^* f$ is a Lie algebra homomorphism. This gives a (not always faithful) representation of $C^\infty(P)$.

1.4 Dirac prequantization

We will unite the results in the previous two paragraphs by using Dirac manifolds. These were introduced by Courant [6] and include both Poisson and presymplectic manifolds as special cases. On the other hand, Jacobi manifolds had already been introduced by Kirillov [20] and Lichnerowicz [24], including Poisson, conformally symplectic, and contact manifolds as special cases. All of these generalizations of Poisson structures were encompassed in the definition by Wade [34] of Dirac-Jacobi[3] manifolds.

To prequantize a Dirac manifold P, we will impose an integrality condition on P which implies the existence of a $U(1)$-bundle $\pi : Q \to P$ with a connection which will be used to construct a Dirac-Jacobi structure on Q. Prequantization of (suitable) functions $g \in C^\infty(P)$ is achieved "Souriau-style" by associating to g the equivalence class of the hamiltonian vector fields of $-\pi^* g$ and by letting this equivalence class act on a suitable subset of the $U(1)$-antiequivariant functions on Q, or equivalently by letting $\pi^* g$ act by the bracket of functions on Q. The same prequantization representation can be realized as an action on sections of a hermitian line bundle over P with an L-connection, where L is the Lie algebroid given by the Dirac manifold P.

We also look at the following very natural example, discovered by Claude LeBrun. Given a contact manifold M with contact distribution $C \subset TM$, the nonzero part of its annihilator C° is a symplectic submanifold of T^*M. When the contact structure is cooriented, we may choose the positive half C°_+ of this submanifold. By adjoining to C°_+ the "the section at infinity of T^*M" we obtain

[3] Wade actually calls them $\mathcal{E}^1(M)$-Dirac manifolds; we will stick to the terminology "Dirac-Jacobi", as introduced in [13].

a manifold with boundary, on which the symplectic structure on C_+° extends to give a Poisson structure. We call this a "LeBrun-Poisson manifold". If now we additionally adjoin the zero section of T^*M we obtain a Dirac manifold P.

First we will describe the prequantization $U(1)$-bundle of P, then we will modify it by collapsing to points the fibers over one of the two boundary components and by applying a conformal change. At the end, restricting this construction to the LeBrun-Poisson manifold (which sits as an open set inside P), we will obtain a contact manifold in which M sits as a contact submanifold.

1.5 Organization of the paper

In Sections 2 and 3 we collect known facts about Dirac and Dirac-Jacobi manifolds. In Section 4 we state our prequantization condition and describe the Dirac-Jacobi structure on the prequantization space of a Dirac manifold. In Section 5 we study the corresponding prequantization representation, and in Section 6 we derive the same representation by considering hermitian line bundles endowed with L-connections. In Section 7 we study the prequantization of LeBrun's examples, and in Section 8 we allow prequantization $U(1)$-bundles to have fixed points, and we endow them with contact structures. We conclude with some remarks in Section 9.

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2 Dirac manifolds

We start by recalling some facts from [6].

Definition 2.1 ([6], Def 1.1.1). A **Dirac structure** on a vector space V is a maximal isotropic subspace $L \subset V \oplus V^*$ with respect to the symmetric pairing

$$\langle X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rangle_+ = \frac{1}{2}(i_{X_2}\xi_1 + i_{X_1}\xi_2). \tag{2.1}$$

L necessarily has the same dimension as V, and denoting by ρ_V and ρ_{V^*} the projections of $V \oplus V^*$ onto V and V^* respectively, we have

$$\rho_V(L) = (L \cap V^*)^\circ \text{ and } \rho_{V^*}(L) = (L \cap V)^\circ \tag{2.2}$$

where the symbol $^\circ$ denotes the annihilator. It follows that L induces (and is equivalent to) a skew bilinear form on $\rho_V(L)$ or a bivector on $V/L\cap V$ ([6], Prop.

1.1.4). If (V, L) is a Dirac vector space and $i : W \to V$ a linear map, then one obtains a pullback Dirac structure on W by $\{Y \oplus i^*\xi : iY \oplus \xi \in L\}$; one calls a map between Dirac vector spaces "backward Dirac map" if it pulls back the Dirac structure of the target vector space to the one on the source vector space [3]. Similarly, given a linear map $p : V \to Z$, one obtains a pushforward Dirac structure on Z by $\{pX \oplus \xi : X \oplus p^*\xi \in L\}$, and one thus has a notion of "forward Dirac map" as well.

On a manifold M, a maximal isotropic subbundle $L \subset TM \oplus T^*M$ is called an **almost Dirac structure** on M. The appropriate integrability condition was discovered by Courant ([6], Def. 2.3.1):

Definition 2.2. A **Dirac structure** on M is an almost Dirac structure L on M whose space of sections is closed under the Courant bracket on sections of $TM \oplus T^*M$, which is defined by

$$(2.3)\quad [X_1 \oplus \xi_1, X_2 \oplus \xi_2] = \big([X_1, X_2] \;\oplus\; \mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 + \frac{1}{2}d(i_{X_2}\xi_1 - i_{X_1}\xi_2)\big).$$

When an almost Dirac structure L is integrable, $(L, \rho_{TM}|_L, [\cdot,\cdot])$ is a Lie algebroid[4] ([6], Thm. 2.3.4). The singular distribution $\rho_{TM}(L)$ is then integrable in the sense of Stefan and Sussmann [28] and gives rise to a singular foliation of M. The Dirac structure induces a closed 2-form (presymplectic form) on each leaf of this foliation ([6], Thm. 2.3.6). The distribution $L \cap V$, called the **characteristic distribution**, is singular in a different way. Its annihilator $\rho_{T^*M}(L)$ is closed in the cotangent bundle, but the distribution itself is not closed unless it has constant rank. It is not always integrable, either. (See Example 2.1 and the beginning of Section 7.)

Next we define hamiltonian vector fields and put a Lie algebra structure on a subspace of $C^\infty(M)$.

Definition 2.3. A function f on a Dirac manifold (M, L) is **admissible** if there exists a smooth vector field X_f such that $X_f \oplus df$ is a section of L. A vector field X_f as above is called a hamiltonian vector field of f. The set of admissible functions forms a subspace $C^\infty_{adm}(M)$ of $C^\infty(M)$.

If f is admissible then $df|_{L\cap TM} = 0$. The converse holds where the characteristic distribution $L \cap TM$ has constant rank, but not in general. In other words, df can be contained in $\rho_{T^*M}(L)$ without being the image of a smooth section of L; see Example 2.1. Since any two hamiltonian vector fields of an admissible function f differ by a characteristic vector field, which annihilates any other admissible function, we can make the following definition.

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[4] Recall that a Lie algebroid is a vector bundle A over a manifold M together with a Lie bracket $[\cdot,\cdot]$ on its space of sections and a bundle map $\rho : A \to TM$ (the "anchor") such that the Leibniz rule $[s_1, fs_2] = \rho s_1(f) \cdot s_2 + f \cdot [s_1, s_2]$ is satisfied for all sections s_1, s_2 of A and functions f on M.

Definition 2.4. The bracket on $C^\infty_{adm}(M)$ is given by $\{f,g\} = X_g \cdot f$.

This bracket differs by a sign from the one in the original paper of Courant [6], but it allows us to recover the usual conventions for presymplectic and Poisson manifolds, as shown below. The main feature of this bracket is the following (see [6], Prop. 2.5.3):

Proposition 2.1. *Let (M,L) be a Dirac manifold. If X_f and X_g are any hamiltonian vector fields for the admissible functions f and g, then $-[X_f, X_g]$ is a hamiltonian vector field for $\{f,g\}$, which is therefore admissible as well. The integrability of L implies that the bracket satisfies the Jacobi identity, so $(C^\infty_{adm}(M), \{\cdot,\cdot\})$ is a Lie algebra.*

We remark that the above can be partially extended to the space $C^\infty_{bas}(M)$ of **basic** functions, i.e. of functions ϕ satisfying $d\phi|_{L\cap TM} = 0$, which contains the admissible functions. (This two spaces of functions coincide when $L \cap TM$ is regular). Indeed, if h is admissible and ϕ is basic, then $\{\phi, h\} := X_h \cdot \phi$ is well defined and basic, since the flow of a hamiltonian vector field X_h induces vector bundle automorphisms of $TM \oplus T^*M$ that preserve $L \cap TM$ (see Section 2.4 in [6]). If f is an admissible function, then the Jacobiator of f, h, and ϕ vanishes (adapt the proof of Prop. 2.5.3 in [6]).

We recall how manifolds endowed with 2-forms or bivectors fit into the framework of Dirac geometry. Let ω be a 2-form on M, $\tilde{\omega} : TM \to T^*M$ the bundle map $X \mapsto \omega(X, \cdot)$. Its graph $L = \{X \oplus \tilde{\omega}(X) : X \in TM\}$ is an almost Dirac structure; it is integrable iff ω is closed. If ω is symplectic, i.e. nondegenerate, then every function f is admissible and has a unique hamiltonian vector field satisfying $\tilde{\omega}(X_f) = df$; the bracket is given by $\{f,g\} = \omega(X_f, X_g)$.

Example 2.1. Let ω be the presymplectic form $x_1^2 dx_1 \wedge dx_2$ on $M = \mathbb{R}^2$, and let L be its graph. The characteristic distribution $L \cap TM$ has rank zero everywhere except along $\{x_1 = 0\}$, where it has rank two, and it is clearly not integrable (compare the discussion following Definition 2.2). The differential of $f = x_1^2$ takes all its values in the range of ρ_{T^*M}, but f is not admissible. This illustrates the remark following Definition 2.3, i.e. it provides an example of a function which is basic but not admissible.

Let Λ be a bivector field on M, $\tilde{\Lambda} : T^*M \to TM$ the corresponding bundle map $\xi \mapsto \Lambda(\cdot, \xi)$. (Note that the argument ξ is in the second position.) Its graph $L = \{\tilde{\Lambda}(\xi) \oplus \xi : \xi \in T^*M\}$ is an almost Dirac structure which is integrable iff Λ is a Poisson bivector (i.e. the Schouten bracket $[\Lambda, \Lambda]_S$ is zero). Every function f is admissible with a unique hamiltonian vector field $X_f = \{\cdot, f\}$, and the bracket of functions is $\{f,g\} = \Lambda(df, dg)$.

3 Dirac-Jacobi manifolds

Dirac-Jacobi structures were introduced by Wade [34] (under a different name) and include Jacobi (in particular, contact) and Dirac structures as special cases. Like Dirac structures, they are defined as maximal isotropic subbundles of a certain vector bundle.

Definition 3.1. A Dirac-Jacobi structure on a vector space V is a subspace $\bar{L} \subset (V \times \mathbb{R}) \oplus (V^* \times \mathbb{R})$ which is maximal isotropic under the symmetric pairing

(3.1)
$$\big\langle (X_1, f_1) \oplus (\xi_1, g_1)\ ,\ (X_2, f_2) \oplus (\xi_2, g_2) \big\rangle_+ = \frac{1}{2}(i_{X_2}\xi_1 + i_{X_1}\xi_2 + g_1 f_2 + g_2 f_1).$$

A Dirac-Jacobi structure on V necessarily satisfies $\dim \bar{L} = \dim V + 1$. Furthermore, Equations (2.2) hold for Dirac-Jacobi structures too:

$$\rho_V(\bar{L}) = (\bar{L} \cap V^*)^\circ \text{ and } \rho_{V^*}(\bar{L}) = (\bar{L} \cap V)^\circ. \tag{3.2}$$

As in the Dirac case, one has notions of pushforward and pullback structures and as well as forward and backward maps. For example, given a Dirac-Jacobi structure $\bar{L}$ on V and a linear map $p : V \to Z$, one obtains a pushforward Dirac-Jacobi structure on Z by $\{(pX, f) \oplus (\xi, g) : (X, f) \oplus (p^*\xi, g) \in \bar{L}\}$.

On a manifold M, a maximal isotropic subbundle $\bar{L} \subset \mathcal{E}^1(M) := (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ is called an **almost Dirac-Jacobi structure** on M.

Definition 3.2 ([34], Def. 3.2). A **Dirac-Jacobi structure** on a manifold M is an almost Dirac-Jacobi structure $\bar{L}$ on M whose space of sections is closed under the extended Courant bracket on sections of $\mathcal{E}^1(M)$, which is defined by

(3.3)
$$\begin{aligned}
[(X_1, f_1) \oplus (\xi_1, g_1)\ ,\ (X_2, f_2) \oplus (\xi_2, g_2)] = \big(&[X_1, X_2], X_1 \cdot f_2 - X_2 \cdot f_1\big) \\
&\oplus \big(\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 + \frac{1}{2}d(i_{X_2}\xi_1 - i_{X_1}\xi_2) \\
&+ f_1\xi_2 - f_2\xi_1 + \frac{1}{2}(g_2 df_1 - g_1 df_2 - f_1 dg_2 + f_2 dg_1), \\
&X_1 \cdot g_2 - X_2 \cdot g_1 + \frac{1}{2}(i_{X_2}\xi_1 - i_{X_1}\xi_2 - f_2 g_1 + f_1 g_2)\big).
\end{aligned}$$

By a straightforward computation (see also Section 4 of [13]) this bracket can be derived from the Courant bracket (2.3), as follows. Denote by U the embedding $\Gamma(\mathcal{E}^1(M)) \to \Gamma(T(M \times \mathbb{R}) \oplus T^*(M \times \mathbb{R}))$ given by

$$(X, f) \oplus (\xi, g) \mapsto (X + f\frac{\partial}{\partial t}) \oplus e^t(\xi + g dt),$$

where t is the coordinate on the $\mathbb{R}$ factor of the manifold $M \times \mathbb{R}$. Then U is a bracket-preserving map from $\Gamma(\mathcal{E}^1(M))$ with the extended bracket (3.3), to $\Gamma(T(M\times\mathbb{R})\oplus T^*(M\times\mathbb{R}))$ with the Courant bracket (2.3) of the manifold $M\times\mathbb{R}$.

Furthermore in Section 5 of [17] it is shown that any Dirac-Jacobi manifold $(M, \bar{L})$ gives rise to a Dirac structure on $M \times \mathbb{R}$ given by

$$\tilde{\bar{L}}_{(x,t)} = \{(X + f\frac{\partial}{\partial t}) \;\oplus\; e^t(\xi + g dt) : (X, f) \oplus (\xi, g) \in \bar{L}_x\},$$

where t is the coordinate on $\mathbb{R}$. This procedure extends the well known symplectization of contact manifolds and Poissonization of Jacobi manifolds, and may be called “Diracization”.

If $\bar{L}$ is a Dirac-Jacobi structure, $(\bar{L}, \rho_{TM}, [\cdot,\cdot])$ is a Lie algebroid ([34], Thm. 3.4), and each leaf of the induced foliation on M has the structure of a precontact manifold (i.e. simply a 1-form) or of a locally conformal presymplectic manifold (i.e. a 2-form Ω and a closed 1-form ω satisfying $d\Omega = \omega \wedge \Omega$). See Section 4.1 for a description of the induced foliation. As in the Dirac case, one can define hamiltonian vector fields and endow a subset of $C^\infty(M)$ with a Lie algebra structure.

Definition 3.3 ([34], Def. 5.1). A function f on a Dirac-Jacobi manifold $(M, \bar{L})$ is **admissible** if there exists a smooth vector field X_f and a smooth function φ_f such that $(X_f, \varphi_f) \oplus (df, f)$ is a section of $\bar{L}$. Pairs (X_f, φ_f) as above are unique up to smooth sections of $\bar{L} \cap (TM \times \mathbb{R})$, and X_f is called a hamiltonian vector field of f. The set of admissible functions is denoted by $C^\infty_{adm}(M)$.

Definition 3.4. The bracket on $C^\infty_{adm}(M)$ is given by $\{f, g\} = X_g \cdot f + f\varphi_g$

This bracket, which differs by a sign from that in [34], enjoys the same properties stated in Proposition 2.1 for Dirac manifolds (see [34], Prop. 5.2 and Lemma 5.3).

Proposition 3.1. *Let $(M, \bar{L})$ be a Dirac-Jacobi manifold. If f and g are admissible functions, then*

$$\begin{aligned} &[(X_f, \varphi_f) \oplus (df, f)\ ,\ (X_g, \varphi_g) \oplus (dg, g)] = \\ &([X_f, X_g], X_f \cdot \varphi_g - X_g \cdot \varphi_f) \oplus (-d\{f, g\}, -\{f, g\}), \end{aligned} \tag{3.4}$$

hence $\{f, g\}$ is again admissible. The integrability of $\bar{L}$ implies that the admissible functions form a Lie algebra.

We call a function ψ on M **basic** if $X \cdot \psi + \psi f = 0$ for all elements $(X, f) \in \bar{L} \cap (TM \times \mathbb{R})$. This is equivalent to requiring $(d\psi, \psi) \in \rho_{T^*M\times\mathbb{R}}(\bar{L})$ at each point of M, so the basic functions contain the admissible ones. As in the case of Dirac structures, we have the following properties:

Lemma 3.1. *If ψ is a basic and h an admissible function, then the bracket $\{\psi, h\} := X_h \cdot \psi + \psi h$ is well-defined and again basic.*

Proof. It is clear that the bracket is well defined. To show that $X_h \cdot \psi + \psi h$ is again basic we reduce the problem to the Dirac case. Let $(X, f) \in \bar{L}_x \cap (TM \times \mathbb{R})$ Fix a choice of (X_h, φ_h) for the admissible function h. The vector field $X_h + \varphi_h \frac{\partial}{\partial t}$ on the Diracization $(M \times \mathbb{R}, \tilde{\bar{L}})$ (which is just a Hamiltonian vector field of $e^t h$) has a flow $\tilde{\phi}_\epsilon$, which projects to the flow ϕ_ϵ of X_h under $pr_1 : M \times \mathbb{R} \to M$. For each ϵ the flow $\tilde{\phi}_\epsilon$ induces a vector bundle automorphism Φ_ϵ of $\mathcal{E}^1(M)$, covering the diffeomorphism ϕ_ϵ of M, as follows:

$$(X, f) \oplus (\xi, g) \in \mathcal{E}^1_x(M) \mapsto (\tilde{\phi}_\epsilon)_*(X \oplus f\frac{\partial}{\partial t})_{(x,0)} \oplus (\tilde{\phi}_\epsilon^{-1})^*(\xi + gdt)_{(x,0)} \cdot e^{-pr_2(\tilde{\phi}_\epsilon(x,0))},$$

where we identify $T_{\tilde{\phi}_\epsilon(x,0)}(M \times \mathbb{R}) \oplus T^*_{\tilde{\phi}_\epsilon(x,0)}(M \times \mathbb{R})$ with $\mathcal{E}^1_{\phi_\epsilon(x)}(M)$ to make sense of the second term. Since the vector bundle maps induced by the flow $\tilde{\phi}_\epsilon$ preserve the Dirac structure $\tilde{\bar{L}}$ (see Section 2.4 in [6]), using the definition of the Diracization $\tilde{\bar{L}}$ one sees that Φ_ϵ preserves $\bar{L}$, and therefore also $\bar{L} \cap (TM \times \mathbb{R})$. Notice that we can pull back sections of $\mathcal{E}^1(M)$ by setting $(\Phi_\epsilon^*((X, f) \oplus (\xi, g)))_x := \Phi_\epsilon^{-1}((X, f) \oplus (\xi, g))_{\phi_\epsilon(x)}$. A computation shows that

$$(0,0) \oplus (d(X_h \cdot \psi + \varphi_h \psi)\,,\, X_h \cdot \psi + \varphi_h \psi) = \frac{\partial}{\partial \epsilon}\Big|_0 \Phi_\epsilon^*((0,0) \oplus (d\psi, \psi)),$$

so that

$$\begin{aligned} &\langle (0,0) \oplus (d(X_h \cdot \psi + \varphi_h \psi), X_h \cdot \psi + \varphi_h \psi)\ ,\ (X, f) \oplus (0,0) \rangle_+ = \\ &\frac{\partial}{\partial \epsilon}\Big|_0 \left[\langle (0,0) \oplus (d\psi, \psi)_{\phi_\epsilon(x)}\ ,\ \Phi_\epsilon((X, f) \oplus (0,0))_x \rangle_+ e^{pr_2(\tilde{\phi}_\epsilon(x,0))} \right] = 0, \end{aligned}$$

as was to be shown. $\square$

Furthermore, the Jacobiator of admissible functions f, h and a basic function ψ is zero. One can indeed check that Wade's proof of the Jacobi identity for admissible functions ([34] Prop. 5.2) applies in this case too. Alternatively, this follows from the analogous statement for the Diracization $M \times \mathbb{R}$, since the map

$$C^\infty_{adm}(M) \to C^\infty_{adm}(M \times \mathbb{R})\ ,\ g \mapsto e^t g \tag{3.5}$$

is a well-defined Lie algebra homomorphism[5] and maps basic functions to basic functions.

[5]For the well-definedness notice that, if $(X_g, \varphi_g) \oplus (dg, g) \in \Gamma(\bar{L})$, then $(X_g + \varphi_g \frac{\partial}{\partial t}) \oplus d(e^t g) \in \Gamma(\tilde{\bar{L}})$. Notice that in particular $X_g + \varphi_g \frac{\partial}{\partial t}$ is a hamiltonian vector field for $e^t g$. Using this, the equation $e^t\{f, g\}_M = \{e^t f, e^t g\}_{M \times \mathbb{R}}$ follows at once from the definitions of the respective brackets of functions.

Now we display some examples of Dirac-Jacobi manifolds.

There is a one-to-one correspondence between Dirac structures on M and Dirac-Jacobi structures on M contained in $TM \oplus (T^*M \times \mathbb{R})$: to each Dirac structure L one associates the Dirac-Jacobi structure $\{(X, 0) \oplus (\xi, g) : X \oplus \xi \in L, g \in \mathbb{R}\}$ ([34], Remark 3.1).

A Jacobi structure on a manifold M is given by a bivector field Λ and a vector field E satisfying the Schouten bracket conditions $[E, \Lambda]_S = 0$ and $[\Lambda, \Lambda]_S = 2E \wedge \Lambda$. When $E = 0$, the Jacobi structure is a Poisson structure. Any skew-symmetric vector bundle morphism $T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ is of the form $\left(\begin{smallmatrix} \tilde{\Lambda} & -E \\ E & 0 \end{smallmatrix}\right)$ for a bivector field Λ and a vector field E, where as in Section 2 we have $\tilde{\Lambda}\xi = \Lambda(\cdot, \xi)$. Graph $\left(\begin{smallmatrix} \tilde{\Lambda} & -E \\ E & 0 \end{smallmatrix}\right) \subset \mathcal{E}^1(M)$ is a Dirac-Jacobi structure iff (M, Λ, E) is a Jacobi manifold ([34], Sect 4.1). In this case all functions are admissible, the unique hamiltonian vector field of f is[6] $X_f = \tilde{\Lambda}df - fE$, $\varphi_f = E \cdot f$ and the bracket is given by $\{f, g\} = \Lambda(df, dg) + fE \cdot g - gE \cdot f$.

Similarly (see [34], Sect. 4.3), any skew-symmetric vector bundle morphism $TM \times \mathbb{R} \to T^*M \times \mathbb{R}$ is of the form $\left(\begin{smallmatrix} \tilde{\Omega} & \sigma \\ -\sigma & 0 \end{smallmatrix}\right)$ for a 2-form Ω and a 1-form σ, and graph $\left(\begin{smallmatrix} \tilde{\Omega} & \sigma \\ -\sigma & 0 \end{smallmatrix}\right) \subset \mathcal{E}^1(M)$ is a Dirac-Jacobi structure iff $\Omega = d\sigma$.

Any contact form σ on a manifold M defines a Jacobi structure (Λ, E) (where E is the Reeb vector field of σ and $\tilde{\Lambda}\tilde{d\sigma}|_{\ker \sigma} = \mathrm{Id}$; see for example [18], Sect. 2.2), and graph $\left(\begin{smallmatrix} \tilde{d\sigma} & \sigma \\ -\sigma & 0 \end{smallmatrix}\right)$ is equal to graph $\left(\begin{smallmatrix} \tilde{\Lambda} & -E \\ E & 0 \end{smallmatrix}\right)$. Further, by considering suitably defined graphs, one sees that locally conformal presymplectic structures and homogeneous Poisson manifolds (given by a Poisson bivector Λ and a vector field Z satisfying $\mathcal{L}_Z\Lambda = -\Lambda$) are examples of Dirac-Jacobi structures ([34], Sect. 4).

4 The prequantization spaces

In this section we determine the prequantization condition for a Dirac manifold (P, L), and we describe its "prequantization space" (i.e. the geometric object that allows us to find a representation of $C^\infty_{adm}(P)$).

We recall the prequantization of a Poisson manifold (P, Λ) by a $U(1)$-bundle as described in [5]. The bundle map $\tilde{\Lambda} : T^*P \to TP$ extends to a cochain map from forms to multivector fields, which descends to a map from de Rham cohomology $H^\bullet_{dR}(P, \mathbb{R})$ to Poisson cohomology $H^\bullet_\Lambda(P)$ (the latter having the set of p-vector fields as p-cochains). The prequantization condition, first formulated in this form in [31], is that $[\Lambda] \in H^2_\Lambda(P)$ be the image under $\tilde{\Lambda}$ of an integral de Rham class, or equivalently that

$$\tilde{\Lambda}\Omega = \Lambda + \mathcal{L}_A\Lambda \qquad (4.1)$$

for some integral closed 2-form Ω and vector field A on P. Assuming this prequantization condition to be satisfied, let $\pi : Q \to P$ be a $U(1)$-bundle with first

[6] Again, this is opposite to the usual sign convention.

Chern class $[\Omega]$, σ a connection on Q with curvature Ω (i.e. $d\sigma = \pi^*\Omega$), and E the generator of the $U(1)$-action (so that $\sigma(E) = 1$ and $\pi_* E = 0$). Then (see Thm. 3.1 in [5])

$$(\Lambda^H + E \wedge A^H, E) \tag{4.2}$$

is a Jacobi structure on Q which pushes down to $(\Lambda, 0)$ on P via π_*. (The superscript ^{H} denotes horizontal lift, with respect to the connection σ, of multivector fields on P.) We say that π is a Jacobi map.

It follows from the Jacobi map property of π that assigning to a function f on P the hamiltonian vector field of $-\pi^* f$, which is $-(\widetilde{\Lambda^H + E \wedge A^H})(\pi^* df) + (\pi^* f)E$, defines a Lie algebra homomorphism from $C^\infty(P)$ to the operators on $C^\infty(Q)$.

Now we carry out an analogous construction on a Dirac manifold (P, L). Recall that L is a Lie algebroid with the restricted Courant bracket and anchor $\rho_{TP} : L \to TP$ (which is just the projection onto the tangent component). This anchor gives a Lie algebra homomorphism from $\Gamma(L)$ to $\Gamma(TP)$ with the Lie bracket of vector fields. The pullback by the anchor therefore induces a map $\rho^*_{TP} : \Omega^\bullet_{dR}(P, \mathbb{R}) \to \Omega^\bullet_L(P)$, descending to a map from de Rham cohomology to the Lie algebroid cohomology $H^2_L(P)$. (We recall from [8] that $\Omega^\bullet_L(P)$ is the graded differential algebra of sections of the exterior algebra of L^*.) There is a distinguished class in $H^2_L(P)$: on $TP \oplus T^*P$, in addition to the natural symmetric pairing (2.1), there is also an anti-symmetric one given by

$$\langle X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rangle_- = \frac{1}{2}(i_{X_2}\xi_1 - i_{X_1}\xi_2). \tag{4.3}$$

Its restriction Υ to L satisfies $d_L \Upsilon = 0$. Our prequantization condition is

$$[\Upsilon] \in \rho^*_{TP}(i_*(H^2(P, \mathbb{Z}))) \tag{4.4}$$

or equivalently

$$\rho^*_{TP}\Omega = \Upsilon + d_L \beta, \tag{4.5}$$

where Ω is a closed integral 2-form and β a 1-cochain for the Lie algebroid L, i.e. a section of L^*.

Remark 4.1. If L is the graph of a presymplectic form ω then $\Upsilon = \rho^*_{TP}(\omega)$. If L is graph$(\tilde{\Lambda})$ for a Poisson bivector Λ and Ω is a 2-form, then $\rho^*_{TP}[\Omega] = [\Upsilon]$ if and only if $\tilde{\Lambda}[\Omega] = [\Lambda]$.[7] This shows that (4.4) generalizes the prequantization conditions for presymplectic and Poisson structures mentioned in the introduction and in formula (4.1).

[7] This is consistent with the fact that, if ω is symplectic, then graph$(\tilde{\omega})$ = graph$(\tilde{\Lambda})$, where the bivector Λ is defined so that the vector bundle maps $\tilde{\omega}$ and $\tilde{\Lambda}$ are inverses of each other (so if $\omega = dx \wedge dy$ on $\mathbb{R}^2$, then $\Lambda = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$).

Remark 4.2. The prequantization condition above can not even be formulated for twisted Dirac structures. We recall the definition of these structures [25]. If ϕ is a closed 3-form on a manifold P, adding the term $\phi(X_1, X_2, \cdot)$ to the Courant bracket (i.e. to the right hand side of Equation (2.3)) determines a new bracket $[\cdot,\cdot]^\phi$ so that $TP \oplus T^*P$, together with this bracket, the original anchor ρ_{TP} and the symmetric pairing $\langle\cdot,\cdot\rangle_+$, form a Courant algebroid. A ϕ-twisted Dirac structure L is then a maximal isotropic subbundle which is closed under $[\cdot,\cdot]^\phi$; it is automatically a Lie algebroid (whose Lie algebroid differential we denote by d_L^ϕ). The orbits of the Lie algebroid carry 2-forms Ω_L given as in the remark following Definition 2.1, satisfying $d\Omega_L = j^*\phi$ where j is the inclusion of a leaf in P and d is the de Rham differential on the leaves. Since

$$d_L^\phi \Upsilon = d_L^\phi \rho_{TP}^* \Omega_L = \rho_{TP}^* d\Omega_L = \rho_{TP}^* j^* \phi,$$

we conclude that Υ is usually not d_L^ϕ-closed, so we cannot expect Ω to be closed in (4.5), and hence we cannot require that it be integral. The correct notion of prequantization should probably involve a gerbe.

Now, assuming the prequantization condition (4.4) and proceeding as in the Poisson case, let $\pi : Q \to P$ be a $U(1)$-bundle with connection form σ having curvature Ω; denote by E the infinitesimal generator of the $U(1)$-action.

Theorem 4.1. *The subbundle $\bar{L}$ of $\mathcal{E}^1(Q)$ given by the direct sum of*

$$\{(X^H + \langle X \oplus \xi, \beta\rangle E, 0) \oplus (\pi^*\xi, 0) : X \oplus \xi \in L\}$$

and the line bundles generated by $(-E, 0) \oplus (0, 1)$ and $(-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$ is a Dirac-Jacobi structure on Q. Here, $A \oplus \alpha$ is an isotropic section of $TP \oplus T^*P$ satisfying $\beta = 2\langle A \oplus \alpha, \cdot\rangle_+|_L$. Such a section always exists, and the subbundle above is independent of the choice of $A \oplus \alpha$.

Proof. Let C be a maximal isotropic (with respect to $\langle\cdot,\cdot\rangle_+$) complement of L in $TP \oplus T^*P$. Such a complement always exists, since the space of complements at each point is contractible (an affine space modeled on a space of skew-symmetric forms). Now extend β to a functional $\tilde{\beta}$ on $TP \oplus T^*P$ by setting $\tilde{\beta}|_C = 0$. There exists a unique section $A \oplus \alpha$ of $TP \oplus T^*P$ satisfying $\beta = 2\langle A \oplus \alpha, \cdot\rangle_+$ since the symmetric pairing is non-degenerate. Since $\langle A \oplus \alpha, \cdot\rangle_+|_C = 0$ and C is maximal isotropic we conclude that $A \oplus \alpha$ belongs to C and is hence isotropic itself. This shows the existence of $A \oplus \alpha$ as above.

Now clearly $A \oplus \alpha + Y \oplus \eta$ satisfies the property stated in the theorem iff $Y \oplus \eta \subset L$, and in this case it is isotropic (i.e. $\langle A+Y, \alpha+\eta\rangle = 0$) iff $Y \oplus \eta \subset \ker\beta$. So a section $A \oplus \alpha$ as in the theorem is unique up to sections $Y \oplus \eta$ of $\ker\beta$. By inspection one sees that replacing $A \oplus \alpha$ by $A \oplus \alpha + Y \oplus \eta$ in the formula for $\bar{L}$ defines the same subbundle.

That $\bar{L}$ is isotropic with respect to the symmetric pairing on $\mathcal{E}^1(Q)$ follows from the fact that L is isotropic, together with the properties of $A \oplus \alpha$. $\bar{L}$ is clearly a subbundle of dimension $\dim P + 2$, so it is an almost Dirac-Jacobi structure.

To show that $\bar{L}$ is integrable, we use the fact that $\bar{L}$ is integrable if and only if $\langle [e_1, e_2], e_3\rangle_+ = 0$ for all sections e_i of $\bar{L}$ and that $\langle [\cdot,\cdot],\cdot\rangle_+$ is a totally skew-symmetric tensor if restricted to sections of $\bar{L}$, i.e. an element of $\Gamma(\wedge^3 \bar{L}^*)$ ([17], Prop. 2.2). Each section of $\bar{L}$ can be written as a $C^\infty(Q)$-linear combination of the following three types of sections of $\bar{L}$: $a =: (X^H + \langle X \oplus \xi, \beta\rangle E, 0) \oplus (\pi^*\xi, 0)$ where $X \oplus \xi \in \Gamma(L)$, $b := (-E, 0) \oplus (0, 1)$ and $c := (-A^H, 1) \oplus (\sigma - \pi^*\alpha, 0)$. We will use subscripts to label more than one section of a given type. It is immediate that brackets of the form $[a, b]$,$[b_1, b_2]$, and $[c_1, c_2]$ all vanish, and a computation shows that $\langle [a_1, a_2], a_3\rangle_+ = 0$ since $L \subset TP \oplus T^*P$ is a Dirac structure. Finally $\langle [a_1, a_2], c\rangle_+ = 0$ using $d\sigma = \pi^*\Omega$ and the prequantization condition (4.5), which when applied to sections $X_1 \oplus \xi_1$ and $X_2 \oplus \xi_2$ of L reads

$$\Omega(X_1, X_2) = \langle \xi_1, X_2\rangle + X_1\langle \beta, X_2 \oplus \xi_2\rangle - X_2\langle \beta, X_1 \oplus \xi_1\rangle - \langle \beta\ ,\ \big[X_1 \oplus \xi_1, X_2 \oplus \xi_2\big]\rangle.$$

By skew-symmetry, the vanishing of these expressions is enough to prove the integrability of $\bar{L}$. □

Remark 4.3. When (P, L) is a Poisson manifold, $\bar{L}$ is exactly the graph of the Jacobi structure (4.2), i.e. it generalizes the construction of [5]. If (P, L) is given by a presymplectic form Ω, then $\bar{L}$ is the graph of $(d\sigma, \sigma)$.

Remark 4.4. The construction of Theorem 4.1 also works for complex Dirac structures (i.e., integrable maximal isotropic complex subbundles of the complexified bundle $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$). It can be adapted to the setting of generalized complex structures [14] (complex Dirac structures which are transverse to their complex conjugate) and generalized contact structures [18] (complex Dirac-Jacobi structures which are transverse to their complex conjugate) as follows. If (P, L) is a generalized complex manifold, assume all of the previous notation and the following prequantization condition:

$$\rho^*_{TP}\Omega = i\Upsilon + d_L\beta, \tag{4.6}$$

where Ω is (the complexification of) a closed integer 2-form and β a 1-cochain for the Lie algebroid L. Then the direct sum of

$$\{(X^H + \langle X \oplus \xi, \beta\rangle E, 0) \oplus (\pi^*\xi, 0) : X \oplus \xi \in L\}$$

and the complex line bundles generated by $(-iE, 0) \oplus (0, 1)$ and $(-A^H, i) \oplus (\sigma - \pi^*\alpha, 0)$ is a generalized contact structure on Q, where $A \oplus \alpha$ is the unique section of the conjugate of L satisfying $\beta = 2\langle A \oplus \alpha,\ \cdot\ \rangle_+|_L$.

4.1 Leaves of the Dirac-Jacobi structure

Given any Dirac-Jacobi manifold $(M, \bar{L})$, each leaf of the foliation integrating the distribution $\rho_{TM}(\bar{L})$ carries one of two kinds of geometric structures [17], as we describe now. $\rho_1 : \bar{L} \to \mathbb{R}, (X, f) \oplus (\xi, g) \mapsto f$ determines an algebroid 1-cocycle, and a leaf $\bar{F}$ of the foliation will be of one kind or the other depending on whether $\ker \rho_1$ is contained in the kernel of the anchor ρ_{TM} or not. (This property is satisfied either at all points of $\bar{F}$ or at none). As with Dirac structures, the Dirac-Jacobi structure $\bar{L}$ determines a field of skew-symmetric bilinear forms $\Psi_{\bar{F}}$ on the image of $\rho_{TM} \times \rho_1$.

If $\ker \rho_1 \not\subset \ker \rho_{TM}$ on $\bar{F}$ then $\rho_{TM} \times \rho_1$ is surjective, hence $\Psi_{\bar{F}}$ determines a 2-form and a 1-form on $\bar{F}$. The former is the differential of the latter, so the leaf $\bar{F}$ is simply endowed with a 1-form , i.e. it is a precontact leaf. If $\ker \rho_1 \subset \ker \rho_{TM}$ on $\bar{F}$ then the image of $\rho_{TQ} \times \rho_1$ projects isomorphically onto $T\bar{F}$, which therefore carries a 2-form $\Omega_{\bar{F}}$. It turns out that $\omega_{\bar{F}}(Y) := -\rho_1(e)$, for any $e \in \bar{L}$ with $\rho_{TM}(e) = Y$, is a well-defined 1-form on $\bar{F}$, and that $(\bar{F}, \Omega_{\bar{F}}, \omega_{\bar{F}})$ is a locally conformal presymplectic manifold, i.e. $\omega_{\bar{F}}$ is closed and $d\Omega_{\bar{F}} = \Omega_{\bar{F}} \wedge \omega_{\bar{F}}$.

On our prequantization $(Q, \bar{L})$ the leaf $\bar{F}$ through $q \in Q$ will carry one or the other geometric structure depending on whether A is tangent to F, where F denotes the presymplectic leaf of (P, L) passing through $\pi(q)$. Indeed one can check that at q we have $\ker \rho_1 \not\subset \ker \rho_{TQ} \Leftrightarrow A \in T_{\pi(q)}F$. When $\ker \rho_1 \not\subset \ker \rho_{TQ}$ on a leaf $\bar{F}$ we hence deduce that $\bar{F}$, which is equal to $\pi^{-1}(F)$, is a precontact manifold, and a computation shows that the 1-form is given by the restriction of

$$\sigma + \pi^*(\xi_A - \alpha)$$

where ξ_A is any covector satisfying $A \oplus \xi_A \in L$.

A leaf $\bar{F}$ on which $\ker \rho_1 \subset \ker \rho_{TQ}$ is locally conformal presymplectic, and its image under π is an integral submanifold of the integrable distribution $\rho_{TP}(L) \oplus \mathbb{R}A$ (hence a one parameter family of presymplectic leaves). A computation shows that the locally conformal presymplectic structure is given by

$$(\omega_{\bar{F}}, \Omega_{\bar{F}}) = \left(\pi^*\tilde{\gamma}\ ,\ (\sigma - \pi^*\alpha) \wedge \pi^*\tilde{\gamma} + \pi^*\tilde{\Omega}_L\right).$$

Here $\tilde{\gamma}$ is the 1-form on $\pi(\bar{F})$ with kernel $\rho_{TP}(L)$ and evaluating to one on A, while $\tilde{\Omega}_L$ is the two form on $\pi(\bar{F})$ which coincides with Ω_L (the presymplectic form on the leaves of (P, L)) on $\rho_{TP}(L)$ and annihilates A.

4.2 Dependence of the Dirac-Jacobi structure on choices

Let (P, L) be a prequantizable Dirac manifold , i.e. one for which there exist a closed integral 2-form Ω and a section of β of L^* such that

$$\rho_{TP}^*\Omega = \Upsilon + d_L\beta. \tag{4.7}$$

The Dirac-Jacobi manifold $(Q, \bar{L})$ as defined in Theorem 4.1 depends on three data: the choice (up to isomorphism) of the $U(1)$-bundle Q, the choice of connection σ on Q whose curvature has cohomology class $i_* c_1(Q)$, and the choice of β, subject to the condition that Equation (4.7) be satisfied. We will explain here how the Dirac-Jacobi structure $\bar{L}(Q, \sigma, \beta)$ depends on these choices.

First, notice that the value of Ω outside of $\rho_{TP}(L)$ does not play a role in (4.7). In fact, different choices of σ agreeing over $\rho_{TP}(L)$ give rise to the same Dirac-Jacobi structure. This is consistent with the following lemma, which is the result of a straightforward computation:

Lemma 4.1. *For any 1-form γ on P the Dirac-Jacobi structures $\bar{L}(Q, \sigma, \beta)$ and $\bar{L}(Q, \sigma + \pi^*\gamma, \beta + \rho_{TP}^*\gamma)$ are equal.*

Two Dirac-Jacobi structures on a given $U(1)$-bundle Q over P give isomorphic quantizations if they are related by an element of the gauge group $C^\infty(P, U(1))$ acting on Q. Noting that the Lie algebroid differential d_L descends to a map $C^\infty(P, U(1)) \to \Omega^1_L(P)$ we denote by $H^1_L(P, U(1))$ the quotient of the closed elements of $\Omega^1_L(P)$ by the space $d_L(C^\infty(P, U(1))$ of $U(1)$-exact forms.

Now we show:

Proposition 4.1. *The set of isomorphism classes of Dirac-Jacobi manifolds prequantizing (P, L) maps surjectively to the space $(\rho_{TP}^* \circ i_*)^{-1}[\Upsilon]$ of topological types of compatible $U(1)$-bundles; the prequantizations of a given topological type are a principal homogeneous space for $H^1_L(P, U(1))$.*

Proof. Make a choice of prequantizing triple (Q, σ, β). With Q and σ fixed, we are allowed to change β by a d_L-closed section of L^*. If we fix only Q, we are allowed to change σ in such a way that the resulting curvature represents the cohomology class $i_* c_1(Q)$, so we can change σ by $\pi^*\gamma$ where γ is a 1-form on P. Now $\bar{L}(Q, \sigma + \pi^*\gamma, \tilde{\beta}) = \bar{L}(Q, \sigma, \tilde{\beta} - \rho_{TP}^*\gamma)$ by Lemma 4.1, so we obtain one of the Dirac-Jacobi structures already obtained above. Now, if we replace β by $\beta + d_L\phi$ for $\phi \in C^\infty(P, U(1))$, we obtain an isomorphic Dirac-Jacobi structure: in fact $\bar{L}(Q, \sigma, \beta)$ is equal to $\bar{L}(Q, \sigma + \pi^* d\phi, \beta + d_L\phi)$ by Lemma 4.1, which is isomorphic to $\bar{L}(Q, \sigma, \beta + d_L\phi)$ because the gauge transformation given by ϕ takes the connection σ to $\sigma + \pi^* d\phi$. So we see that the difference between two prequantizing Dirac-Jacobi structures on the fixed $U(1)$-bundle Q corresponds to an element of $H^1_L(P, U(1))$. □

In Dirac geometry, a B-field transformation (see for example [25]) is an automorphism of the Courant algebroid $TM \oplus T^*M$ arising from a closed 2-form B and taking each Dirac structure into another one with an isomorphic Lie algebroid. There is a similar construction for Dirac-Jacobi structures. Given any 1-form γ on any manifold M, the vector bundle endomorphism of $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ that acts on $(X, f) \oplus (\xi, g)$ by adding $(0, 0) \oplus \begin{pmatrix} \tilde{d\gamma} & \gamma \\ -\gamma & 0 \end{pmatrix} (X, f)$

preserves the extended Courant bracket and the symmetric pairing. Thus, it maps each Dirac-Jacobi structure to another one. We call this operation an extended B-field transformation.

Lemma 4.2. *Let γ be a closed 1-form on P. Then $\bar{L}(Q, \sigma + \pi^*\gamma, \beta)$ is obtained from $\bar{L}(Q, \sigma, \beta)$ by the extended B-field transformation associated to γ.*

In the statements that follow, until the end of this subsection, we assume that the distribution $\rho_{TP}(L)$ has constant rank, and we denote by $\mathcal{F}$ the regular distribution integrating it.

Corollary 4.1. *Assume that $\rho_{TP}(L)$ has constant rank. Then the isomorphism classes of prequantizing Dirac-Jacobi structures on the fixed $U(1)$-bundle Q, up to extended B-field transformations, form a principal homogeneous space for*

$$H^1_L(P, U(1))/H^1_{\rho_{TP}(L)}(P, U(1)),$$

where $H^\bullet_{\rho_{TP}(L)}(P)$ denotes the foliated (i.e. tangential de Rham) cohomology of $\rho_{TP}(L)$.

Proof. We saw in the proof of Prop. 4.1 that, if (P, L) is prequantizable, the prequantizing Dirac-Jacobi structures on a fixed $U(1)$-bundle Q are given by $\bar{L}(Q, \sigma, \beta + \beta')$ where Q, σ, β are fixed and β' ranges over all d_L-closed sections of L^*. Consider $\rho^*_{TP}\gamma$ for a closed 1-form γ. Then $\bar{L}(Q, \sigma, \beta + \rho^*_{TP}\gamma) = L(Q, \sigma - \pi^*\gamma, \beta)$ by Lemma 4.1, and this is related to $\bar{L}(Q, \sigma, \beta)$ by an extended B-field transformation because of Lemma 4.2. To finish the argument, divide by the $U(1)$-exact forms. □

We will now give a characterization of the β's appearing in a prequantization triple.

Lemma 4.3. *Let (P, L) be a Dirac manifold for which $\rho_{TP}(L)$ is a regular foliation. Given a section β' of L^*, write $\beta' = \langle A' \oplus \alpha', \cdot \rangle|_L$. Then $d_L\beta' = \rho^*_{TP}\Omega'$ for some 2-form along $\mathcal{F}$ iff the vector field A' preserves the foliation $\mathcal{F}$. In this case, $\Omega' = d\alpha' - \mathcal{L}_{A'}\Omega_L$ where Ω_L is the presymplectic form on the leaves of $\mathcal{F}$ induced by L.*

Proof. For all sections $X_i \oplus \xi_i$ of L we have

$$d_L\beta'(X_1 \oplus \xi_1, X_2 \oplus \xi_2) = d\alpha'(X_1, X_2) + (\mathcal{L}_{A'}\xi_2)X_1 - (\mathcal{L}_{A'}\xi_1)X_2 + A' \cdot \langle \xi_1, X_2 \rangle.$$

Clearly $d_L\beta'$ is of the form $\rho^*_{TP}\Omega'$ iff $L \cap T^*P \subset \ker d_L\beta'$ (and in this case Ω' is clearly unique). Using the constant rank assumption to extend appropriately elements of $L \cap T^*P$ to some neighborhood in P, one sees that this is equivalent to $(\mathcal{L}_{A'}\xi)X = 0$ for all sections ξ of $L \cap T^*P = (\rho_{TP}(L))^\circ$ and vectors X in $\rho_{TP}(L)$, i.e. to A' preserving the foliation.

The formula for Ω' follows from a computation manipulating the above expression for $d_L\beta'$ by means of the Leibniz rule for Lie derivatives. □

We saw in the proof of Prop. 4.1 that, if (P, L) is prequantizable, the prequantizing Dirac-Jacobi structures on a fixed $U(1)$-bundle Q are given by $\bar{L}(Q, \sigma, \beta+\beta')$ where Q, σ, β are fixed and β' ranges over all d_L-closed sections of L^*. Since $\Upsilon = \rho_{TP}^* \Omega_L$, it follows from (4.7) that $d_L\beta$ is the pullback by ρ_{TP} of some 2-form along $\mathcal{F}$. So, by the above lemma, $\beta = \langle A \oplus \alpha, \cdot\rangle|_L$ for some vector field A preserving the regular foliation $\mathcal{F}$. Also, $\beta' = \langle A' \oplus \alpha', \cdot\rangle|_L$ where A' is a vector field preserving $\mathcal{F}$ and $d\alpha' - \mathcal{L}_{A'}\Omega_L = 0$, and conversely every d_L-closed β' arises this way (but choices of $A' \oplus \alpha'$ differing by sections of L will give rise to the same β').

Example 4.1. Let F be an integrable distribution on a manifold P (tangent to a regular foliation $\mathcal{F}$), and $L = F \oplus F^\circ$ the corresponding Dirac structure. By Lemma 4.3 (or by a direct computation) one sees that the d_L-closed sections β of L^* are sums of sections of TP/F preserving the foliation and closed 1-forms along F. By Prop. 4.1, the set of isomorphism classes of prequantizing Dirac-Jacobi structures maps surjectively to the set $\ker(\rho_{TP}^* \circ i_*)$ of topological types; the inverse image of a given type is a principal homogeneous space for

$$\{\text{Sections of } TP/F \text{ preserving the foliation}\} \times H_F^1(P, U(1)),$$

where the Lie algebroid cohomology $H_F^\bullet(P)$ is the tangential de Rham cohomology of $\mathcal{F}$ (and $\ker(\rho_{TP}^* \circ i_*)$ denotes the kernel in degree two).

5 The prequantization representation

In this section, assuming the prequantization condition (4.5) for the Dirac manifold (P, L) and denoting by $(Q, \bar{L})$ its prequantization as in Theorem 4.1, we construct a representation of the Lie algebra $C^\infty_{adm}(P)$. We will do so by first mapping this space of functions to a set of "equivalence classes of vector fields" on Q and then by letting these act on $C^\infty_{bas}(Q, \mathbb{C})_{P-loc}$, a sheaf over P. Here $C^\infty_{bas}(Q, \mathbb{C})$ denotes the complex basic[8] functions on $(Q, \bar{L})$, as defined in Section 3, which in the case at hand are exactly the functions whose differentials annihilate $\bar{L} \cap TQ$. The subscript "$_{P-loc}$" indicates that we consider functions which are defined on subsets $\pi^{-1}(U)$ of Q, where U ranges over the open subsets[9] of P. We will decompose this representation and make some comments on the faithfulness of the resulting subrepresentations.

Let $\tilde{L} = \{(X, 0) \oplus (\xi, g) : X \oplus \xi \in L, g \in \mathbb{R}\}$ be the Dirac-Jacobi structure associated to the Dirac structure L on P. It is immediate that $\tilde{L}$ is the pushforward of $\bar{L}$ via $\pi : Q \to P$, i.e. $\tilde{L} = \{(\pi_* Y, f) \oplus (\xi, g) : (Y, f) \oplus (\pi^*\xi, g) \in \bar{L}\}$.

[8]We use basic instead of admissible functions in order to obtain the same representation as in Section 6.

[9]We use *P-local* instead of *global* basic functions because the latter could be too small for certain injectivity statements. See Proposition 5.2 below and the remarks following it, as well as Section 9.

From this it follows that if functions f, g on P are admissible then their pullbacks $\pi^* f, \pi^* g$ are also admissible[10] and

$$\{\pi^* f, \pi^* g\} = \pi^*\{f, g\}. \tag{5.1}$$

Proposition 5.1. *The map*

$$\begin{array}{ccl} (C^\infty_{adm}(P), \{\cdot,\cdot\}) & \to & Der(C^\infty_{bas}(Q, \mathbb{C})_{P-loc}) \\ g & \mapsto & \{\pi^* g, \cdot\} \end{array} \tag{5.2}$$

determines a representation on $C^\infty_{bas}(Q, \mathbb{C})_{P-loc}$.

Proof. Recall that the expression $\{\pi^* g, \phi\}$ for $\phi \in C^\infty_{bas}(Q, \mathbb{C})_{P-loc}$ was defined in Section 3 as $-X_{\pi^* g}(\phi) - \phi \cdot 0 = -X_{\pi^* g}(\phi)$, for any choice $X_{\pi^* g}$ of hamiltonian vector field for $\pi^* g$. The proposition follows from the versions of the following statements for basic functions (see Lemma 3.1 and the remark following it). First: the map (5.2) is well-defined since the set of admissible functions on the Dirac-Jacobi manifold Q is closed under the bracket $\{\cdot,\cdot\}$. Second: it is a Lie algebra homomorphism because of Equation (5.1) and because the bracket of admissible functions on Q satisfies the Jacobi identity. Alternatively, for the second statement we can make use of the relation $[-X_{\pi^* f}, -X_{\pi^* g}] = -X_{\{\pi^* f, \pi^* g\}}$ (see Proposition 3.1). □

Since the Dirac-Jacobi structure on Q is invariant under the $U(1)$ action, the infinitesimal generator E is a derivation of the bracket. We can decompose $C^\infty_{bas}(Q, \mathbb{C})_{P-loc}$ into the eigenspaces H^n_{bas} of E corresponding to the eigenvalues $2\pi i n$, where n must be an integer, and similarly for H^n_{adm}. The derivation property implies that $\{H^n_{adm}, H^{n'}_{bas}\} \subseteq H^{n+n'}_{bas}$. The Lie algebra of admissible functions on P may be identified with the real-valued global functions in H^0_{adm}, which acts on each H^n_{bas} by the bracket, i.e. by the representation (5.2). In particular, the action on H^{-1}_{bas} is the usual prequantization action. The classical limit is obtained by letting $n \to -\infty$. Clearly all of the above applies if we restrict the representation (5.2) to $C^\infty_{adm}(Q, \mathbb{C})_{P-loc}$, i.e. if we replace "$H^n_{bas}$" by "$H^n_{adm}$" above.

Now we will comment on the faithfulness of the above representations. The map that assigns to an admissible function g on P the equivalence class of hamiltonian vector fields of $-\pi^* g$ depends on the choices of Ω and β in Equation (4.5) as well as on the prequantizing $U(1)$ bundle Q and connection σ. In general, there is no choice for which it is injective, as the following example shows. It follows that the prequantization representation on H^n_{bas} or H^n_{adm} (given by restricting suitably the representation (5.2)) is generally not faithful for any n.

[10] To show the smoothness of the hamiltonian vector fields of $\pi^* f$ and $\pi^* g$, we actually have to use the particular form of $\bar{L}$.

Example 5.1. Consider the Poisson manifold $(S^2 \times \mathbb{R}^+, \Lambda = t\Lambda_{S^2})$ where t is the coordinate on $\mathbb{R}^+$ and Λ_{S^2} is the product of the Poisson structure on S^2 corresponding to the standard symplectic form ω_{S^2} and the zero Poisson structure on $\mathbb{R}^+$. (This is isomorphic to the Lie-Poisson structure on $\mathfrak{su}(2)^* - \{0\}$.) We first claim that for all choices of Ω and A in (4.1) (which, as pointed out in Remark 4.1, is equivalent to (4.5)), the $\frac{\partial}{\partial t}$-component of the vector field A has the form $(ct^2 - t)\frac{\partial}{\partial t}$ for some real constant c.

Indeed, notice that $\Lambda + [-t\frac{\partial}{\partial t}, \Lambda] = 0$, so

$$\tilde{\Lambda} cp^*\omega_{S^2} = ct^2\Lambda_{S^2} = \Lambda + [A, \Lambda] \tag{5.3}$$

where $A = ct^2\frac{\partial}{\partial t} - t\frac{\partial}{\partial t}$. Now any vector field B satisfying $[B, \Lambda] = 0$ must map symplectic leaves to symplectic leaves, and since all leaves have different areas, B must have no $\frac{\partial}{\partial t}$-component. Hence any vector field satisfying Equation (5.3) has the same $\frac{\partial}{\partial t}$-component as A above. Now any closed 2-form Ω on $S^2 \times \mathbb{R}^+$ is of the form $cp^*\omega_{S^2} + d\beta$ for some 1-form β, where $p : S^2 \times \mathbb{R}^+ \to S^2$. Since $\tilde{\Lambda} d\beta = -[\tilde{\Lambda}\beta, \Lambda]$ and $-\tilde{\Lambda}\beta$ has no $\frac{\partial}{\partial t}$ component, our first claim is proved.

Now, for any choice of Q and σ, let g be a function on $S^2 \times \mathbb{R}^+$ such that $X_{\pi^*g} = X_g^H + (\langle dg, A\rangle - g)E$ vanishes. This means that g is a function of t only, satisfying $(ct^2 - t)g' = g$. For any real number c, there exist non-trivial functions satisfying these conditions, for example $g = \frac{ct-1}{t}$, therefore for all choices the homomorphism $g \mapsto -X_{\pi^*g}$ is not injective.

This example also shows that one can not simply omit the vector field A from the definition of prequantizability, since no choice of c makes A vanish here.

Even though the prequantization representation for functions acting on H^n_{adm} and H^n_{bas} is usually not faithful for any integer n, we still have the following result, which shows that hamiltonian vector fields do act faithfully.

Proposition 5.2. *For each integer $n \neq 0$, the map that assigns to an equivalence class of hamiltonian vector fields X_{π^*g} the corresponding operator on H^n_{adm} or H^n_{bas} is injective.*

Proof. Since $H^n_{adm} \subset H^n_{bas}$, it is enough to consider the H^n_{adm} case. Since the hamiltonian vector field of any function on Q is determined up to smooth sections of the singular distribution $F := \bar{L} \cap TQ = \{X^H + \langle \alpha, X\rangle E : X \in L \cap TP\}$, we have to show that, if a $U(1)$-invariant vector field Y on Q annihilates all functions in H^n_{adm}, then Y must be a section of F.

We start by characterizing the functions in H^n_{adm} on neighborhoods where a constant rank assumption holds:

Lemma 5.1. *Let U be an open set in P on which the rank of $L \cap TP$ is constant and $\bar{U} = \pi^{-1}(U)$. Then a function ϕ on $\bar{U}$ is admissible iff ϕ is constant along the leaves of F. Further $\cap_{\phi \in H^n_{adm}} \ker d\phi = F$.*

Proof. We have

$$\phi \text{ admissible } \Leftrightarrow (d\phi, \phi) \subset \rho_{T^*Q\times\mathbb{R}}(\bar{L}) \Leftrightarrow d\phi \subset \rho_{T^*Q}(\bar{L}), \tag{5.4}$$

where the first equivalence follows from the formula for $\bar{L}$, the remark following Definition 2.3 and the fact that $\dim(L \cap TP)$ is constant. For any Dirac-Jacobi structure one has $\rho_{T^*Q}(\bar{L}) = (\bar{L} \cap TQ)^\circ$, so the first statement follows.

Now consider the regular foliation of $\bar{U}$ with leaves equal to $U(1) \cdot \mathcal{F}$, where $\mathcal{F}$ ranges over the leaves of $F|_{\bar{U}}$[11]. Fix $p \in \bar{U}$ and choose a submanifold S through p which is transverse to the foliation $U(1) \cdot \mathcal{F}$. Given any covector $\xi \in T_p^*S$ we can find a function ϕ on S with differential ξ at p, and we extend ϕ to $\bar{U}$ so that it is constant on the leaves of F and equivariant with respect to the n-th power of the standard $U(1)$ action on $\mathbb{C}$. Then ϕ will lie in H^n_{adm} and $d_p\phi$ will be equal to ξ on T_pS, equal to $2\pi in$ on E_p, and will vanish on F_p. Since we can construct such a function $\phi \in H^n_{adm}$ for any choice of ξ, it is clear that a vector at p annihilated by all functions in H^n_{adm} must lie in F_p, so $\cap_{\phi\in H^n_{adm}} \ker d\phi \subset F$. The other inclusion is clear. $\square$

Now we make use of the fact that for any open subset V of P there exists a nonempty open subset $U \subset V$ on which $\dim(L \cap TP)$ is constant[12], and prove Proposition 5.2.

End of proof of Proposition 5.2. Suppose now the $U(1)$-invariant vector field Y on Q annihilates all functions in H^n_{adm} but is not a section of F. Then $Y \notin F$ at all points of some open set $\bar{U}$. By the remark above, we can assume that on $\bar{U}$ $\dim(L \cap TP)^H = \dim F$ is constant. By Lemma 5.1 on $\bar{U}$ the vector field Y must be contained in F, a contradiction. $\square$

If we modified the representation (5.2) to act on *global* admissible or basic functions, the injectivity statement of Proposition 5.2 could fail, as the following example shows.

Example 5.2. Let P be $(\mathbb{T}^2 \times \mathbb{R}, d\varepsilon)$, where $\varepsilon = x_3(dx_1 + x_3 dx_2)$ with (x_1, x_2) and x_3 standard coordinates on the torus and $\mathbb{R}$ respectively. This is a regular presymplectic manifold, so by Lemma 5.1 all basic functions on any prequantization Q are admissible. P is clearly prequantizable, and we can choose $\Omega = 0$ and $\beta = -\rho^*_{TP}\varepsilon$ in the prequantization condition (4.5). Therefore Q is the trivial $U(1)$ bundle over P, with trivial connection $\sigma = d\theta$ (where θ is the standard fiber coordinate). The distribution F on Q, as defined at the beginning of the proof of Proposition 5.2, is one dimensional, spanned by $2x_3\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - x_3^2\frac{\partial}{\partial \theta}$. The coefficients $2x_3$,

[11]The distribution $F = \bar{L} \cap TQ$ is clearly involutive; see Definition 3.2.

[12]Indeed, if q is a point of V where $\dim(L \cap TP)$ is minimal among all points of V, in a small neighborhood of q $\dim(L \cap TP)$ can not decrease, nor it can increase because $L \cap TP$ is an intersection of subbundles.

-1, and $-x_3^2$ are linearly independent over $\mathbb{Z}$ unless x_3 is a quadratic algebraic integer, so the closures of the leaves of F will be of the form $\mathbb{T}^2 \times \{x_3\} \times U(1)$ for a dense set of x_3's. Therefore $C^\infty_{adm}(Q, \mathbb{C}) = C^\infty_{bas}(Q, \mathbb{C})$ consists exactly of complex functions depending only on x_3. For similar reasons, the admissible functions on P are exactly those depending only on x_3. But the vector field $X_{\pi^* g}$ on Q associated to such a function g has no $\frac{\partial}{\partial x_3}$ component, so it acts trivially on $C^\infty_{adm}(Q, \mathbb{C})$.

Next we illustrate how the choices involved in the prequantization representation affect injectivity.

Example 5.3. Let $P = S^2 \times \mathbb{R} \times S^1$, with coordinate t on the $\mathbb{R}$-factor and s on the S^1-factor. Endow P with the Poisson structure Λ which is the product of the zero Poisson structure on $\mathbb{R} \times S^1$ and the inverse of an integral symplectic form ω_{S^2} on S^2. This Poisson manifold is prequantizable; in Equation (4.1) we can choose $\Omega = p^*\omega_{S^2}$ (where $p : P \to S^2$) and as A any vector field that preserves the Poisson structure. Each $g \in C^\infty(P)$ is prequantized by the action of the negative of its hamiltonian vector field $X_{\pi^* g} = (\tilde{\Lambda} dg)^H + (A(g) - g)E$. Therefore the kernel of the prequantization representation is given by functions of t and s satisfying $A(g) = g$. It is clear that if A is tangent to the symplectic leaves the representation will be faithful. If A is not tangent to the symplectic leaves, then $A(g) = g$ is an honest first order differential equation. However, even in this case the representation might be faithful: it is faithful if we choose $A = \frac{\partial}{\partial t}$, but not if $A = \frac{\partial}{\partial s}$.

Remark 5.1. Let (P, Λ) be a Poisson manifold such that its symplectic foliation $\mathcal{F}$ has constant rank, and assume that (P, Λ) is prequantizable (i.e. (4.1), or equivalently (4.5), is satisfied). It follows from the discussion following Lemma 4.3 that, after we fix a prequantizing $U(1)$-bundle Q, the prequantizing Dirac-Jacobi structures on Q are given by $\bar{L}(Q, \sigma, A)$ where σ is fixed and A is unique up to vector fields A' preserving $\mathcal{F}$ such that $\mathcal{L}_{A'}\Omega_L = 0$, i.e. up to vector fields whose flows are symplectomorphisms between the symplectic leaves. If the topology and geometry of the symplectic leaves of P "varies" sufficiently from one leaf to another (as in Example 5.1 above), then the projection of the A's as above to $TP/T\mathcal{F}$ will all coincide. Therefore the kernels of the prequantization representations (5.2), which associate to $g \in C^\infty(P)$ the negative of the hamiltonian vector field $X_{\pi^* g} = (\tilde{\Lambda} dg)^H + (A(g) - g)E$, will coincide for all representations arising from prequantizing Dirac-Jacobi structures over Q.

We end this section with two remarks linked to Kostant's work [22].

Remark 5.2. Kostant ([22], Theorem 0.1) has observed that the prequantization of a symplectic manifold can be realized by the Poisson bracket of a symplectic manifold two dimensions higher, i.e. that prequantization is "classical mechanics two dimensions higher". In the general context of Dirac manifolds we have seen in

(5.2) that prequantization is given by a Jacobi bracket[13]; we will now show that Kostant's remark applies in this context too.

Let (P, L) be a prequantizable Dirac manifold, $(Q, \bar{L})$ its prequantization and $(Q \times \mathbb{R}, \bar{\bar{L}})$ the "Diracization" of $(Q, \bar{L})$. To simplify the notation, we will denote pullbacks of functions (to Q or $Q \times \mathbb{R}$) under the obvious projections by the same symbol. Using the homomorphism (3.5) we can re-write the representation (5.2) of $C^\infty_{adm}(P)$ on $C^\infty_{adm}(Q, \mathbb{C})_{P-loc}$ (or $C^\infty_{bas}(Q, \mathbb{C})_{P-loc}$) as

$$g \mapsto e^{-t}\{e^t g, e^t \cdot\}_{Q\times\mathbb{R}} = \{e^t g, \cdot\}_{Q\times\mathbb{R}},$$

i.e. g acts by the Poisson bracket on $Q \times \mathbb{R}$.

Remark 5.3. Kostant [22] also shows that a prequantizable symplectic manifold (P, Ω) can be recovered by reduction from the symplectization $(Q \times \mathbb{R}, d(e^t\sigma))$ of its prequantization (Q, σ). More precisely, the inverse of the natural $U(1)$ action on $Q \times \mathbb{R}$ is hamiltonian with momentum map e^t, and symplectic reduction at $t = 0$ delivers (P, Ω). We will show now how to extend this construction[14] to prequantizable Dirac manifolds.

Let (P, L), $(Q, \bar{L})$ and $(Q \times \mathbb{R}, \bar{\bar{L}})$ be as in Remark 5.2. Since $-E \oplus de^t \in \bar{\bar{L}}$ we see that e^t is a "momentum map" for the inverse $U(1)$ action on $Q \times \mathbb{R}$, and by Dirac reduction [2] at the regular value 1 we obtain L: indeed, the pullback of $\bar{\bar{L}}$ to $Q \times \{0\}$ is easily seen to be $\{(X^H + (\langle X \oplus \xi, \beta\rangle - g)E) \oplus \pi^*\xi : X \oplus \xi \in L\}$, and its pushforward via $\pi : Q \to P$ is exactly L.

6 The line bundle approach

In this section we will prequantize a Dirac manifold P by letting its admissible functions act on sections of a hermitian line bundle K over P. This approach was first taken by Kostant for symplectic manifolds and was extended by Huebschmann [16] and Vaisman [31] to Poisson manifolds. The construction of this section generalizes Vaisman's and turns out to be equivalent to the one we described in Sections 4 and 5.

Definition 6.1. [11] Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid over the manifold M and K a real vector bundle over M. An A-connection on the vector bundle $K \to M$ is a map $D : \Gamma(A) \times \Gamma(K) \to \Gamma(K)$ which is $C^\infty(M)$-linear in the $\Gamma(A)$ component and satisfies

$$D_e(h \cdot s) = h \cdot D_e s + \rho e(h) \cdot s,$$

[13] The bracket on functions on the prequantization $(Q, \bar{L})$ of a Dirac manifold makes $C^\infty_{adm}(Q)$ into a Jacobi algebra. See Section 5 of [34], which applies because the constant functions are admissible for the Dirac-Jacobi structure $\bar{L}$.

[14] Kostant calls the procedure of taking the symplectization of the prequantization "symplectic induction"; the term seems to be used here in a different sense from that in [19].

for all $e \in \Gamma(A)$, $s \in \Gamma(K)$ and $h \in C^\infty(M)$. The curvature of the A-connection is the map $\Lambda^2 A^* \to End(K)$ given by

$$R_D(e_1, e_2)s = D_{e_1} D_{e_2} s - D_{e_2} D_{e_1} s - D_{[e_1,e_2]} s.$$

If K is a complex vector bundle, we define an A-connection on K as above, but with $C^\infty(M)$ extended to the complex-valued smooth functions.

Remark 6.1. When $A = TM$ the definitions above specialize to the usual notions of covariant derivative and curvature. Moreover, given an ordinary connection ∇ on K, we can pull it back to a A-connection by setting $D_e = \nabla_{\rho e}$.

With this definition we can easily adapt Vaisman's construction [31] [32], extending it from the case where $L = T^*P$ is the Lie algebroid of a Poisson manifold to the case where L is a Dirac structure. We will act on locally defined, basic sections.

Lemma 6.1. *Let (P, L) be a Dirac manifold and K a hermitian line bundle over P endowed with an L-connection D. Then $R_D = 2\pi i \Upsilon$, where $\Upsilon = \langle \cdot, \cdot \rangle_-|_L$, iff the correspondence*

$$\hat{g}s = -(D_{X_g \oplus dg} s + 2\pi i g s)$$

defines a Lie algebra representation of $C^\infty_{adm}(P)$ on $\{s \in \Gamma(K)_{loc} : D_{Y\oplus 0} s = 0$ for $Y \in L \cap TP\}$, where X_g is any choice of hamiltonian vector field for g.

Proof. If $\hat{g}$ and s are as above, then clearly $\hat{g}s$ is a well-defined section of K. We will now show that $\hat{g}s \in \{s \in \Gamma(K)_{loc} : D_{Y\oplus 0} s = 0$ for $Y \in L \cap TP\}$, so that the above "representation" is well-defined. The case where $Y \in L \cap T_pP$ can be locally extended to a smooth section of $L \cap TP$ is easy, whereas the techniques (see Section 2.5 of [11]) needed for general case are much more involved.

The section $X_g \oplus dg$ of L induces a flow ϕ_t on P (which is just the flow of the vector field X_g) and a one-parameter family of bundle automorphisms Φ_t on $TP \oplus T^*P$ which (see Section 2.4 in [6]) preserves L, and which takes L-paths to L-paths[15]. Further, Φ_t acts on the sections s of the line bundle K too, as follows: $(\Phi_t^* s)_p$ is the parallel translation of $s_{\phi_t(p)}$ along the L-path $\Phi_\bullet(X_g \oplus dg)_p = (X_g \oplus dg)_{\phi_\bullet(p)}$. Now $(D_{(X_g \oplus dg)} s)_p = \frac{\partial}{\partial t}|_0 (\Phi_t^* s)_p$, and $(D_{(Y\oplus 0)} D_{X_g\oplus dg} s)_p = \frac{\partial}{\partial t}|_0 (D_{(Y\oplus 0)} \Phi_t^* s)_p$. For every t, since ϕ_t preserves $L \cap TP$, we have

$$0 = (D_{(\phi_{t*} Y \oplus 0)} s)_{\phi_t(p)} = \frac{\partial}{\partial \epsilon}\Big|_0 \mathbin{\|}_0^\epsilon s_{\phi_t(\gamma(\epsilon))} \tag{6.1}$$

where Γ is an L-path starting at $(Y \oplus 0) \in L_p$, γ is its base path, and $\|_0^\epsilon$ is parallel translation along the L-path $\Phi_t(\Gamma(\bullet))$. (This notation denotes the path

[15] For any algebroid A over P an A-path is a defined as a path $\Gamma(t)$ in A such that the anchor maps $\Gamma(t)$ to the velocity of the base path $\pi(\Gamma(t))$.

$\epsilon \mapsto \Phi_t(\Gamma(\epsilon))$.) Now we parallel translate the element (6.1) of $K_{\phi_t(p)}$ to p using the L-path $\Phi_\bullet(X_g \oplus dg)_p$, and compare the result with

$$(6.2) \qquad (D_{(Y\oplus 0)}\Phi_t^* s)_p = \frac{\partial}{\partial \epsilon}\Big|_0 \backslash\!\backslash_0^\epsilon \backslash\!\backslash_0^t s_{\phi_t(\gamma(\epsilon))},$$

where the parallel translation is taken first along $\Phi_\bullet(X_g \oplus dg)_{\gamma(\epsilon)}$ and then along $\Gamma(\bullet)$.

The difference between (6.2) and the parallel translation to p of (6.1) lies only in the order in which the parallel translations are taken. Now applying $\frac{\partial}{\partial t}|_0$ to this difference (and recalling that $\Phi_t(X_g \oplus dg)_p = (X_g \oplus dg)_{\phi_t(p)}$) we obtain the evaluation at p of

$$D_{\Phi_t\Gamma(\epsilon)}D_{(X_g\oplus dg)}s - D_{(X_g\oplus dg)}D_{\Phi_t\Gamma(\epsilon)}s,$$

which by the definition of curvature is just

$$(D_{[\Phi_t\Gamma(\epsilon),X_g\oplus dg]}s)_p + \Upsilon(Y \oplus 0, (X_g \oplus dg)_p)s.$$

The second term vanishes because $Y \in L \cap T_pP$, and using the fact that Φ_t is the flow generated by X_g one sees that the Courant bracket in the first term is also zero. Altogether we have proven that $(D_{(Y\oplus 0)}D_{X_g\oplus dg}s)_p$ vanishes, and from this is follows easily that the "representation" in the statement of the lemma is well defined.

Since

$$[\hat{f}, \hat{g}] = D_{X_f\oplus df}D_{X_g\oplus dg} - D_{X_g\oplus dg}D_{X_f\oplus df} + 2\pi i(X_f(g) - X_g(f)),$$

using $-[X_f \oplus df, X_g \oplus dg] = X_{\{f,g\}} \oplus d\{f, g\}$ ([6], Prop. 2.5.3) we see that the condition on R_D holds iff $[\hat{f}, \hat{g}] = \widehat{\{f, g\}}$. □

Now assume that the prequantization condition (4.5) is satisfied, i.e. that there exists a closed integral 2-form Ω and a Lie algebroid 1-cochain β for such that

$$\rho_{TP}^*\Omega = \Upsilon + d_L\beta.$$

Then we can construct an L-connection D satisfying the property of the previous lemma:

Lemma 6.2. *Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid over the manifold M, Ω a closed integral 2-form on M, and ∇ a connection (in the usual sense) on a hermitian line bundle K with curvature $R_\nabla = 2\pi i\Omega$. If $\rho^*\Omega = \Upsilon + d_L\beta$ for a 2-cocycle Υ and a 1-cochain β on A, then the A-connection D defined by*

$$D_e = \nabla_{\rho e} - 2\pi i\langle e, \beta\rangle$$

has curvature $R_D = 2\pi i\Upsilon$.

Proof. An easy computation shows

$$R_D(e_1, e_2) = R_\nabla(\rho e_1, \rho e_2) + 2\pi i(-\rho e_1\langle e_2, \beta\rangle + \rho e_2\langle e_1, \beta\rangle + \langle [e_1, e_2], \beta\rangle),$$

which using $\rho^*\Omega = \Upsilon + d_L\beta$ reduces to $2\pi i\Upsilon(e_1, e_2)$. □

Altogether we obtain that

$$\hat{g} = -[\nabla_{X_g} - 2\pi i(\langle X_g \oplus dg, \beta\rangle - g)]$$

determines a representation of $C^\infty_{adm}(P)$ on $\{s \in \Gamma(K)_{loc} : \nabla_Y s - 2\pi i\langle Y \oplus 0, \beta\rangle s = 0 \text{ for } Y \in L \cap TP\}$. Notice that, when P is symplectic, we recover Kostant's prequantization mentioned in the introduction. Now let $Q \to P$ be the $U(1)$-bundle corresponding to K, with the connection form σ corresponding to ∇. If $\bar{s}$ is the $U(1)$-antiequivariant complex valued function on Q corresponding to the section s of K, then $X^H(\bar{s})$ corresponds to $\nabla_X s$ and $E(\bar{s})$ to $-2\pi i s$. Here $X \in TP$, $X^H \in \ker\sigma$ its horizontal lift to Q, and E is the infinitesimal generator of the $U(1)$ action on Q (so $\sigma(E) = 1$). Translating the above representation to the $U(1)$-bundle picture, we see that $\hat{g} = -[X_g^H + (\langle X_g \oplus dg, \beta\rangle - g)E]$ defines a representation of $C^\infty_{adm}(P)$ on

$$\{\bar{s} \in C^\infty(Q, \mathbb{C})_{P-loc} : \bar{s} \text{ is } U(1)\text{-antiequivariant and}$$
$$(Y^H + \langle Y \oplus 0, \beta\rangle E)\bar{s} = 0 \text{ for } Y \in L \cap TP\},$$

which is nothing else than H^{-1}_{bas} as defined in Section 5. Since $X_g^H + (\langle X_g \oplus dg, \beta\rangle - g)E$ is the hamiltonian vector field of $\pi^* g$ (with respect to the Dirac-Jacobi structure $\bar{L}$ on Q as in Theorem 4.1), we see that this is exactly our prequantization representation given by Equation (5.2) restricted to H^{-1}_{bas}.

6.1 Dependence of the prequantization on choices: the line bundle point of view

In Subsection 4.2 we gave a classification the Dirac-Jacobi structures induced on the prequantization of a given Dirac manifold, and hence also a classification of the corresponding prequantization representations. Now we will see that the line bundle point of view allows for an equivalent but clearer classification.

Recall that, given a Dirac manifold satisfying the prequantization condition (4.5), we associated to it a hermitian line bundle K and a representation as in Lemma 6.1, where the L-connection D is given as in Lemma 6.2

Proposition 6.1. *Fix a line bundle K over P with $(\rho^*_{TP} \circ i_*)c_1(K) = [\Upsilon]$. Then all the hermitian L-connections of K with curvature $2\pi i\,\Upsilon$ are given by the L-connections constructed in Lemma 6.2. Therefore there is a surjective map from the set of isomorphism classes of prequantization representations of (P, L) to the space $(\rho^*_{TP} \circ i_*)^{-1}[\Upsilon]$ of topological types; the set with a given type is a principal homogeneous space for $H^1_L(P, U(1))$.*

Proof. Exactly as in the case of ordinary connections one shows that the difference of two hermitian L-connections on K is a section of L^*, whose d_L-derivative is the difference of the curvatures. Fix a choice of L-connection D as in Lemma 6.2, say given by $D_{(X\oplus\xi)} = \nabla_X - 2\pi i\langle X \oplus \xi, \beta\rangle$. Another L-connection D' with curvature $2\pi i\,\Upsilon$ is given by $D'_{(X\oplus\xi)} = \nabla_X - 2\pi i\langle X \oplus \xi, \beta + \beta'\rangle$ for some d_L-closed section β' of L^*, hence it arises as in Lemma 6.2. This shows the first claim of the proposition. Since, as we have just seen, the L-connections with given curvature differ by d_L-closed sections of L^* and since $U(1)$-exact sections of L^* give rise to gauge equivalences of hermitian line bundles with connections, the second claim follows as well. □

Using Lemma 4.1 it is easy to see that choices of (σ, β) giving rise to the same L-connection (as in Lemma 6.2) also give rise to the same Dirac-Jacobi structure $\bar{L}$, in accord with the results of Section 4.2. Given this, it is natural to try to express the Dirac-Jacobi structure $\bar{L}$ intrinsically in terms of the L-connection to which it corresponds; this is subject of work in progress.

7 Prequantization of Poisson and Dirac structures associated to contact manifolds

We have already mentioned in Remark 5.3 the symplectization construction, which associates to a manifold M with contact form σ the manifold $M \times \mathbb{R}$ with symplectic form $d(e^t\sigma)$. The construction may also be expressed purely in terms of the cooriented contact distribution C annihilated by σ. In fact, given any contact distribution, its nonzero annihilator C° is a (locally closed) symplectic submanifold of T^*M. When C is cooriented, we can select the positive component C°_+. Either of these symplectic manifolds is sometimes known as the symplectization of (M, C). It is a bundle over M for which a trivialization (which exists in the cooriented case) corresponds to the choice of a contact form σ and gives a symplectomorphism between this "intrinsic" symplectization and $(M \times \mathbb{R}, d(e^t\sigma))$. The contact structure on M may be recovered from its symplectization along with the conformally symplectic $\mathbb{R}$ action generated by $\partial/\partial t$.

One may partially compactify C°_+ (we stick to the cooriented case for simplicity) at either end to get a manifold with boundary diffeomorphic to M. The first, and simplest way, is simply to take its closure $C^\circ_{0,+}$ in the cotangent bundle by adjoining the zero section. The result is a presymplectic manifold with boundary, diffeomorphic to $M \times [0, \infty)$ with the exact 2-form $d(s\sigma) = ds \wedge \sigma + s d\sigma$, where s is the exponential of the coordinate t in $\mathbb{R}$. For positive s, this is symplectic; the characteristic distribution of $C^\circ_{0,+}$ lives along the boundary $M \times \{0\}$, where it may be identified with the contact distribution C. This is highly nonintegrable even though $d(s\sigma)$ is closed, so we have another example of the phenomenon alluded to in the discussion after Definition 2.2.

We also note that the basic functions on $C^\circ_{0,+}$ are just those which are constant on $M \times \{0\}$. One can prove that all of these functions are admissible as well, even though the characteristic distribution is singular. It would be interesting to characterize the Dirac structures for which these two classes of functions coincide.

To compactify the other end of C°_+, we begin by identifying C°_+ with the positive part of its dual $(TM/C)_+$, using the "inversion" map j which takes $\phi \in C^\circ_+$ to the unique element $X \in (TM/C)_+$ for which $\phi(X) = 1$. We then form the union $C^\circ_{+,\infty}$ of C°_+ with the zero section in TM/C and give it the topology and differentiable structure induced via j from the closure of $(TM/C)_+$. It was discovered by LeBrun [23] that the Poisson structure on C°_+ corresponding to its symplectic structure extends smoothly to $C^\circ_{+,\infty}$. We call $C^\circ_{+,\infty}$ with this Poisson structure the **LeBrun-Poisson manifold** corresponding to the contact manifold (M, C).

To analyze the LeBrun-Poisson structure more closely, we introduce the inverted coordinate $r = 1/s$, which takes values in $[0, \infty)$ on $C^\circ_{+,\infty}$. In suitable local coordinates on M, the contact form σ may be written as $du + \sum p_i dq^i$. On the symplectization, we have the form $d(r^{-1}(du + \sum p_i dq^i))$. The corresponding Poisson structure turns out to be

$$\Lambda = r\left[\left(r\frac{\partial}{\partial r} + \sum p_i \frac{\partial}{\partial p_i}\right) \wedge \frac{\partial}{\partial u} + \sum \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}\right].$$

From this formula we see not only that Λ is smooth at $r = 0$ but also that its linearization

$$r \sum \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$$

at the origin (which is a "typical" point, since M looks the same everywhere) encodes the contact subspace in terms of the symplectic leaves in the tangent Poisson structure.

We may take the union of the two compactifications above to get a manifold $C^\circ_{0,+,\infty}$ diffeomorphic to M times a closed interval. It is presymplectic at the 0 end and Poisson at the ∞ end, so it can be treated globally only as a Dirac manifold. In what follows, we will simply denote this Dirac manifold as (P, L).

To prequantize (P, L), we first notice that its Dirac structure is "exact" in the sense that the cohomology class $[\Upsilon]$ occurring in the condition (4.4) is zero. In fact, on the presymplectic end, L is isomorphic to TP, and Υ is identified with the form $d(s\sigma)$, so we can take the cochain β to be the section of L^* which is identified with $-s\sigma$. To pass to the other end, we compute the projection of this section of L^* into TP and find that it is just the Euler vector field $A = s\frac{\partial}{\partial s}$. In terms of the inverse coordinate r, $A = -r\frac{\partial}{\partial r}$. (The reader may check that the Poisson differential of this vector field is $-\Lambda$, either by direct computation or using the degree 1 homogeneity of Λ with respect to r.) On the Poisson end, L^* is isomorphic to TP, so $-r\frac{\partial}{\partial r}$ defines a smooth continuation of β to all of P.

Continuing with the prequantization, we can take the 2-form Ω to be zero and the $U(1)$-bundle Q to be the product $P \times U(1)$ with the trivial connection $d\theta$, where θ is the (2π-periodic) coordinate on $U(1)$. On the presymplectic end, the Dirac-Jacobi structure is defined by the 1-form $\sigma = s\sigma + \theta$, which is a contact form when $s \neq 0$.

On the Poisson end, we get the Jacobi structure $(\Lambda^H + E \wedge A^H, E)$ which in coordinates becomes

$$(7.1) \qquad \left(r \left[\left(r\frac{\partial}{\partial r} + \sum p_i \frac{\partial}{\partial p_i} \right) \wedge \frac{\partial}{\partial u} + \sum \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} - \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial r} \right], \frac{\partial}{\partial \theta} \right).$$

8 Prequantization by circle actions with fixed points

Inspired by a construction of Engliš [10] in the complex setting, we modify the prequantization in the previous section by "pinching" the boundary component $M \times U(1)$ at the Poisson end and replacing it by a copy of M. To do this, we identify $U(1)$ with the unit circle in the plane $\mathbb{R}^2$ with coordinates (x, y). In addition, we make a choice of contact form on M so that P is identified with $M \times [0, \infty]$, with the coordinate r on the second factor. Next we choose a smooth nonnegative real valued function $f : [0, \infty] \to \mathbb{R}$ such that, for some $\epsilon > 0$, $f(r) = r$ on $[0, \epsilon]$ and $f(r)$ is constant on $[2\epsilon, \infty]$. Let Q' be the submanifold of $P \times \mathbb{R}^2$ defined by the equation $x^2 + y^2 = f(r)$.

Radial projection in the (x, y) plane determines a map $F : Q \to Q'$ which is smooth, and in fact a diffeomorphism, where $r > 0$. The boundary $M \times U(1)$ of Q is projected smoothly to $M \times (0, 0)$ in Q', but F itself is not smooth along the boundary. We may still use F to transport the Jacobi structure on Q to the part of Q' where $r > 0$. For small r, we have $x = \sqrt{r}\cos\theta$ and $y = \sqrt{r}\sin\theta$, so $r = x^2+y^2$, $r\frac{\partial}{\partial r} = \frac{1}{2}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$, and $\frac{\partial}{\partial \theta} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Using these substitutions to write the Jacobi structure (7.1) with polar coordinates (r, θ) replaced by rectangular coordinates (x, y), we see immediately that the structure extends smoothly to a Jacobi structure on the Poisson end of Q' and to a Dirac-Jacobi structure on all of Q', and that the projection $Q' \to P$, like $Q \to P$ pushes the Dirac-Jacobi structure on Q' to the Dirac structure on P. (Thus, the projection is a "forward Dirac-Jacobi map"; see the beginning of Section 3.) The essential new feature of Q' is that the vector field $E' = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ of the Jacobi structure on Q' vanishes along the locus $x = y = 0$ where the projection is singular.

The vanishing of E' at some points means that the Jacobi structure on Q' does not arise from a contact form, even on the Poisson end, where $r < \infty$. However, it turns out that we can turn it into a contact structure by making a conformal change, i.e. by multiplying the bivector by $1/f$ and replacing E' by $E'/f + X_{1/f}$. The resulting Jacobi structure still extends smoothly over Q', and now comes from a contact structure over the Poisson end; the price we pay is that the projection to P is now a conformal Jacobi map rather than a Jacobi map.

Remark 8.1. Looking back at the construction above, we see that we have embedded any given contact manifold M as a codimension 2 submanifold in another contact manifold. Our construction depended only on the choice of a contact form. On the other hand, Eliashberg and Polterovich [9] construct a similar embedding in a canonical way, without the choice of a contact form. It is not hard to show that the choice of a contact form defines a canonical isomorphism between our contact manifold and theirs.

Example 8.1. Let M be the unit sphere in $\mathbb{C}^n$, with the contact structure induced from the Cauchy-Riemann structure on the boundary of the disc D^{2n}. It turns out that a neighborhood U of M in the disc can be mapped diffeomorphically to a neighborhood V of M at the Poisson end in its LeBrun-Poisson manifold P so that the symplectic structure on the interior of V pulls back to the symplectic structure on U coming from the Kähler structure on the open disc, viewed as complex hyperbolic space. If we now pinch the end of the prequantization Q, as above, the part of the contact manifold Q' lying over V can be glued to the usual prequantization of the open disc so as to obtain a compact contact manifold Q'' projecting by a "conformal Jacobi map" to the closed disc. The fibres of the map are the orbits of a $U(1)$-action which is principal over the open disc. In fact, Q'' is just the unit sphere in $\mathbb{C}^{n+1}$ with *its* usual contact structure. All this is the symplectic analogue of the complex construction by Engliš [10], who enlarges a bounded pseudoconvex domain D in $\mathbb{C}^n$ to one in $\mathbb{C}^{n+1}$ with a $U(1)$ action on its boundary which degenerates just over the boundary of D.

The "moral" of the story in this section is that, in prequantizing a Poisson manifold P whose Poisson structure degenerates along a submanifold, one might want to allow the prequantization bundle to be a Jacobi manifold Q whose vector field E generates a $U(1)$ action having fixed points and for which the quotient projection $Q \to P$ is a Jacobi map.

9 Final remarks and questions

We conclude with some suggestions for further research along the lines initiated in this paper.

9.1 Cohomological prequantization

Cohomological methods have already been used in geometric quantization of symplectic manifolds: rather than the space of global polarized sections, which may be too small or may have other undesirable properties, one looks at the higher cohomology of the sheaf of local polarized sections. (An early reference on this approach is [26].) When we deal with Dirac (e.g. presymplectic) manifolds, it may already be interesting to introduce cohomology at the prequantization stage. There are two ways in which this might be done.

The first approach, paralleling that which is done with polarizations, is to replace the Lie algebra of global admissible functions on a Dirac manifold P by the cohomology of the sheaf of Lie algebras of local admissible functions. Similarly, one would replace the sheaf of P-local functions on Q by its cohomology. The first sheaf cohomology should then act on the second.

The other approach, used by Cattaneo and Felder [4] for the deformation quantization of coisotropic submanifolds of Poisson manifolds, would apply to Dirac manifolds P whose characteristic distribution is regular. Here, one introduces the "longitudinal de Rham complex" of differential forms along the leaves of the characteristic foliation on P. The zeroth cohomology of this foliation is just the admissible functions, so it is natural to consider the full cohomology, or even the complex itself. It turns out that, if one chooses a transverse distribution to the characteristic distribution, the transverse Poisson structure induces the structure of an L_∞ algebra on the longitudinal de Rham complex. Carrying out a similar construction on a prequantization Q should result in an L_∞ representation of this algebra.

9.2 Noncommutative prequantization

If the characteristic distribution of a Dirac structure P is regular, we may consider the groupoid algebra associated to the characteristic foliation as a substitute for the admissible functions. By adding some extra structure, as in [1][29][36], we can make this groupoid algebra into a noncommutative Poisson algebra. This means that the Poisson bracket is not a Lie algebra structure, but rather a class with degree 2 and square 0 in the Hochschild cohomology of the groupoid algebra. It should be interesting to define a notion of representation for an algebra with such a cohomology class, and to construct such representations from prequantization spaces. Such a construction should be related to the algebraic quantization of Dirac manifolds introduced in [30].

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