

# Poisson-Dirac branes in Poisson-Sigma models

by

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## Abstract

We analyse the general boundary conditions (branes) consistent with the Poisson-sigma model and study the structure of the phase space of the model defined on the strip with these boundary conditions. Finally, we discuss the perturbative quantization of the model on the disc with a Poisson-Dirac brane and relate it to Kontsevich's formula for the deformation quantization of the Dirac bracket induced on the brane.

## 1 Introduction

Poisson-Sigma models ([10], [13]) are topological field theories based on a bundle map from the tangent bundle of a surface  $\Sigma$  to the cotangent bundle of a Poisson manifold  $M$ . A number of two-dimensional gauge theories such as pure gravity, WZW  $G/G$  (locally) and Yang-Mills (up to addition of a non-topological term containing the Casimir of the Poisson structure) are particular cases of Poisson-Sigma models.

The mathematical interest of the Poisson-Sigma model resides in the fact that it naturally encodes the Poisson Geometry of the target and allows to unravel it by means of techniques from Classical and Quantum Field Theory. This feature has allowed to shed considerable light on previous mathematical results, such as the interpretation of Kontsevich's star product in terms of Feynman diagrams by Cattaneo and Felder in [4]. Moreover, the same authors showed in [5] the connection between the structure of the phase space of the model and the symplectic groupoid (when the latter exists) integrating the target Poisson manifold. Their ideas inspired to a large extent the crucial work [7] of Crainic and Fernandes on the integrability of Lie algebroids.

The exposition below (based on [3]) is concerned with the boundary conditions (BC) allowed in the Poisson-Sigma model. We generalize the results of Cattaneo and Felder ([6]) and show that the base map can be consistently restricted at the

boundary to almost arbitrary submanifolds (branes) of the target. A thorough analysis of the model with general BC is provided in the Lagrangian as well as in the Hamiltonian formalism. It turns out that the structure of the phase space is related to the Poisson bracket on the brane obtained by reduction rather than to the original Poisson bracket on  $M$ .

In ref. [6] the claim was that only coisotropic branes were admissible. In those cases the reduced Poisson bracket on the brane is just the original Poisson bracket restricted to the subset of gauge invariant functions.

Our general setup allows in particular the other extreme situation, i.e. no gauge transformations at all. We will use the term *Poisson-Dirac* branes following [8]. This case leads to a nontrivial reduction of the Poisson bracket of  $M$  to the brane (Dirac bracket).

In these pages we want to stress the interest of this class of branes for the quantization of the model. We devote the last section to the discussion on the perturbative quantization on the disc with a Poisson-Dirac brane and conjecture that, after a suitable choice of the unperturbed part of the action, it yields the Kontsevich's star product for the Dirac bracket induced on the brane.

## 2 Poisson-Sigma models

The Poisson-Sigma model is a two-dimensional topological Sigma model defined on a surface  $\Sigma$  and with a finite dimensional Poisson manifold  $(M, \Gamma)$  as target.

The fields of the model are given by a bundle map  $(X, \psi) : T\Sigma \rightarrow T^*M$  consisting of a base map  $X : \Sigma \rightarrow M$  and a 1-form  $\psi$  on  $\Sigma$  with values in the pullback by  $X$  of the cotangent bundle of  $M$ . If  $X^i$  are local coordinates in  $M$ ,  $\sigma^\mu$ ,  $\mu = 1, 2$  local coordinates in  $\Sigma$ ,  $\Gamma^{ij}$  the components of the Poisson structure in these coordinates and  $\psi_i = \psi_{i\mu} d\sigma^\mu$ , the action reads

$$(2.1) \quad S_{P\sigma}(X, \psi) = \int_{\Sigma} dX^i \wedge \psi_i - \frac{1}{2} \Gamma^{ij}(X) \psi_i \wedge \psi_j$$

It is straightforward to work out the equations of motion in the bulk:

$$(2.2) \quad \begin{aligned} dX^i + \Gamma^{ij}(X) \psi_j &= 0 \\ d\psi_i + \frac{1}{2} \partial_i \Gamma^{jk}(X) \psi_j \wedge \psi_k &= 0 \end{aligned}$$

One can show ([1]) that for solutions of (2.2) the image of  $X$  lies within one of the symplectic leaves of the foliation of  $M$ .

Under the infinitesimal transformation

$$(2.3) \quad \begin{aligned} \delta_\epsilon X^i &= \Gamma^{ji}(X) \epsilon_j \\ \delta_\epsilon \psi_i &= d\epsilon_i + \partial_i \Gamma^{jk}(X) \psi_j \epsilon_k \end{aligned}$$

where  $\epsilon = \epsilon_i dX^i$  is a section of  $X^*(T^*M)$ , the action (2.1) transforms by a boundary term

$$(2.4) \quad \delta_\epsilon S_{P\sigma} = - \int_\Sigma d(dX^i \epsilon_i).$$

### 3 Boundary conditions

In surfaces with boundary (in this section we restrict ourselves to a boundary with one connected component) a new term appears in the variation of the action under a change of  $X$  when performing the integration by parts:

$$(3.1) \quad \delta_X S = \int_{\partial\Sigma} \delta X^i \psi_i - \int_\Sigma \delta X^i (d\psi_i + \frac{1}{2} \partial_i \Gamma^{jk}(X) \psi_j \wedge \psi_k)$$

Let us take the field  $X|_{\partial\Sigma} : \partial\Sigma \rightarrow C$  for an arbitrary (for the moment) closed embedded submanifold  $C$  of  $M$  (brane, in a more stringy language). Then  $\delta X \in T_X C$  at every point of the boundary and, if the surface term is to vanish, the contraction of  $\psi = \psi_i dX^i$  with vectors tangent to the boundary (that we will denote by  $\psi_t = \psi_{it} dX^i$ ) must belong to  $N_X^* C$ . Here  $N^* C$  is the conormal bundle of  $C$ , i.e. the subbundle of  $T_C^* M$  whose fibers are the covectors that kill all vectors tangent to  $C$ .

On the other hand, by continuity, the equations of motion in the bulk must be satisfied also at the boundary. In particular,  $\partial_t X = \Gamma_X^\sharp \psi_t$ , where by  $\partial_t$  we denote the derivative along the vector on  $\Sigma$  tangent to the boundary and  $\Gamma^\sharp : T^* M \rightarrow TM$  is given by the contraction with the first factor of  $\Gamma$ . As  $\partial_t X$  belongs to  $T_X C$  it follows that  $\psi_t \in \Gamma_X^{\sharp-1}(T_X C)$ .

Both conditions for  $\psi_t$  imply that

$$(3.2) \quad \psi_t(m) \in \Gamma_{X(m)}^{\sharp-1}(T_{X(m)} C) \cap N_{X(m)}^* C, \text{ for any } m \in \partial\Sigma$$

which is the boundary condition we shall take for  $\psi_t$ .

Now, we must find out how these BC restrict the gauge transformations (2.3) at the boundary. In order to cancel the boundary term (2.4)  $\epsilon|_{\partial\Sigma}$  must be a smooth section of  $N^* C$  and if (2.3) is to preserve the boundary condition of  $X$ ,  $\epsilon|_{\partial\Sigma}$  must belong to  $\Gamma^{\sharp-1}(TC)$ . Hence,

$$(3.3) \quad \epsilon(m) = \Gamma_{X(m)}^{\sharp-1}(T_{X(m)} C) \cap N_{X(m)}^* C, \forall m \in \partial\Sigma.$$

Now we have the following

**Theorem 3.1.** *If*

$$\dim(\Gamma_p^\sharp(N_p^* C) + T_p C) = k, \forall p \in C$$

*then the gauge transformations satisfying (3.3) also preserve the BC (3.2) for  $\psi$ , i.e. we have consistent branes for the Poisson-Sigma model.*

*Proof.* The proof can be found in ref. [3].  $\square$

Some particular cases that have been considered in the literature are the free boundary conditions  $C = M$  and the coisotropic one  $\Gamma^\sharp(N^*C) \subset TC$ . Both fulfill the constant dimension requirement of theorem 3.1.

As mentioned in the introduction, a novel kind of brane particularly interesting in the context of quantization is obtained when  $C$  is a *constant rank Poisson-Dirac submanifold*:  $\Gamma_p^\sharp(N_p^*C) \cap T_pC = 0$  and  $\dim(\Gamma_p^\sharp(N_p^*C) + T_pC) = k$ ,  $\forall p \in C$ . In this case there is no gauge transformation acting on the brane.

Before going to the Hamiltonian analysis of the theory we will need some results on the algebraic reduction of Poisson manifolds that will be summarized in the next section.

## 4 Reduction of Poisson manifolds.

Let  $C$  be a closed submanifold of  $(M, \Gamma)$ . We adopt the notation  $\mathcal{A} = C^\infty(M)$  and take the ideal (with respect to the pointwise product of functions in  $\mathcal{A}$ ),  $\mathcal{I} = \{f \in \mathcal{A} | f(p) = 0, \forall p \in C\}$ .

Define  $\mathcal{F} \subset \mathcal{A}$  as the set of *first-class functions*, also called the *normalizer* of  $\mathcal{I}$ ,

$$\mathcal{F} := \{f \in \mathcal{A} | \{f, \mathcal{I}\} \subset \mathcal{I}\}$$

$\mathcal{F}$  is a Poisson subalgebra of  $\mathcal{A}$  and  $\mathcal{F} \cap \mathcal{I}$  is a Poisson ideal of  $\mathcal{F}$ . Then, we have canonically defined a Poisson bracket in the quotient  $\mathcal{F}/(\mathcal{F} \cap \mathcal{I})$ . We would like to find a Poisson bracket on  $C^\infty(C) \cong \mathcal{A}/\mathcal{I}$  (or, at least, in a subset of it). To that end we define an injective map

$$(4.1) \quad \begin{aligned} \phi : \mathcal{F}/(\mathcal{F} \cap \mathcal{I}) &\longrightarrow \mathcal{A}/\mathcal{I} \\ f + \mathcal{F} \cap \mathcal{I} &\longmapsto f + \mathcal{I} \end{aligned}$$

$\phi$  is an homomorphism of abelian, associative algebras with unit and then induces a Poisson algebra structure  $\{.,.\}_C$  on the image that will be denoted by  $\mathcal{C}(\Gamma, M, C)$ ,

$$(4.2) \quad \{f_1 + \mathcal{I}, f_2 + \mathcal{I}\}_C = \{f_1, f_2\} + \mathcal{I}. \quad f_1, f_2 \in \mathcal{F}.$$

In some cases it is possible to give a simple characterization of the image of  $\phi$  as shows the following

**Theorem 4.1.** *If  $\dim(\Gamma_p^\sharp(N_p^*C) + T_pC) = k$ ,  $\forall p \in C$ , then*

$$\phi(\mathcal{F}/\mathcal{F} \cap \mathcal{I}) = \{f + \mathcal{I} | \{f, \mathcal{F} \cap \mathcal{I}\} \subset \mathcal{I}\}$$

*In other words,  $\mathcal{C}(\Gamma, M, C)$  is the set of gauge invariant functions on  $C$ .*

*Proof.* See [3].  $\square$

For the constant rank Poisson-Dirac submanifold described at the end of the previous section there are not gauge transformations acting on it and the map  $\phi$  is onto.

## 5 Hamiltonian analysis

We proceed to the hamiltonian study of the model when  $\Sigma = [0, \pi] \times \mathbb{R}$  (open string). The fields in the hamiltonian formalism are a smooth map  $X : [0, \pi] \rightarrow M$  and a 1-form  $\psi$  on  $[0, \pi]$  with values in the pull-back  $X^*(T^*M)$ ; in coordinates,  $\psi = \psi_{i\sigma} dX^i d\sigma$ .

Consider the infinite dimensional manifold of smooth maps  $(X, \psi)$  with canonical symplectic structure  $\Omega$ . The action of  $\Omega$  on two vector fields (denoted for shortness  $\delta, \delta'$ ) reads

$$(5.1) \quad \Omega(\delta, \delta') = \int_0^\pi (\delta X^i \delta' \psi_{i\sigma} - \delta' X^i \delta \psi_{i\sigma}) d\sigma$$

The phase space  $P(M; C_0, C_\pi)$  of the theory is defined by the constraint:

$$(5.2) \quad \partial_\sigma X^i + \Gamma^{ij}(X) \psi_{j\sigma} = 0$$

and BC as in section 3 with  $X(0) \in C_0$  and  $X(\pi) \in C_\pi$  for two closed submanifolds  $C_u \subset M$ ,  $u = 0, \pi$ .

This geometry, with a boundary consisting of two connected components, raises the question of the relation between the BC at both ends. Note that due to eq. (5.2)  $X$  varies in  $[0, \pi]$  inside a symplectic leaf of  $M$ . This implies that in order to have solutions the symplectic leaf must have non-empty intersection both with  $C_0$  and  $C_\pi$ . In other words, only points of  $C_0$  and  $C_\pi$  that belong to the same symplectic leaf lead to points of  $P(M; C_0, C_\pi)$ . In the following we will assume that this condition is met for every point of  $C_0$  and  $C_\pi$  and correspondingly for the tangent spaces. That is, if we define the maps

$$(5.3) \quad \begin{aligned} J_0 : P(M, C_0, C_\pi) &\longrightarrow C_0 \\ (X, \psi) &\longmapsto X(0). \end{aligned}$$

and

$$(5.4) \quad \begin{aligned} J_\pi : P(M, C_0, C_\pi) &\longrightarrow C_\pi \\ (X, \psi) &\longmapsto X(\pi). \end{aligned}$$

we assume that both maps are surjective submersions.

The canonical symplectic 2-form is only presymplectic when restricted to  $P(M; C_0, C_\pi)$ . The kernel is given by:

$$(5.5) \quad \begin{aligned} \delta_\epsilon X^i &= \epsilon_j \Gamma^{ji}(X) \\ \delta_\epsilon \psi_{i\sigma} &= \partial_\sigma \epsilon_i + \partial_i \Gamma^{jk}(X) \psi_{j\sigma} \epsilon_k \end{aligned}$$

where  $\epsilon$ , a section of  $X^*(T^*M)$ , is subject to the BC

$$\epsilon(u) \in \Gamma_{X(u)}^{\sharp-1}(T_{X(u)} C_u) \cap N_{X(u)}^* C_u, \text{ for } u = 0, \pi$$

which is consistent with the results of section 3 in the lagrangian formalism.

The presymplectic structure induces a Poisson algebra  $\mathcal{P}$  in a subset of the functions on the phase space  $P(M, C_0, C_\pi)$ . On the other hand, we have the Poisson algebras  $\mathcal{C}(\Gamma, M, C_0)$  and  $\mathcal{C}(\Gamma, M, C_\pi)$ . What is the relation among them?

We first analyse when a function  $F(X, \psi) = (J_0^* f)(X, \psi) = f(X(0))$ ,  $f \in C^\infty(M)$  belongs to  $\mathcal{P}$ , i.e. when it has a hamiltonian vector field  $\delta_F$ . Solving the corresponding equation we see that the general solution is of the form (5.5) with

$$(5.6) \quad \epsilon(0) - df_{X(0)} \in N_{X(0)}^* C_0, \quad \epsilon(0) \in \Gamma_{X(0)}^{\sharp-1}(T_{X(0)} C_0)$$

and

$$\epsilon(\pi) \in \Gamma_{X(\pi)}^{\sharp-1}(T_{X(\pi)} C_\pi) \cap N_{X(\pi)}^* C_\pi.$$

Assuming  $\dim(\Gamma^\sharp(N_p^* C_0) + T_p C_0) = \text{const.}$ , equation (5.6) can be solved in  $\epsilon(0)$  if and only if  $F$  is a gauge invariant function (i.e. it is invariant under (5.5)). This is equivalent to saying that  $f + \mathcal{I}_0$  belongs to the Poisson algebra  $\mathcal{C}(\Gamma, M, C_0)$ . (Here  $\mathcal{I}_0$  is the ideal of functions that vanish on  $C_0$ ).

Now, given two such functions  $F_1$  and  $F_2$  associated to  $f_1 + \mathcal{I}_0, f_2 + \mathcal{I}_0 \in \mathcal{C}(\Gamma, M, C_0)$  and with gauge field  $\epsilon_1$  and  $\epsilon_2$  respectively, one immediately computes the Poisson bracket  $\{F_1, F_2\}_P = \Omega(\delta_{F_1}, \delta_{F_2})$  and obtains

$$(5.7) \quad \{F_1, F_2\}_P = \Gamma^{ij} \epsilon_{1i}(0) \epsilon_{2j}(0)$$

This coincides with the restriction to  $C_0$  of  $\{f_1 + \mathcal{I}_0, f_2 + \mathcal{I}_0\}_{C_0}$  and defines a Poisson homomorphism between  $\mathcal{C}(\Gamma, M, C_0)$  and the Poisson algebra of  $P(M, C_0, C_\pi)$ . This homomorphism is  $J_0^*$ , the pull-back defined by  $J_0$ , and the latter turns out to be a Poisson map. In an analogous way we may show that  $J_\pi$  is an anti-Poisson map and besides

$$\{f_0 \circ J_0, f_\pi \circ J_\pi\} = 0 \quad \text{for any } f_u \in \mathcal{C}(\Gamma, M, C_u), \quad u = 0, \pi$$

These results can be summarized in the following diagram

$$(5.8) \quad \mathcal{C}(\Gamma, M, C_0) \xrightarrow{J_0^*} \mathcal{P} \xleftarrow{J_\pi^*} \mathcal{C}(\Gamma, M, C_\pi)$$

in which  $J_0^*$  is a Poisson homomorphism,  $J_\pi^*$  antihomomorphism and the image of each map is the commutant (with respect to the Poisson bracket) of the other. This can be considered as a generalization of the symplectic dual pair to the context of Poisson algebras.

## 6 Quantization

Since the work of Cattaneo and Felder [4] we know that the perturbative expansion of certain Green's functions of the model defined on the disc with free boundary

conditions ( $C = M$ ) yields the deformation quantization of the Poisson bracket on  $M$  proposed by Kontsevich in ref. [12], namely

$$(6.1) \quad \langle f(X(0))g(X(1)) \rangle_{X(\infty)=x} = f * g(x)$$

where 0, 1 and  $\infty$  are three different (ordered) points at the boundary of the disc.

The Hamiltonian analysis and the topological nature of the model suggest that for a general  $C$  the perturbative expansion of (6.1) should reproduce Kontsevich's formula for the reduced Poisson bracket on the brane.

In a recent paper [6] the same authors have computed the Green's functions for coisotropic branes. Their result indicates that under some technical assumptions about the gauge transformations at the boundary one obtains again a deformation of the algebra of functions invariant under the deformed gauge transformations. We want to stress that the gauge transformations at the boundary introduce technical difficulties for the consistent quantization of the theory and make the conclusion somewhat involved.

A different scenario in which one could have a cleaner derivation is the case of constant rank Poisson-Dirac branes. Here there are not gauge transformations acting at the boundary and the technical requirements disappear. The natural guess is that the perturbative quantization will give Kontsevich's formula for the Dirac bracket (see [3] to check that the reduced Poisson bracket on a Poisson-Dirac brane is the standard Dirac bracket).

At first sight things do not seem to work, as the propagator corresponding to the perturbative expansion around the zero Poisson structure does not exist with these BC. However, this is not very surprising since to compute the Dirac bracket one has to invert the matrix of Poisson brackets of constraints defining the constant rank Poisson-Dirac brane. The appropriate expansion in this case is not likely to be around the zero Poisson structure.

We are currently working on this problem and we can announce some preliminary results. Consistent perturbative quantization of the theory can be defined by changing the decomposition of the action into the unperturbed part and the perturbation. Proceeding in this way we can show that at least in coordinates of the target adapted to the brane such that the components of the Poisson structure are constant, the perturbative quantization of the model produces the Kontsevich's formula for the Dirac bracket on the brane. Details of the calculation will appear elsewhere.

On the light of these facts it is natural to conjecture that the same result on the deformation of the Dirac bracket (or some other equivalent to it) holds for a general Poisson bracket on  $M$  and a general Poisson-Dirac brane. This will be the subject of further research.

## References

- [1] Bojowald, M. and Strobl, T.: Classical Solutions for Poisson Sigma Models on a Riemman surface. [arXiv:hep-th/0304252](#).
- [2] Calvo, I., Falceto, F. and García-Álvarez, D.: Topological Poisson-Sigma models on Poisson-Lie groups. *JHEP* 10 (2003) 033. [arXiv:hep-th/0307178](#)
- [3] Calvo, I. and Falceto, F.: Poisson reduction and branes in Poisson-Sigma models. *Lett. Math. Phys.* **70** (2004) 231-247 [arXiv:hep-th/0405176](#)
- [4] Cattaneo, A. S. and Felder G.: A path integral approach to the Kontsevich quantization formula. *Commun. Math. Phys.* **212** (2000) 591-611. [arXiv:math.QA/9902090](#).
- [5] Cattaneo, A. S. and Felder G.: Poisson sigma models and symplectic groupoids. in: *Quantization of Singular Symplectic Quotients*, Progr. in Math. 198, (2001) 61-93. [arXiv:math.SG/0003023](#).
- [6] Cattaneo, A. S. and Felder G.: Coisotropic submanifolds in Poisson geometry and branes in the Poisson sigma model. *Lett. Math. Phys.* **69** (2004) 157-175 [arXiv:math.QA/0309180](#).
- [7] Crainic, C. and Fernandes, R. L.: Integrability of Lie brackets. *Annals of Mathematics*, **157** (2003), 575-620. [arXiv:math.DG/0105033](#).
- [8] Crainic, C. and Fernandes, R. L.: Integrability of Poisson Brackets. [arXiv:math.DG/0210152](#).
- [9] Falceto, F. and Gawedzki, K.: Boundary G/G Theory and Topological Poisson-Lie Sigma Model. *Lett. Math. Phys.* **59** (2002), 61-79. [arXiv:hep-th/0108206](#).
- [10] Ikeda, N.: Two-dimensional gravity and nonlinear gauge theory. *Ann. Phys.*, **235** (1994), 435-464. [arXiv:hep-th/9312059](#).
- [11] Kimura, T: Generalized classical BRST cohomology and reduction of Poisson manifolds. *Commun. Math. Phys.*, **151** (1993), 155-182.
- [12] Kontsevich, M.: Deformation Quantization of Poisson Manifolds. *Lett. in Math. Phys.* **66** (2003) 157-216. [arXiv:q-alg/9709040](#).
- [13] Schaller, P. and Strobl, T.: Poisson Structure Induced (Topological) Field Theories. *Mod. Phys. Lett.* **A9** (1994), 3129-3136. [arXiv:hep-th/9405110](#).

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