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Primary and flat secondary characteristic classes for Lie algebroids, review and problems

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Characteristic homomorphisms of principal fibre bundles (primary and secondary) are invariants of Lie algebroids of principal fibre bundles and can be constructed for all regular Lie algebroids.

In this short note we give a survey of the problems for primary characteristic classes of Lie algebroids and a part of problems for secondary characteristic classes which concern "flat" classes. The paper stresses the fact the right approach to the theory of algebroid characteristic classes **is not** an adaptation of the classical theory of invariant polynomials of a given Lie group and of their representations and then enumeration of the primary and secondary characteristic classes **but** the study of bundles of polynomials and their invariant cross-sections. We also present some open problems.

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1 Primary characteristic classes

1.1 Classical theory of primary characteristic classes for principal fibre bundles

The classical theory of primary characteristic classes for principal fibre bundles is well known. Let (P, π, M, G) be a G-pfb on M with projection $\pi : P \to M$ and structural Lie group G acting on P from the right. The domain of the Chern-Weil homomorphism for P is the space $I(G) := (\bigvee \mathfrak{g}^*)_I = (\bigvee \mathfrak{g}^*)_{I(\mathrm{Ad}_G)}$ of symmetric multilinear functions (equivalently polynomials) on $\mathfrak{g} = \mathfrak{gl}(G)$ invariant with respect to the adjoint representation $\mathrm{Ad}_G : G \to GL(\mathfrak{g})$ of G. The Chern-Weil homomorphism for P, see [G-H-V, Vol. II, Ch.VI], $h_p : (\bigvee \mathfrak{g}^*)_I \to H_{dR}(M)$ can be defined by $h_P(\Gamma) = [\chi_P(\Gamma)]$ where for invariant k-polynomial $\Gamma \in \bigvee^k \mathfrak{g}^*$ the differential form $\chi_P(\Gamma) \in \Omega^{2k}(M)$ is such that $\pi^*(\chi_P(\Gamma)) = \frac{1}{k!}\Gamma(\Omega^k)$ where $\Omega \in \Omega^2(P, \mathfrak{g})$ is the curvature form of any connection ω in P and $\Gamma(\Omega^k) = \langle \Gamma, \Omega \vee ... \vee \Omega \rangle$ is the pairing defined via the permanent ($\Omega \vee ... \vee \Omega$ is the usual multiplication of differential forms and values of which are multiplied under the symmetric multilinear mapping $\mathfrak{g}\times ... \times \mathfrak{g} \to \bigvee^k \mathfrak{g}$). The image $\operatorname{Im}(h_P) \subset H_{dR}(M)$ is called the *Pontryagin algebra* of P. The algebra $I(G)$ is well known for standard Lie groups.

1.2 Algebroid nature of primary characteristic classes

Theorem 1.1 *If (P,π,M,G) and (P',π',M,G') are principal fibre bundles on the same manifold M with connected total spaces P and P' such that their Lie algebroids $A(P)$ and $A(P')$ are isomorphic (G and G' can be nonconnected), then $h_P = h_{P'}$, i.e. the Pontryagin algebra of P depends only on the Lie algebroid of P.*

We recall that by a *Lie algebroid* on a manifold M [P] we mean a triple $(A, [\![\cdot,\cdot]\!], \#_A)$ where A is a vector bundle on M, $[\![\cdot,\cdot]\!]$ is a real Lie algebra structure in the space of global cross-sections $\operatorname{Sec} A$ and $\#_A : A \to TM$ is a linear homomorphism fulfilling the Leibniz condition $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + \#_A(\xi)(f)\cdot\eta$. $\operatorname{Sec}\#_A : \operatorname{Sec} A \to \mathfrak{X}(M)$ is a homomorphism of Lie algebras [He], [B-K-W]. There are many geometric sources of Lie algebroids: differential groupoids, principal fibre bundles, vector bundles, transversely complete foliations, nonclosed Lie subgroups, Poisson manifolds, Jacobi manifolds, locally conformal symplectic structures etc.

There are non-isomorphic *principal fibre fibre bundles* having isomorphic Lie algebroids (for example P equal to the trivial $\operatorname{Spin}(3)$ pfb on $\mathbb{RP}^5$ and P' equal to the nontrivial $\operatorname{Spin}(3)$ structure on $\mathbb{RP}^5$ [K3]).

The Lie algebroid $A(P)$ of a principal fibre bundle P can be constructed in three different but equivalent ways [P], [K3]

— the Lie algebroid of the Ehresmann Lie groupoid PP^{-1},

— the Atiyah vector bundle TP/G according to the observation $\operatorname{Sec}(TP/G) \cong \mathfrak{X}^r(P)$, $\mathfrak{X}^r(P)$ is the Lie algebra of G-right invariant vector fields on P,

— as an associated bundle $W^1(P)\times_{G_n^1}(\mathbb{R}^n\times\mathfrak{g})$ with the first-order prolongation of P.

Take the Atiyah short exact sequence [A] ($\mathfrak{g}$ is the **right** Lie algebra of G)

$$0 \to P\times_G \mathfrak{g} \to TP/G \overset{[\pi_*]}{\to} TM \to 0.$$

$P\times_G\mathfrak{g}$ is a Lie algebra bundle, for any $z\in P_x$, $\hat{z} : \mathfrak{g} \cong (P\times_G\mathfrak{g})_x$, $v \longmapsto [(z,v)]$, is an isomorphism of Lie algebras. We notice that having only the Lie algebroid $A(P)$ we can not reconstruct the structural Lie group, but only its Lie algebra! The Lie algebroid $A(P)$ acts on the Lie algebra bundle $P\times_G\mathfrak{g}$ by $\operatorname{ad}_{A(P)}(\xi)(\nu) = [\![\xi,\nu]\!]$. This actions can be extended to the actions $\operatorname{ad}^{\vee}_{A(P)}$ of $A(P)$ on the symmetric power of the dual of $P\times_G\mathfrak{g}$, $\bigvee^k(P\times_G\mathfrak{g})^*$ - i.e. on the vector bundle of polynomials. We take the algebra of invariant cross-sections $I(A(P)) := \bigoplus^k \operatorname{Sec}(\bigvee^k(P\times_G\mathfrak{g})^*)_{I(ad)}$ of vector bundles $\bigvee^k(P\times_G\mathfrak{g})^*$. We have

$\Gamma \in \mathrm{Sec}(\bigvee^k (P \times_G \mathfrak{g})^*)_{I(ad)}$ if and only if for every $\xi \in \mathrm{Sec}(A(P))$ and $\nu_1, ..., \nu_k \in \mathrm{Sec}(P \times_G \mathfrak{g})$

$$[\pi_*](\xi)\langle\Gamma, \nu_1 \vee ... \vee \nu_k\rangle = \sum_{i=1}^{k} \langle\Gamma, \nu_1 \vee ... \vee [\![\xi, \nu_i]\!] \vee ... \vee \nu_k\rangle.$$

Theorem 1.2 *If P is a* ***connected*** *pfb (G can be non-connected) then there exists an isomorphism of algebras $\rho : I(G) \to I(A(P))$. (In general, we have always a monomorphism).*

To understand this fact we take the Ad_G-homomorphism of principal fibre bundles $\mathrm{Ad}_P : P \to L\boldsymbol{g}$, $z \mapsto [(z, \cdot)]$, ($L\boldsymbol{g}$ is the $GL(\mathfrak{g})$-principal fibre bundle of frames of $\boldsymbol{g}{:=}P \times_G \mathfrak{g}$) which is called the *adjoint representation of a principal fibre bundle* and consider its differential $(\mathrm{Ad}_P)' : A(P) \to A(\boldsymbol{g})$ which is equal to the adjoint representation $\mathrm{ad}_{A(P)}$ of the Lie algebroid $A(P)$, $\mathrm{ad}_{A(P)} = (\mathrm{Ad}_P)'$.

Remark 1.3 We recall [M1], [K8] that for every vector bundle $\mathfrak{f}$ the Lie algebroid $A(\mathfrak{f}) := A(L\mathfrak{f})$ of the principal fibre bundle $L\mathfrak{f}$ of all frames can be constructed directly as the Lie algebroid whose global cross-sections are covariant derivative operators, $\mathcal{L} \in \mathrm{Sec}(A(\mathfrak{f})) \Leftrightarrow \mathcal{L}$ is an first order operator $\mathcal{L} : \mathrm{Sec}\,\mathfrak{f} \to \mathrm{Sec}\,\mathfrak{f}$ with the anchor, i.e. for which there exists a vector field X (denoted by $\#(\mathcal{L})$) such that $\mathcal{L}(f \cdot \nu) = f \cdot \mathcal{L}(\nu) + X(f) \cdot \nu$, $f \in C^\infty(M)$, $\nu \in \mathrm{Sec}\,\mathfrak{f}$. For the Lie algebroid of a vector bundle with given transition functions see [K5].

The representation Ad_P and $\mathrm{ad}_{A(P)}$ of P in the vector bundle $\boldsymbol{g}$ can be lifted standardly to the representations of P in $\bigvee^k \boldsymbol{g}^*$, $\mathrm{Ad}_P^\vee : P \to L(\bigvee^k \boldsymbol{g}^*)$ and $\mathrm{ad}_{A(P)}^\vee : A(P) \to A(\bigvee^k \boldsymbol{g}^*)$. The analogous property $(\mathrm{Ad}_P^\vee)' = \mathrm{ad}_{A(P)}^\vee$ holds.

For invariant cross-sections of the vector bundle $\bigvee^k \boldsymbol{g}^*$ with respect to the representations $\mathrm{Ad}_P^\vee$ and $\mathrm{ad}_{A(P)}^\vee$ [K4] we have a canonical monomorphism

$$\rho : I^k(G) \cong (\mathrm{Sec} \textstyle\bigvee^k \boldsymbol{g}^*)_{I(\mathrm{Ad}_P^\vee)} \subset (\mathrm{Sec} \textstyle\bigvee^k \boldsymbol{g}^*)_{I(\mathrm{ad}_{A(P)}^\vee)} = I^k(A(P)) \tag{1.1}$$

which is an isomorphism when P is connected, $I^k(G) \cong I^k(A(P))$. This fact generalize standard results concerning spaces of invariant vectors for a representation of a Lie group in a finite-dimensional vector space and its differential to representations of principal fibre bundles and Lie algebroids [K4, s.5.5]:

— Let $\mu : G \to GL(V)$ be a representation of a Lie group G in a finite-dimensional vector space V and $\mathfrak{f}$ a vector bundle with the typical fibre V. Let $T : P \to L\mathfrak{f}$ be a μ-homomorphism of principal fibre bundles. A cross-section $\xi \in \mathrm{Sec}\,\mathfrak{f}$ is called *T-invariant* if there exists a vector $v \in V$ such that $T(z)(v) = \xi_{\pi z}$ for all $z \in P$. Denote by $(\mathrm{Sec}\,\mathfrak{f})_{I(T)}$ the space of all T-invariant cross-sections of $\mathfrak{f}$.

— Denote by $V_{I(\mu)}$ the subspace of V of μ-invariant vectors. Then, for $v \in V_{I(\mu)}$, the function $\xi_v : M \to \mathfrak{f}$, $\quad x \longmapsto T(z)(v)$ where $z \in P_x$, is a correctly defined smooth cross-section of $\mathfrak{f}$ and

$$V_{I(\mu)} \overset{\cong}{\to} (\operatorname{Sec}\mathfrak{f})_{I(T)}, \quad v \longmapsto \xi_v, \tag{1.2}$$

is an isomorphism. Therefore applying to the representation $\operatorname{Ad}_P^\vee : P \to L(\bigvee^k \boldsymbol{g}^*)$ we have

$$I^k(G) = (\textstyle\bigvee^k \mathfrak{g}^*)_{I(\operatorname{Ad}_G)} \cong (\operatorname{Sec}\textstyle\bigvee^k \boldsymbol{g}^*)_{I(\operatorname{Ad}_P^\vee)}.$$

— Let $S : A \to A(\mathfrak{f})$ be a homomorphism of Lie algebroids (S is called a *representation* of A in a vector bundle $\mathfrak{f}$). A cross-section $\xi \in \operatorname{Sec}\mathfrak{f}$ is called S-*invariant* (or S-*parallel*) if $S(v)(\xi) = 0$ for all $v \in A$. Denote by $(\operatorname{Sec}\mathfrak{f})_{I(S)}$ the space of all S-invariant cross-sections of $\mathfrak{f}$.

Theorem 1.4 *[K4] Let $T : P \to L\mathfrak{f}$ be a μ-homomorphism of principal fibre bundles and $T' : A(P) \to A(\mathfrak{f})$ its differential. The spaces of invariant cross-sections $(\operatorname{Sec}\mathfrak{f})_{I(T)}$ and $(\operatorname{Sec}\mathfrak{f})_{I(T')}$ under T and its differential T' are related by*

$$(\operatorname{Sec}\mathfrak{f})_{I(T)} \subset (\operatorname{Sec}\mathfrak{f})_{I(T')}.$$

*If P is **connected** (nothing is assumed about the connectedness of G), then*

$$(\operatorname{Sec}\mathfrak{f})_{I(T)} = (\operatorname{Sec}\mathfrak{f})_{I(T')}. \tag{1.3}$$

In consequence, applying to the representation $\operatorname{ad}^\vee_{A(P)} : A(P) \to A(\bigvee^k \boldsymbol{g}^*)$ we have a monomorphism (1.1) which is an isomorphism where P is connected.

1.3 Chern-Weil homomorphism for Lie algebroids.

Let A be a transitive Lie algebroid A with the Atiyah sequence

$$0 \to \boldsymbol{g} \to A \overset{\#_A}{\to} TM \to 0, \tag{1.4}$$

$\boldsymbol{g} = \ker \#_A$ is a Lie algebra bundle. The Lie algebroid A acts on $\boldsymbol{g}$ by $\operatorname{ad}_A : A \to A(\boldsymbol{g})$, $\operatorname{ad}_A(\xi)(\nu) = [\![\xi, \nu]\!]$, $\xi \in \operatorname{Sec} A$, $\nu \in \operatorname{Sec}\boldsymbol{g}$. This action can be extended on the bundle $\bigvee^k \boldsymbol{g}^*$ to the action $\operatorname{ad}_A : A \to A(\bigvee^k \boldsymbol{g}^*)$. Let $I^k(A)$ denotes the space of invariant cross-sections of $\bigvee^k \boldsymbol{g}^*$. $I(A) := \bigoplus I^k(A)$ is an algebra. The value $\Gamma_x \in \bigvee^k \boldsymbol{g}_x^*$ at x of any invariant cross-section is a vector invariant with respect to the adjoint representation $\operatorname{ad}_{\boldsymbol{g}_x}$ of the isotropy Lie algebra $\boldsymbol{g}_x$. Each invariant vector $v \in \bigvee^k \boldsymbol{g}_x^*$ can be extended uniquely to some invariant cross-section of the bundle $\bigvee^k \boldsymbol{g}^*$ over an open subset containing x (for example if this neighbourhood is contractible). The vectors $v \in \bigvee^k \boldsymbol{g}_x^*$ which can be extended onto the whole manifold M are described by formula [M1, Th. IV, 1.19].

Take a connection $\omega : TM \to A$ (i.e. a linear homomorphism ω such that $\#_A \circ \omega = \mathrm{id}_{TM}$) and its curvature form $\Omega \in \Omega^2(M; \boldsymbol{g})$, $\Omega(X,Y) = [\![\omega(X), \omega(Y)]\!] - \omega([X,Y])$. The Chern-Weil homomorphism of A is defined by [K4]

$$(1.5) \qquad h_A : I(A) \to H_{dR}(M), \quad I^k(A) \ni \Gamma \longmapsto \frac{1}{k!}[\langle \Gamma, \Omega \vee ... \vee \Omega \rangle].$$

If there exists a flat connection in A then $h_A^+ = 0$. The analogous construction can be made analogously for regular Lie algebroid over a regular foliated manifold (M,F), we must use the algebra of tangential differential forms $\Omega(F)$ and its cohomology $H(F)$ instead of $\Omega(M)$ and $H_{dR}(M)$ [K4].

Theorem 1.5 *If $A(P)$ is a Lie algebroid of a principal fibre bundle (P, π, M, G), then*

$$h_P = h_{A(P)} \circ \rho : I(G) \to I(A(P)) \to H_{dR}(M).$$

If P is connected, then under the identification $I(G) = I(A(P))$ we have $h_P = h_{A(P)}$.

This gives Th. 1.1.

There exists Lie algebroids which are not integrable, i.e. which do not come from principal bundles, but have nontrivial Chern-Weil homomorphisms.

Theorem 1.6 *[K4] (1) Let $H \subset G$ be any connected Lie subgroup of G and let h, $\bar{h}$ and $\mathfrak{g}$ be the Lie algebras of H, of its closure $\bar{H}$ and of G, respectively. Let $A(G;H)$ be the Lie algebroid of the foliation of left cosets of G by H. Denote by $h_P : (\mathsf{V}\bar{\mathfrak{h}}^*)_I \to H_{dR}(G/\bar{H})$ the Chern-Weil homomorphism of the $\bar{H}$-principal bundle $P = \left(G \to G/\bar{H}\right)$. Then there exists an isomorphism of algebras κ such that the following diagram commutes:*

$$\begin{array}{ccc} I(A(G;H)) & \overset{h_{A(G;H)}}{\longrightarrow} & H_{dR}(G/\bar{H}) \\ \cong\downarrow \kappa & & \uparrow h_P \\ (\mathsf{V}(\bar{\mathfrak{h}}/\mathfrak{h})^*) & \rightarrowtail & (\mathsf{V}\bar{\mathfrak{h}}^*)_I \end{array}$$

(2) If G is a connected, compact and semisimple Lie group and $H \subset G$ is a nonclosed Lie subgroup than $h_{A(G;H)}$ is nontrivial. Adding the simple connectedness to the assumption about G, we get, according to the Almeida-Molino theorem [A-M], some nonintegrable transitive Lie algebroid having the nontrivial Chern-Weil homomorphism.

Concerning the primary Chern-Weil homomorphism, the following papers seems to be first on this subject: Teleman (1972) [T1], [T2], Kubarski (1986) [K1], [K2], Mackenzie (1988) [M2], Moore, Schochet (1988) [M-S], Kubarski (1991) [K4], Belko (1994) [B], Kubarski (1994) [K5], [K6], Vaisman (1994) [V2], Itskov, Karasev, Vorobjev (1998) [I-K-U], Huebschmann (1999) [Hu2], Fernandes (2000) [F1], [F2].

1.4 The Chern-Weil homomorphism for pairs of Lie algebroids

Take a pair of Lie algebroids (A, L) on a manifold M and assume that A is transitive (we may assume less, that A is regular). Let (1.4) be the Atiyah sequence of A. By an *L-connection* in A we mean a linear homomorphism $\nabla : L \to A$ compatible with the anchors $\#_A \circ \nabla = \#_L$. By a *curvature form* of ∇ we shall mean the 2-form $\Omega_\nabla \in \Omega^2(L; \boldsymbol{g})$ defined by $\Omega_\nabla(\xi, \eta) = [\![\nabla\circ\xi, \nabla\circ\eta]\!] - \nabla\circ[\![\xi, \eta]\!]$. If $L = TM$ then ∇ is a usual connection in A. If $L = T^*M$ is the Lie algebroid of a Poisson manifold $(M, \{\cdot,\cdot\})$ and $A = A(P)$ we have the so-called *contravariant connection* in a principal fibre bundle P [V1], if $A = A(\mathfrak{f})$ is the Lie algebroid of a vector bundle $\mathfrak{f}$ we have the so-called *contravariant connection* in a vector bundle $\mathfrak{f}$ [F1]. If $0 \to L' \to L \to L'' \to 0$ is an extension of Lie algebroids, then any splitting $\nabla : L'' \to L$ is a L''-connection in L [Hu1]. Let us remark that an L-connection in $A(\mathfrak{f})$ is the same as L-covariant derivative $\nabla_\xi \nu$ in a vector bundle $\mathfrak{f}$, $\xi \in \operatorname{Sec} L$, $\nu \in \operatorname{Sec}\mathfrak{f}$, i.e. an operator $\nabla_\xi \nu$ fulfilling the usual Koszul axioms with the following difference: $\nabla_\xi(f\nu) = f \cdot \nabla_\xi \nu + \#_L(\xi)(f) \cdot \nu$.

By the Chern-Weil homomorphism of the pair (A, L) we mean $h_{L,A} : I(A) \to H(L)$ defined by the formula analogous to (1.5), see [B-K-W]. The image of $h_{L,A}$ is the Pontryagin algebra of the pair (L, A), $\operatorname{Pont}(L, A) := \operatorname{Im} h_{L,A}$. The comparison with h_A is given thanks to the commutativity of the diagram

$$\begin{tikzcd} & H_{dR}(M) \arrow[dr, "(\#_L)^\#"] & \\ I(A) \arrow[ur, "h_A"] \arrow[rr, "h_{L,A}"'] & & H(L) \end{tikzcd}$$

It follows from the fact that the superposition $L \overset{\#_L}{\to} TM \overset{\omega}{\to} A$ (where $\omega : TM \to A$ is a connection in A) is an example of an L-connection in A. In particular $(\#_L)^\#[\operatorname{Pont} A] = \operatorname{Pont}(L, A)$.

Consider $L = A$, $\nabla = \mathrm{id}_A : A \to A$ is a flat A-connection in A, so $h^+_{A,A} = 0$. Therefore $\operatorname{Pont}^+ A \subset \ker(\#_A)^*$. In this way we have a simply proof of the well-known fact concerning principal fibre bundle $\pi : P \to M$, $\operatorname{Pont}^+(P) \subset \ker \pi^\#$.

1.5 Problem

Let $(A, [\![\cdot,\cdot]\!], \#_A)$ and $\mathfrak{f}$ be a transitive Lie algebroid and a vector bundle on a manifold M, respectively. Assume that $F \subset TM$ is a C^∞ regular involutive distribution and $\mathcal{F}$ – the foliation determined by F. We recall that A and F give rise to the regular Lie algebroid A^F over (M, F), where we put $A^F := (\#_A)^{-1}[F] \subset A$. Its Atiyah sequence is $0 \longrightarrow \boldsymbol{g} \hookrightarrow A^F \overset{\#_A^F}{\longrightarrow} F \longrightarrow 0$, where $\boldsymbol{g}$ is the Lie algebra bundle adjoint of A, and $\#_A^F := \#_A | A^F$. Any representation $T : A \to A(\mathfrak{f})$ of A on

$\mathfrak{f}$ restricts to the representation $T^F = T|A^F : A^F \longrightarrow A(\mathfrak{f})$ of A^F on $\mathfrak{f}$. For F-basic functions $f^i \in \Omega_b^\circ(M,F)$ and T-invariant cross-sections $\nu_i \in \operatorname{Sec}\mathfrak{f}$, $\sum_i f^i \cdot \nu_i$ is a T^F-invariant cross-section, in other words,

$$\Omega_b^\circ(M,\mathcal{F}) \cdot (\operatorname{Sec}\mathfrak{f})_{I(T)} \subset (\operatorname{Sec}\mathfrak{f})_{I(T^F)} \,.$$

In general, the above inclusion cannot be replaced by the equality, which means that not every T^F-invariant cross-section is of the form $\sum_i f^i \cdot \nu_i$ for $\mathcal{F}$-basic functions f^i and T-invariant cross-sections ν_i [K5]. Each T^F-invariant cross-section $\nu \in (\operatorname{Sec}\mathfrak{f})_{I(T^F)}$ not belonging to $\Omega_b^\circ(M,\mathcal{F}) \cdot (\operatorname{Sec}\mathfrak{f})_{I(T)}$ will be called *singular*. By the *tangential Chern-Weil homomorphism* of a transitive Lie algebroid A over (M,F) we mean the Chern-Weil homomorphism h_{A^F} of the regular Lie algebroid A^F. Applying it to the Lie algebroid $A(P)$ of a G-pfb P on M, and a foliation $F \subset TM$, we obtain the *tangential Chern-Weil homomorphism* of P over (M,F). Let $G_\circ$ be the connected component containing the unit of G. If each $G_\circ$-invariant element of $\mathsf{V}\mathfrak{g}^\star$ is G-invariant, then the domain of the homomorphism $h_{A(P)^F}$ is equal to $\Omega_b^\circ(M,\mathcal{F}) \cdot I(A(P))$ ($\cong \Omega_b^\circ(M,\mathcal{F}) \cdot (\mathsf{V}\mathfrak{g}^\star)_{I(\mathrm{Ad}_G)}$ when P is connected). The case $(\mathsf{V}\mathfrak{g}^\star)_{I(\mathrm{Ad}_G)} \subsetneq (\mathsf{V}\mathfrak{g}^\star)_{I(\mathrm{Ad}_{G_o})}$ can be the source of the strong inclusion $\Omega_b^\circ(M,\mathcal{F}) \cdot I\left(A\left(P\right)\right) \subsetneq I(A\left(P\right)^F)$ and then the nontriviality of singular primary characteristic classes corresponding to singular invariant sections from $I(A\left(P\right)^F) \backslash (\Omega_b^\circ(M,\mathcal{F}) \cdot I\left(A\left(P\right)\right))$. Examples can be constructed in the following way. We can consider a connected G-pfb P with a nonconnected Lie group G and a foliation F on M such that the restriction of P to each leaf L of F possesses a G_o-reduction $P^o_{|L}$. Invariants cross-sections from $I(A(P^o_{|L}))$ corresponding to vectors from $(\mathsf{V}\mathfrak{g}^\star)_{I(\mathrm{Ad}_{G_o})} \setminus (\mathsf{V}\mathfrak{g}^\star)_{I(\mathrm{Ad}_G)}$ after gluing them via a suitable basic function gives an invariant singular cross-section.

Problem 1.7 Find a nontrivial singular characteristic class. Does there exist a transitive Lie algebroid A and a foliation F such that $\operatorname{Pont}^+ A = 0$ and $\operatorname{Pont}^+ A^F \neq 0$?

For other open problems concerning comparison of Chern-Weil homomorphisms for single Lie algebroids, for pairs and for extensions see [B-K-W].

2 Secondary flat characteristic classes for Lie algebroids

2.1 Classical theory for principal fibre bundles

Consider the triple (P,P',ω) where $P = (P,\pi,M,G)$ is a G-pfb, P' is its G'-reduction ($G' \subset G$ is a closed Lie subgroup of G), and $\omega \subset TP$ is a flat connection in P with the connection form $\breve{\omega} : TP \to \mathfrak{g}$ ($\mathfrak{g}$ is the Lie algebra of

G). Equivalently (according to Lehmann's approach [L]) we can consider two ideals J_1 and J_2 in the algebra of invariant polynomials $I(G)$, $J_1 = I^+(G)$, $J_2 = \ker(I(G) \to I(G'))$. The characteristic homomorphism

$$\Delta_{\#(P,P',\omega)} : H(\mathfrak{g}, G') \longrightarrow H_{dR}(M)$$

is one of the most important notion in the differential geometry of principal fibre bundles [K-T]. The cohomology classes from the image of the homomorphism $\Delta_{\#(P,P',\omega)}$ are called the *secondary* [or *exotic*] *flat characteristic classes* of (P, P', ω). The homomorphism $\Delta_{\#(P,P',\omega)}$ has functoriality property and $\Delta_{\#(P,P',\omega)}$ is an invariant of the class of homotopic G'-reductions. The nontriviality of $\Delta_{\#(P,P',\omega)}$ implies that there is no homotopic change of P' containing the connection ω.

We recall that $H(\mathfrak{g}, G')$, called the *relative Lie algebra cohomology,* is the cohomology space of the complex $((\bigwedge(\mathfrak{g}/\mathfrak{g}')^*)_{I_{G'}}, d^{G'})$ where $\mathfrak{g}'$ is the Lie algebra of G' and $(\bigwedge(\mathfrak{g}/\mathfrak{g}')^*)_{I_{G'}}$ is the space of invariant elements with respect to the adjoint representation of the Lie group G' and the differential $d^{G'}$ is defined via the formula

$$\left\langle d^{G'}(\psi), [w_1] \wedge \ldots \wedge [w_k] \right\rangle = \sum_{i<j} (-1)^{i+j} \langle \psi, [[w_i, w_j]] \wedge [w_1] \wedge \ldots \hat{\imath} \ldots \hat{\jmath} \ldots \wedge [w_k] \rangle$$

for $\psi \in \bigwedge^k (\mathfrak{g}/\mathfrak{g}')^*_{I_{G'}}$ and $w_i \in \mathfrak{g}$. The homomorphism $\Delta_{\#(P,P',\omega)}$ on the level of forms is given by the following direct formula

$$(\Delta\psi)_x (w_1 \wedge \ldots \wedge w_k) = \langle \psi, [\breve{\omega}_z(\tilde{w}_1)] \wedge \ldots \wedge [\breve{\omega}_z(\tilde{w}_k)] \rangle \tag{2.1}$$

where $x \in M$, $z \in P'$, $\pi z = x$, $w_i \in T_x M$, $\tilde{w}_i \in T_z P'$, $\pi'_* \tilde{w}_i = w_i$.

The relative Lie algebra cohomology $H(\mathfrak{g}, G')$ is well known for many pairs $(\mathfrak{g}, G')$ ([K-T], [G], [G-H-V, Vol III]).

2.2 Algebroid's generalization

For details concerning this section see [B-K], [K7], [K9]. Consider a triple

$$(A, B, \nabla)$$

consisting of transitive Lie algebroid A and its transitive Lie subalgebroid B and an arbitrary Lie algebroid L on M (irregular, in general) and a flat L-connection $\nabla : L \to A$ (without any trouble we can assume less: A and B are regular Lie algebroids over the same foliated manifold). In the diagram below $\lambda_B : TM \to B$ means an arbitrary auxiliary connection in B. Then $j \circ \lambda_B : TM \to A$ is a connection in A. Let $\omega^{j\circ\lambda_B} : A \to \boldsymbol{g}$ be its connection form.

(2.2)
$$
\begin{tikzcd}
0 \arrow[r] & \boldsymbol{g} \arrow[r, hook, shift left] & A \arrow[l, shift left, "\omega^{j\circ\lambda_B}"] \arrow[dr, "\#_A"] & & L \arrow[ll, "\nabla"'] \arrow[d, "\#_L"] \\
0 \arrow[r] & \boldsymbol{h} \arrow[u, hook] \arrow[r, hook] & B \arrow[u, hook, "j"] \arrow[r, shift left, "\#_B"] & TM \arrow[l, shift left, "\lambda_B"] \arrow[r, phantom, "\supset"] & F
\end{tikzcd}
$$

The constructed characteristic homomorphism for the triple (A, B, ∇) is measuring the incompatibility of the flat structure with a given subalgebroid and has homotopic properties in analogy to the classical case of principal fibre bundles.

Example 2.1 *1.* For $L = TM$ we obtain the case in which the connection ∇ is a usual connection in A [K7],

2. For $L = TM$ and $A = TP/G$ and $B = TP'/G'$ (P' is an G'-reduction of P) we obtain the classical case equivalent to the standard case of principal fibre bundles [K-T].

3. For $L = A$ and $\nabla = \mathrm{id}_A$ we consider only the Lie algebroid A and its Lie subalgebroid B. This case produces a characteristic homomorphism for the inclusion $B \subset A$, in particular for the inclusion of Lie algebras $\mathfrak{h} \subset \mathfrak{g}$, and in particular for inclusion of principal bundles $P' \subset P$ which finally produces a new theorem for principal bundles probably not mentioned earlier in the literature.

4. Let $\mathfrak{f}$ be a vector bundle equipped with a Riemannian metric h. If $A = A(\mathfrak{f})$ and $B := A(\mathfrak{f}, \{h\}) \subset A$ is a Riemannian reduction [K5], (more precisely B is the Lie algebroid of the principal bundle of orthogonal frames) we obtain the case equivalent to the one considered by M. Crainic [C] of the characteristic exotic characteristic classes for a representation of any Lie algebroid L in a vector bundle $\mathfrak{f}$. However we must add that Crainic have lost one of the characteristic classes for oriented vector bundle of even rank.

To construct the characteristic homomorphism for (A, B, ∇) we notice that for a general connection $\nabla : L \to A$ does not exist a suitable notion of a connection form. The connection form was used in the direct formula (2.1) for the classical case. We must produce a characteristic homomorphism for (P, P', ω) without the connection form $\breve{\omega}$. Take auxiliarily a connection λ' in P' and extend it to a connection λ in P. Let $\breve{\lambda} : TP \to \mathfrak{g}$ be the connection form of λ. Then it appears that the characteristic homomorphism for (P, P', ω) can be equivalently defined on the level of differential forms via

$$
(\Delta\psi)_x \,(w_1 \wedge ... \wedge w_k) = \left\langle \psi, [-\breve{\lambda}\,(\hat{w}_1)] \wedge ... \wedge [-\breve{\lambda}\,(\hat{w}_k)] \right\rangle \tag{2.3}
$$

$\hat{w}_i \in T_z P$ being the ω-horizontal lifting of w_i, $z \in P_{|x}$.

In the general case (A, B, ∇) we define the homomorphism

$$\omega_{B,\nabla} : L \longrightarrow \boldsymbol{g}/\boldsymbol{h}, \quad w \longmapsto [-\omega^{j\circ\lambda_B}(\nabla w)],$$

see diagram (2.2). It is an important observation that $\omega_{B,\nabla}$ does not depend on the choice of an auxiliary connection λ_B and $\omega_{B,\nabla} = 0$ if ∇ takes values in B.

Define the homomorphism of algebras

$$\Delta : \mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^* \longrightarrow \Omega(L)$$

$$(\Delta\Psi)(x; w_1 \wedge ... \wedge w_k) = \langle \Psi_x, \omega_{B,\nabla}(w_1) \wedge ... \wedge \omega_{B,\nabla}(w_k) \rangle ,$$

$\Psi \in \mathrm{Sec} \bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*$, $x \in M$, $w_i \in L_{|x}$. Observe that Δ can be written as the superposition $\Delta = \nabla^* \circ \Delta_o$,

$$\Delta : \mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^* \xrightarrow{\Delta_o} \Omega(A) \xrightarrow{\nabla^*} \Omega(L)$$

where ∇^* is the pullback of forms and Δ_o is the homomorphism given for particular case of flat connection $\nabla = \mathrm{id}_A$, so that

$$(\Delta_o \Psi)_x (\upsilon_1 \wedge ... \wedge \upsilon_k) = \left\langle \Psi_x, [-\omega^{j\circ\lambda_B}(\upsilon_1)] \wedge ... \wedge [-\omega^{j\circ\lambda_B}(\upsilon_1)] \right\rangle .$$

In the algebra $\mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^*$ we distinguish a subalgebra $(\mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^*)_{I_B}$ of invariant cross-sections with respect to the adjoint representation of B in $\bigwedge (\boldsymbol{g}/\boldsymbol{h})^*$. $\Psi \in \mathrm{Sec} \bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*$ is invariant if and only if

$$(\#_B \circ \xi) \langle \Psi, [\nu_1] \wedge ... \wedge [\nu_k] \rangle = \sum_{j=1}^{k} (-1)^{j-1} \langle \Psi, [[j \circ \xi, \nu_j]] \wedge [\nu_1] \wedge ...\hat{\jmath}... \wedge [\nu_k] \rangle .$$

for all $\xi \in \mathrm{Sec}\, B$ and $\nu_j \in \mathrm{Sec}\, \boldsymbol{g}$. In the space $(\mathrm{Sec} \bigwedge (\boldsymbol{g}/\boldsymbol{h})^*)_{I_B}$ of invariant cross–sections we have a differential $\bar{\delta}$ defined by

$$\left\langle \bar{\delta}\Psi, [\nu_1] \wedge ... \wedge [\nu_k] \right\rangle = -\sum_{i<j} (-1)^{i+j} \langle \Psi, [[\nu_i, \nu_j]] \wedge [\nu_1] \wedge ...\hat{\imath}...\hat{\jmath}... \wedge [\nu_k] \rangle .$$

Let $H(\boldsymbol{g}, B)$ denote the cohomology algebra of the complex obtained. (Remark: The difference here, in comparison with the classical formula for principal fibre bundles — the sign "−" — has its roots in the fact that the Lie algebra of the structure Lie group in the principal fibre bundle considered there is taken left, not right.) The homomorphism Δ commutes with the differentials $\bar{\delta}$ and d_L. In consequence, Δ and Δ_o induce the homomorphisms in cohomology

$$\Delta_{\#(A,B,\nabla)} : H(\boldsymbol{g}, B) \xrightarrow{\Delta_{o\#}} H(A) \xrightarrow{\nabla^{\#}} H(L) .$$

The map $\Delta_{\#(A,B,\nabla)}$ is called the *characteristic homomorphism* of the triple (A, B, ∇) and the characteristic classes from the image are called the *secondary* [*exotic*] *characteristic classes* of the triple (A, B, ∇). Of course, $\Delta_{o\#}$ is the characteristic homomorphism of the pair (A, B), $B \subset A$.

Remark 2.2 We see that for a pair of transitive Lie algebroids (A, B), $B \subset A$, [analogously for both regular over the same foliation] and for an arbitrary element $\zeta \in H(\boldsymbol{g}, B)$ there exists a "universal" cohomology class $\Delta_{o\#}(\zeta) \in H(A)$ such that for any (irregular in general) Lie algebroid L on M and a flat L-connection $\nabla : L \to A$ it holds the equality

$$\Delta_{\#(A,B,\nabla)}(\zeta) = \nabla^{\#}(\Delta_{o\#}(\zeta)).$$

Problem 2.3 Is the characteristic homomorphism $\Delta_{o\#} : H(\boldsymbol{g}, B) \longrightarrow H(A)$ *a* monomorphism for a given $B \subset A$? (The answer "YES" holds in many important cases [B-K]).

The characteristic homomorphism $\Delta_{\#(A,B,\nabla)} : H(\boldsymbol{g}, B) \longrightarrow H(L)$ has functoriality property and is invariant under homotopic subalgebroids and homotopic connections [B-K]. We recall that [K6] two transitive Lie subalgebroids B_0, $B_1 \subset A$ are said to be *homotopic* if there exists a transitive Lie subalgebroid $B \subset T\mathbb{R} \times A$ such that $\upsilon_x \in B_{t|x} \Longleftrightarrow (\theta_t, \upsilon_x) \in B_{|(t,x)}$ for $t \in \{0, 1\}$, $\theta_t \in T_t\mathbb{R}$ are null vectors. B is called a *subalgebroid joining* B_0 *with* B_1. This relation is closely related to the relation of homotopic subbundles of a principal fibre bundle [K7] and can be used to Lie algebras, i.e. to Lie algebroids over a point.

Theorem 2.4 (The first homotopy independence) *If B_0, $B_1 \subset A$ are homotopic subalgebroids of A and $\nabla : L \to A$ is a flat L-connection in A, then the characteristic homomorphisms $\Delta_{\#(A,B_t,\nabla)} : H(\boldsymbol{g}, B_t) \to H(L)$, $t = 0, 1$, are equivalent in the sense that there exists an isomorphism $\alpha : H(\boldsymbol{g}, B_0) \xrightarrow{\simeq} H(\boldsymbol{g}, B_1)$ of algebras such that $\Delta_{\#(A,B_1,\nabla)} \circ \alpha = \Delta_{\#(A,B_0,\nabla)}$.*

Let H_0, $H_1 : L' \to L$ be homomorphisms of Lie algebroids. By a *homotopy joining* H_0 to H_1 we mean [K6] a (nonstrong) homomorphism of Lie algebroids $H : T\mathbb{R} \times L' \longrightarrow L$ such that $H(\theta_0, \cdot) = H_0$ and $H(\theta_1, \cdot) = H_1$. We say that H_0 and H_1 are *homotopic* if there exists a homotopy joining H_0 to H_1. The homotopy $H : T\mathbb{R} \times L' \longrightarrow L$ determines a chain homotopy operator [K6] which implies that $H_0^{\#} = H_1^{\#} : H(L) \to H(L')$. The relation can be used to homomorphisms of Lie algebras.

Theorem 2.5 (The second homotopy independence) *If ∇_0, $\nabla_1 : L \to A$ are homotopic flat L-connections in A, then the characteristic homomorphisms are equal $\Delta_{\#(A,B,\nabla_0)} = \Delta_{\#(A,B,\nabla_1)}$.*

A finite-dimensional Lie algebra is a Lie algebroid over a point. For a pair $(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{h} \subset \mathfrak{g}$, of finite-dimensional Lie algebras, we have a characteristic homomorphism

$$\Delta_{o\#} : H(\mathfrak{g}, \mathfrak{h}) = H((\bigwedge(\mathfrak{g}/\mathfrak{h})^*)_I, \bar{\delta}) \to H(\mathfrak{g})$$

$$(\Delta_o\psi)\,(w_1\wedge ...\wedge w_k) = (-1)^k\,\langle\psi,[w_1]\wedge ...\wedge[w_k]\rangle$$

$(H\,(\mathfrak{g},\mathfrak{h}) = H\,(\mathfrak{g},G')$ for arbitrary connected Lie group G' having $\mathfrak{h}$ as its Lie algebra). The homomorphism $\Delta_{o\#}$ can be nontrivial in general. For example for $\mathfrak{g}=\mathfrak{gl}\,(n,\mathbb{R})$ and $\mathfrak{h}=\mathfrak{so}\,(n)$ $(=\mathrm{Sk}\,(n,\mathbb{R}))$ just "the trace" $\mathrm{tr}:\mathfrak{g}/\mathfrak{h}\to\mathbb{R}$ is invariant and gives a nontrivial element in the cohomology and $\Delta_{o\#}\,([\mathrm{tr}])\neq 0$. If $\mathfrak{h}$ is reductive in $\mathfrak{g}$ and $\mathfrak{h}$ is noncohomologous to zero in $\mathfrak{g}$ the homomorphism $\Delta_{o\#}$ is a monomorphism, see [G-H-V, Vol. III, Th. IX, X]. Tables I-III in [G-H-V, Vol. III, Sec.XI] contain many n.c.z. pairs, for example $(\mathfrak{gl}\,(2m+1,\mathbb{R})\,,\mathfrak{so}\,(2m+1))\,,$ $(\mathfrak{so}\,(n,\mathbb{C})\,,\mathfrak{so}\,(k,\mathbb{C}))$ for $k<n$, $(\mathfrak{so}\,(2m+1)\,,\mathfrak{so}\,(2k+1))$ and $(\mathfrak{so}\,(2m)\,,\mathfrak{so}\,(2k+1))$ for $k<m$. If $\mathfrak{h}=0$ then $\Delta_{o\#}=-\,\mathrm{id}_{H(\mathfrak{g})}\,.$

The "partially flat" characteristic homomorphism for the triple (A,B,λ'), where $B\subset A$ are transitive Lie algebroids and $\lambda':TM\to A$ is a connection in A, partially flat over a regular foliation F were (in particular) considered in [K9]. The triple (A,B,λ') determines an object investigated in our paper, $(A^F,B^F,\lambda'|F)$, in which $A^F=\#_A^{-1}\,[F]\,,\ B^F=\#_B^{-1}\,[F]$ are regular Lie algebroids over (M,F) and $\lambda'|F:F\to A^F$ is a flat connection in A^F. With the objects (A,B,λ') and $(A^F,B^F,\lambda'|F)$ we have associated two (secondary) characteristic homomorphisms: $\Delta_{q'\#}:H(\mathcal{W}(\boldsymbol{g},\boldsymbol{h})_{q',I_B})\to H_{dR}\,(M)\,,\ q'\geq\operatorname{codim}F$, [K9, s.4.7] and $\Delta_{\#}:H(\boldsymbol{g},B')\to H\,(F)$ (see also [K7, Prop.3.3]). On the level of forms we have a simple relations between them, see [K9], described by the following diagram:

$$\begin{tikzcd}
(\mathrm{Sec}\textstyle\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*)_{I_B} \arrow[r, hook] \arrow[d] & \mathcal{W}(\boldsymbol{g},\boldsymbol{h})_{q',I_B} \arrow[r, "\Delta_{q'*}"] & \Omega(M) \arrow[d] \\
(\mathrm{Sec}\textstyle\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*)_{I_{B'}} \arrow[rr, "\Delta_*"'] & & \Omega_F(M').
\end{tikzcd}$$

Problem 2.6 What does the relation above look like on the level of cohomology?

The above can be applied to the partially flat Bott connection ω for any regular foliation F which is of course globally flat if we consider it in the regular Lie algebroid $A\,(TM/F)^F$ over $(M,F)\,.$

2.3 Application to principal fibre bundles

Taking a **connected** principal fibre bundle $P=(P,\pi,M,G)$ with a structure Lie group G and a **connected** G'-reduction $P'\subset P$ and using the isomorphism of algebras κ (i.e. the superposition of (1.2) and (1.3) for suitable representations) we define the homomorphism $\Delta_{\#(P,P')}$ by the commutative diagram

$$
\begin{array}{ccccc}
H(\mathfrak{g},G') & \overset{\Delta_{\#(P,P')}}{\longrightarrow} & H^r_{dR}(P) & \longrightarrow & H_{dR}(P) \\
\cong\downarrow\kappa & & \| & & \\
H(\boldsymbol{g},A(P')) & \overset{\Delta_{o\#}}{\longrightarrow} & H(A(P)) & &
\end{array}
\tag{2.4}
$$

and we obtain

Theorem 2.7 *If G is a compact connected group and P' is a connected G'-reduction in an G-pfb P, then there exists a "universal" characteristic homomorphism $\Delta_{\#(P,P')} : H(\mathfrak{g},G') \longrightarrow H_{dR}(P)$ acting from the algebra $H(\mathfrak{g},G')$ to the total cohomology $H_{dR}(P)$. In the case of a flat principal fibre bundle P the characteristic homomorphism $\Delta_{\#(P,P',\omega)} : H(\mathfrak{g},G') \longrightarrow H_{dR}(M)$ for every flat connection ω in P is factorized by $\Delta_{\#(P,P')}$, i.e. the diagram below commutes*

$$
\begin{tikzcd}
 & H_{dR}(P) \arrow[dr, "\omega^{\#}"] & \\
H(\mathfrak{g},G') \arrow[ur, "\Delta_{\#(P,P')}"] \arrow[rr, "\Delta_{\#(P,P',\omega)}"'] & & H_{dR}(M)
\end{tikzcd}
$$

where $\omega^{\#}$ on the level of right-invariant forms Ω^r is given as the pullback of forms, $\omega^* : \Omega^r(P) \longrightarrow \Omega(M)$, $\omega^*(\phi)(x;u_1\wedge\ldots\wedge u_k) = \phi(z;\tilde{u}_1\wedge\ldots\wedge\tilde{u}_k)$ where $z \in P_{|x}$, $\tilde{u}_i$ is the horizontal lift of u_i [recall that $H^r_{dR}(P) := H(\Omega^r(P)) \simeq H_{dR}(P)$]. In the general case (noncompact or nonconnected Lie group G) there exists a homomorphism $\Delta_{o\#} : H(\mathfrak{g},G') \longrightarrow H^r_{dR}(P)$ of algebras which factorizes the characteristic homomorphism for every flat connection. The homomorphism $\Delta_{o\#}$ on the level of forms is given by the following direct formula

$$
(\Delta_o\psi)_z(w_1\wedge\ldots\wedge w_k) = \left\langle \psi, [-\breve{\lambda}_z(w_1)]\wedge\ldots\wedge[-\breve{\lambda}_z(w_k)] \right\rangle,
$$

where $\breve{\lambda}$ is the connection form of a connection λ on P extending an arbitrary connection on P'.

The following questions seems to be interesting:

— Is the homomorphism $\Delta_{o\#(P,P')} : H(\mathfrak{g},G') \longrightarrow H^r_{dR}(P)$ a monomorphism?

2.4 Crainic characteristic classes

Take a vector bundle $\mathfrak{f}$ and its Lie algebroid $A(\mathfrak{f})$ as well as a Riemannian metric h in $\mathfrak{f}$. The metric h yields the Lie subalgebroid $B = A(\mathfrak{f},\{h\})$. We recall that $\mathcal{L} \in \mathrm{Sec}(A(\mathfrak{f},\{h\})) \Longleftrightarrow \mathcal{L} \in \mathrm{Sec}(A(\mathfrak{f}))$ and for each cross-sections $\xi,\eta \in \mathrm{Sec}\,\mathfrak{f}$ the formula holds $h(\mathcal{L}(\xi),\eta) = (\#\mathcal{L})(h(\xi,\eta)) - h(\xi,\mathcal{L}(\eta))$. Two Lie subalgebroids

$B_i = A(\mathfrak{f}, \{h_i\})$, $i = 1, 2$, corresponding to Riemannian metrics h_i are homotopic Lie subalgebroids. The Atiyah sequences for $A(\mathfrak{f})$ and $A(\mathfrak{f}, \{h\})$ are

$$0 \longrightarrow \mathrm{End}(\mathfrak{f}) \longrightarrow A(\mathfrak{f}) \longrightarrow TM \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{Sk}(\mathfrak{f}) \longrightarrow A(\mathfrak{f}, \{h\}) \longrightarrow TM \longrightarrow 0.$$

$\mathrm{Sk}(\mathfrak{f})$ denotes the vector bundle of skew symmetric endomorphisms with respect to the metric h.

(A) If the vector bundle $\mathfrak{f}$ is nonorientable, then the characteristic homomorphism $\Delta_{\#} : H(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h\})) \to H(L)$ for the triple $(A(\mathfrak{f}), A(\mathfrak{f}, \{h\}), \nabla)$ produces the Crainic characteristic classes [C]. Indeed, using the isomorphism κ, see (2.4), and the classical relations ([K-T], [G]) we have

$$H(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h\})) \overset{\kappa}{\cong} H(\mathfrak{gl}(n, \mathbb{R}), O(n)) \cong \bigwedge(y_1, y_3, ..., y_{n'})$$

where n' is the largest odd integer $\leq n = \mathrm{rank}\,\mathfrak{f}$, and by definition

$$y_{2k-1} \in H^{4k-3}(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h\}))$$

are represented by the multilinear trace form $\tilde{y}_k \in \mathrm{Sec}(\bigwedge(\mathrm{End}\,\mathfrak{f}/\,\mathrm{Sk}\,\mathfrak{f})^*)$

$$\tilde{y}_{2k-1}([A_1], ..., [A_{4k-3}]) = \sum_{\sigma \in S_{4k-3}} \mathrm{sgn}\,\sigma \cdot \mathrm{tr}(\tilde{A}_{\sigma(1)} \circ ... \circ \tilde{A}_{\sigma(4k-3)}) \tag{2.5}$$

where $\tilde{A}_i = \frac{1}{2}(A_i + A_i^*)$ is the symmetrization of A_i with respect to the inner scalar product induced by the metric h.

(B) In the case of oriented vector bundle with a metric volume v, the metric h and v induce an $SO(n, \mathbb{R})$-reduction $L(\mathfrak{f}, \{h, \mathrm{v}\})$ of the frames bundle $L\mathfrak{f}$ of $\mathfrak{f}$. Consider the characteristic homomorphism $\Delta_{\#} : H(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h, \mathrm{v}\})) \to H(L)$ corresponding to $(A(\mathfrak{f}), A(\mathfrak{f}, \{h, \mathrm{v}\}), \nabla)$. Therefore

— if n is odd, then

$$H(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h, \mathrm{v}\})) \cong H(\mathfrak{gl}(n, \mathbb{R}), SO(n)) = H(\mathfrak{gl}(n, \mathbb{R}), O(n)),$$

— if n is even, $n = 2m$, then

$$H(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h, \mathrm{v}\})) \cong H(\mathfrak{gl}(2m, \mathbb{R}), SO(2m)) \cong \bigwedge(y_1, y_3, ..., y_{n'}, \chi_{2m})$$

where $n' = 2m - 1$, $y_{2k-1} \in H^{4k-3}(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h, \mathrm{v}\}))$ are represented by the multilinear trace form $\tilde{y}_k \in \mathrm{Sec}(\bigwedge(\mathrm{End}\,\mathfrak{f}/\,\mathrm{Sk}\,\mathfrak{f})^*)$ defined by (2.5), and $\chi_{2m} \in H^{2m}(\mathrm{End}\,\mathfrak{f}, A(\mathfrak{f}, \{h, \mathrm{v}\}))$ is represented by the form $\tilde{y}_{2m} \in \mathrm{Sec}(\bigwedge^{2m}(\mathrm{End}\,\mathfrak{f}/\,\mathrm{Sk}\,\mathfrak{f})^*)$

$$\tilde{y}_{2m}([A_1], ..., [A_{2m}]) = d(z_{2m-1})(\widetilde{A}_1, ..., \widetilde{A}_{2m}).$$

d is the differential on the algebra $\bigwedge(\operatorname{End}\mathfrak{f})^*$,

$$d(\phi)(A_1,...,A_n)=\sum_{1\le p<q\le n}(-1)^{p+q}\,\phi([A_p,A_q],A_1,...\hat{p}...\hat{q}...,A_n)$$

and $z_{2m-1}\in\operatorname{Sec}(\bigwedge^{2m-1}(\operatorname{End}\mathfrak{f})^*)$ is described by the formula

$$\begin{aligned}&z_{2m-1}(A_1,....,A_{2m-1})\\&=\sum_{\sigma\in S_{2m-1}}\operatorname{sgn}\sigma\cdot(e,\alpha(A_{\sigma(1)})\wedge\alpha[A_{\sigma(2)},A_{\sigma(3)}]\wedge...\wedge\alpha[A_{\sigma(2m-2)},A_{\sigma(2m-1)}]),\end{aligned}$$

e is a non-zero cross-section of $\bigwedge^{2m}\mathfrak{f}$, $\alpha:\operatorname{End}\mathfrak{f}\to\bigwedge^2\mathfrak{f}$ is given by

$$(\alpha(A),X\wedge Y)=\frac{1}{2}((AX,Y)-(X,AY)),$$

$A\in\operatorname{End}\mathfrak{f}$, $X,Y\in\operatorname{Sec}\mathfrak{f}$; observe that $\alpha|\operatorname{Sym}\mathfrak{f}=0$ and $\alpha|\operatorname{Sk}\mathfrak{f}:\operatorname{Sk}\mathfrak{f}\xrightarrow{\cong}\bigwedge^2\mathfrak{f}$ $((\alpha(A),X\wedge Y)=(AX,Y))$ is an isomorphism; see [G-H-V, Vol.III, p.257 and Appendix A].

Theorem 2.8

$$\Delta_{\#}(\tilde{y}_{2k-1})=(-1)^k\cdot\frac{(4k-3)!}{2^{4k-3}\cdot(2k-1)!\cdot(2k-2)!}\cdot[u_{4k-3}(\mathfrak{f},\nabla)]$$

where $u_{4k-3}(\mathfrak{f},\nabla)$ *represent the Crainic characteristic classes.*

An explicit formula uses any metric h in $\mathfrak{f}$ and the symmetric-values form $\theta=\nabla^h-\nabla$ where ∇ is any flat L-connection in $\mathfrak{f}$ and ∇^h is the adjoint L-connection induced by the metric h, $u_{2p-1}(\mathfrak{f},\nabla)=(-1)^{\frac{p+1}{2}}\operatorname{cs}_p(\nabla,\nabla^h)$, p is odd (let us remark that only odd p gives nontrivial classes for real $\mathfrak{f}$) and

$$\operatorname{cs}_p(\nabla,\nabla^h)=\int_0^1\operatorname{ch}_p(\nabla^{\text{aff}})=(-1)^{p-1}\frac{p!\cdot(p-1)!}{(2p-1)!}\cdot\operatorname{tr}(\underbrace{\theta\wedge...\wedge\theta}_{2p-1})\in\Omega^{2p-1}(L)$$

for the affine combination $\nabla^{\text{aff}}=(1-t)\cdot\nabla+t\cdot\nabla^h$ and $\operatorname{ch}_p(\nabla^{\text{aff}})=\operatorname{tr}(R^{\nabla^{\text{aff}}})^p$. We must add that M.Crainic [C] has lost the class χ_{2m} for oriented vector bundles of even rank $2m$.

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