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Lie Algebroid Associated with an Almost Dirac Structure

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Abstract

We show that to an almost Dirac structure of a manifold, there associates a Lie algebroid. In the case of a Poisson manifold, this Lie algebroid coincides with the usual cotangent Lie algebroid with Lie algebra bracket on the space of one-forms.

1 Introduction

Let π be an arbitrary 2-vector field on M, i.e. a smooth section of $\wedge^2(\mathrm{T}M)$. We denote by $\tilde{\pi}$, the bundle homomorphism $\mathrm{T}^*M \to \mathrm{T}M$ defined by $\alpha_x \mapsto \pi(\alpha_x, \cdot)$ ($x \in M$). By an abuse of notations, we denote by the same letter $\tilde{\pi}$, the homomorphism $\Gamma(\mathrm{T}^*M) \to \Gamma(\mathrm{T}M)$ between sections. For the Schouten bracket $[\pi, \pi]$ of π, which is a 3-vector field, we define $\ker[\pi, \pi] = \{\alpha \in \mathrm{T}^*M \mid [\pi, \pi](\alpha, \cdot, \cdot) = 0\}$. If $\ker[\pi, \pi]$ forms a bundle of constant rank, it was proved in [7] that $\ker[\pi, \pi]$ becomes a Lie algebroid with respect to the bracket $L_{\tilde{\pi}(\alpha)}\beta - L_{\tilde{\pi}(\beta)}\alpha - d(\pi(\alpha, \beta))$ and the anchor $\rho(\alpha) = \tilde{\pi}(\alpha)$. Clearly, it coincides with the usual Lie algebroid structure of T^*M of a Poisson manifold (M, π), where $[\pi, \pi] = 0$. On the other hand, the graph of $\tilde{\pi} : \mathrm{T}^*M \to \mathrm{T}M$ defines a sub-bundle of $\mathrm{T}M \oplus \mathrm{T}^*M$, which is an almost Dirac structure (see Section 1), and $\ker[\pi, \pi]$ can be identified with a subset of this almost Dirac structure. The aim of this paper is, generalizing the above result, to show that a certain sub-bundle $\mathcal{L}_0$ of an almost Dirac structure is a Lie algebroid with respect to the bracket and the anchor, which are naturally defined on the almost Dirac structure (Theorem 2.1). The sub-bundle $\mathcal{L}_0$ is given as the kernel of the 3-tensor field T restricted to the almost Dirac structure, introduced in [1] (see Definition 2.2).

In Section 1, we review some basic facts on Dirac structures and prove that $\mathcal{L}_0$ is a Lie algebroid. In Section 2, in order to clarify the conditions under which an element belongs to $\ker T$, we use the description of an almost Dirac structure by means of a "2-vector field on a sub-bundle of T^*M". In Section 3, we give a 'dual' description of Dirac structures in which we use "2-forms" defined on a sub-bundle of $\mathrm{T}M$. We also give simple examples.

It is possible to generalize our result in the case of deformed bracket in [4] or [3], and also seems highly possible in the case of the twisted Poisson structures

[8]. However, we restricted ourselves to the case of the ordinary Dirac structures in order to make the arguments and the computations clear. We hope interesting examples will come about from the further generalizations.

2 Dirac Structures

Let $\mathrm{T}(M)$ and $\mathrm{T}^*(M)$ be the tangent and the cotangent bundle of M, respectively. Let $\langle\cdot,\cdot\rangle_+$ be the symmetric pairing on $\mathrm{T}(M)\oplus\mathrm{T}^*(M)$ defined by

$$\langle(X_x,\alpha_x),(Y_x,\beta_x)\rangle_+ = \alpha_x(Y_x)+\beta_x(Y_x),\quad (X_x,\alpha_x),(Y_x,\beta_x)\in \mathrm{T}_xM\oplus\mathrm{T}^*_xM.$$

Definition 2.1 (T. Courant). A smooth sub-bundle $\mathcal{L}\subset\mathrm{T}(M)\oplus\mathrm{T}^*(M)$ is an almost Dirac structure if $\mathcal{L}$ is maximally isotropic with respect to the pairing $\langle\cdot,\cdot\rangle_+$. This means $\mathcal{L}$ is a sub-bundle of rank $n(=\dim M)$ and the restriction of $\langle\cdot,\cdot\rangle_+$ to $\mathcal{L}\times\mathcal{L}$ vanishes identically.

Remark 2.1. In [1], an almost Dirac structure is called a Dirac structure, however we use the word *Dirac structure* to mean the one which was called an *integrable* Dirac structure in [1].

On $\Gamma\left(\mathrm{T}(M)\oplus\mathrm{T}^*(M)\right)$, we have a bracket defined by
(2.1)
$$[\![(X_1,\alpha_1),(X_2,\alpha_2)]\!] = \left([X_1,X_2]\ ,\ L_{X_1}\alpha_2 - L_{X_2}\alpha_1 + \frac{1}{2}d\left(\alpha_1(X_2)-\alpha_2(X_1)\right)\right)$$

where $[X_1,X_2]$ is the usual Lie bracket of vector fields and $L_X\alpha$ is the Lie derivative of 1-form α with respect to the vector field X.

The bracket $[\![(X_1,\alpha_1),(X_2,\alpha_2)]\!]$ is skew-symmetric but does not satisfy the Jacobi identity. Indeed, let (J_1,J_2) denote the Jacobiator

$$(J_1,J_2) = [\![[\![(X_1,\alpha_1),(X_2,\alpha_2)]\!],(X_3,\alpha_3)]\!] + c.p.\ .$$

Clearly $J_1=0$. As for J_2, however, we have

Proposition 2.1. *The second component J_2 of the above Jacobiator is given by*

$$J_2 = \frac{1}{4}d\left(2\alpha_1([X_2,X_3]) + L_{X_1}\left(\alpha_2(X_3)-\alpha_3(X_2)\right)\right) + c.p.\ .$$

Especially, the restriction of J_2 to an almost Dirac structure $\mathcal{L}$ is

$$\frac{1}{2}d\big(\alpha_1([X_2,X_3]) + L_{X_1}(\alpha_2(X_3))\big) + c.p.\ .$$

Proof. This is shown directly from the definitions of $\langle\cdot,\cdot\rangle_+$ and $[\![\cdot,\cdot]\!]$. □

Definition 2.2. An almost Dirac structure $\mathcal{L}$ is called a(n) (*integrable*) *Dirac structure* if $\Gamma(\mathcal{L})$ is closed under the bracket $[\![\cdot,\cdot]\!]$.

In [1], Courant introduced the $\boldsymbol{R}$-tri-linear map on $\mathrm{T}(M) \oplus \mathrm{T}^*(M)$ to $\boldsymbol{R}$ defined by $T((X_1,\alpha_1),(X_2,\alpha_2),(X_3,\alpha_3)) = \langle [\![(X_1,\alpha_1),(X_2,\alpha_2)]\!],(X_3,\alpha_3)\rangle_+$ for $(X_i,\alpha_i) \in \mathrm{T}(M)\oplus\mathrm{T}^*(M)$ $(i = 1,2,3)$, and showed that an almost Dirac structure is integrable if and only if T restricted to $\mathcal{L}$ vanishes: $T|_{\mathcal{L}} \equiv 0$. We note that the restriction $T|_{\mathcal{L}}$ has the tensor property. That is $T|_{\mathcal{L}}$ is tri-linear over $C^\infty(M)$.

Proposition 2.2. *Let $\mathcal{L}$ be an almost Dirac structure. Then $T|_{\mathcal{L}}$, T restricted to $\mathcal{L}$, is computed as*

$$T|_{\mathcal{L}}((X_1,\alpha_1),(X_2,\alpha_2),(X_3,\alpha_3)) = \big(\alpha_1([X_2,X_3]) + L_{X_1}(\alpha_2(X_3))\big) + c.p.$$

and

$$J_2|_{\mathcal{L}}((X_1,\alpha_1),(X_2,\alpha_2),(X_3,\alpha_3)) = \frac{1}{2}d\left(T|_{\mathcal{L}}((X_1,\alpha_1),(X_2,\alpha_2),(X_3,\alpha_3))\right).$$

Proof. On $\mathcal{L}$, we have

$$\begin{aligned}
&\langle [\![(X_1,\alpha_1),(X_2,\alpha_2)]\!],(X_3,\alpha_3)\rangle_+ \\
&\quad = (L_{X_1}\alpha_2)(X_3) - (L_{X_2}\alpha_1)(X_3) - \frac{1}{2}L_{X_3}(\alpha_2(X_1) - \alpha_1(X_2)) + \alpha_3([X_1,X_2]) \\
&\quad = L_{X_1}(\alpha_2(X_3)) - \alpha_2([X_1,X_3]) - L_{X_2}(\alpha_1(X_3)) + \alpha_1([X_2,X_3]) \\
&\qquad\qquad - L_{X_3}(\alpha_2(X_1)) + \alpha_3([X_1,X_2]).
\end{aligned}$$

This together with Proposition 2.1 shows Proposition 2.2. □

Let $\mathcal{L}$ be an almost Dirac structure. We consider the 'sub-bundle' $\mathcal{L}_0$ of $\mathcal{L}$ consisting of the elements in $\ker T|_{\mathcal{L}}$. More precisely, we put

$$\mathcal{L}_0 = \{e = (Z,\gamma) \in \mathcal{L} \ \mid\ T(e_1,e_2,e) = 0,\ e_1,e_2 \in \mathcal{L}\}.$$

Since T restricted to $\mathcal{L}$, is skew-symmetric with respect to all the arguments, $\mathcal{L}_0$ can be considered as the kernel of the bundle map $T : \mathcal{L} \to \wedge^2\mathcal{L}^*,\ e \mapsto T(\cdot,\cdot,e)$. Since the fiber dimension of $\mathcal{L}_0$ may change from point to point, to get a Lie algebroid, we have to restrict $\mathcal{L}_0$ to a submanifold of M where $\mathcal{L}_0$ is of constant rank. Hereafter, for simplicity, *we assume that $\mathcal{L}_0$ is a bundle of constant rank* on whole M. The following proposition is obvious from Proposition 2.2.

Proposition 2.3. *If one of e_1,e_2,e_3 in $\Gamma(\mathcal{L})$ is an element in $\Gamma(\mathcal{L}_0)$, we have the Jacobi identity:*

$$[\![[\![e_1,e_2]\!],e_3]\!] + [\![[\![e_2,e_3]\!],e_1]\!] + [\![[\![e_3,e_1]\!],e_2]\!] = 0.$$

The following proposition is used to show that $\Gamma(\mathcal{L}_0)$ is closed under the bracket $[\![\cdot,\cdot]\!]$.

Proposition 2.4. *For* $e = (Z, \gamma) \in \Gamma(\mathcal{L}_0)$ *and* $e_1 = (Y, \beta) \in \Gamma(\mathcal{L})$*, we have* $[\![e, e_1]\!] \in \Gamma(\mathcal{L})$.

Proof. Since T restricted to $\mathcal{L}$ is skew symmetric, we have $\langle [\![e, e_1]\!], e_2\rangle_+ = T(e, e_1, e_2) = T(e_1, e_2, e) = 0$, for any $e_1, e_2 \in \Gamma(\mathcal{L})$. By the maximality of $\mathcal{L}$, we can conclude $[\![e, e_1]\!]$ is in $\Gamma(\mathcal{L})$. □

By the above propositions, we obtain the following theorem.

Theorem 2.1. *Let* $\mathcal{L}$ *be an almost Dirac structure and* $\mathcal{L}_0$ *the kernel of* T*, which we assume a sub-bundle of* $\mathcal{L}$*. Then* $\mathcal{L}_0$ *is a Lie algebroid with respect to the bracket* $[\![\cdot,\cdot]\!]$ *and the anchor* $\rho_{\mathcal{L}_0}$*, which is the natural projection* $\rho : \mathrm{T}(M) \oplus \mathrm{T}^*(M) \to \mathrm{T}(M)$ *restricted to* $\mathcal{L}_0$*.*

Proof. Let e_1, e_2 be two elements of $\Gamma(\mathcal{L}_0)$. Then for any e_3 and e_4 in $\Gamma(\mathcal{L})$, we have

$$\begin{aligned} T([\![e_1, e_2]\!], e_3, e_4) &= \langle [\![[\![e_1, e_2]\!], e_3]\!], e_4\rangle_+ = \langle [\![[\![e_1, e_3]\!], e_2]\!] + [\![e_1, [\![e_2, e_3]\!]]\!], e_4\rangle_+ \\ &= T([\![e_1, e_3]\!], e_2, e_4) + T(e_1, [\![e_2, e_3]\!], e_4) = 0. \end{aligned}$$

The second equality holds because of the Jacobi identity (Proposition 2.3) for e_1, e_2, e_3 and the last one is true because $[\![e_1, e_3]\!], [\![e_2, e_3]\!]$ are both in $\Gamma(\mathcal{L})$ by Proposition 2.4. This shows that $\Gamma(\mathcal{L}_0)$ is closed under the bracket. Since the Jacobi identity is obvious for the elements in $\Gamma(\mathcal{L}_0)$(Proposition 2.3), $[\![\cdot,\cdot]\!]$ is a Lie algebra bracket on $\Gamma(\mathcal{L}_0)$. That $\rho_{\mathcal{L}_0}$ satisfies the condition of an anchor map is also verified directly from the definition (2.1) of $[\![\cdot,\cdot]\!]$. □

3 An alternative description of a Dirac structure

In this section and the next, we give alternative descriptions of an almost Dirac structure and give more explicit conditions for an element in $\mathcal{L}$ to be in the kernel of 3-tensor $T|_{\mathcal{L}}$.

Let $\mathcal{L}$ be an almost Dirac structure on M and $\rho_{\mathcal{L}}$ and $\rho^*_{\mathcal{L}}$ denote the restriction of the natural projections $\mathrm{T}(M) \oplus \mathrm{T}^*(M) \to \mathrm{T}(M)$ and $\mathrm{T}(M) \oplus \mathrm{T}^*(M) \to \mathrm{T}^*(M)$ to $\mathcal{L}$, respectively. We put $\mathcal{E} = \mathrm{Im}\rho_{\mathcal{L}}$ and $\mathcal{A} = \mathrm{Im}\rho^*_{\mathcal{L}}$. As was remarked before, the fiber rank of $\mathcal{E}$ as well as $\mathcal{A}$, may not be constant. To justify our computations we only treat the case when $\mathcal{E}$ and $\mathcal{A}$ are bundles of constant rank.

Take an element $\alpha \in \mathcal{A}_x$ (the fiber over $x \in M$), then for some $X \in \mathrm{T}(M)$, (X, α) lies in $\mathcal{L}$. That $\mathcal{L}$ is isotropic implies that the restriction $X|_{\mathcal{A}_x}$ of X considered as an element in $\mathcal{A}^*_x$ (dual space) depends only on α and we obtain an well-defined fiber map $\pi : \mathcal{A} \to \mathcal{A}^*$ (see [1]). We may consider π as a '2-vector field' defined on (each fiber of) $\mathcal{A}$. It is skew-symmetric since for $\alpha, \beta \in \mathcal{A}$, we have

$$\pi(\alpha, \beta) = X(\beta) = -Y(\alpha) = -\pi(\beta, \alpha), \quad \text{where } (Y, \beta) \in \mathcal{L}.$$

From the sub-bundle $\mathcal{A} = \mathrm{Im}\rho^*_{\mathcal{L}}$ and a skew-symmetric 2-field π on $\mathcal{A}$, we can recover $\mathcal{L}$ as a bundle given by

$$\mathcal{L}' = \{(X, \theta) \in \mathrm{T}(M) \oplus \mathrm{T}^*(M) \mid \ \theta \in \mathcal{A},\ \tilde{\pi}(\theta) = X|_{\mathcal{A}}\}.$$

Indeed it is easy to see that $\mathcal{L}'$ is a vector bundle of rank $n(= \dim M)$. That $\mathcal{L}'$ is isotropic with respect to $\langle \cdot, \cdot \rangle_+$ follows from the skewness of π. If $(X', \alpha') \in \mathcal{L}$, we see

$$\begin{aligned} \langle (X', \alpha'), (X, \theta) \rangle_+ =& \alpha'(X) + \theta(X') = \pi(\theta, \alpha') + \theta(X') \\ =& - \pi(\alpha', \theta) + \theta(X') = -X'(\theta) + \theta(X') = 0 \end{aligned}$$

for $(X, \theta) \in \mathcal{L}'$. This together with the maximality of $\mathcal{L}$ implies $\mathcal{L} = \mathcal{L}'$.

Now we are going to characterize the element of $\mathcal{L}_0$ in terms of π and $\mathcal{A}$, where $\mathcal{L}_0$ is the sub-bundle $\ker T|_{\mathcal{L}}$ of $\mathcal{L}$. First, we observe that $\mathcal{A}^*$ is a quotient bundle of $\mathrm{T}(M)$ by the sub-bundle $\mathcal{A}^\circ$, where $\mathcal{A}^\circ$ is the bundle consisting of the annihilators of $\mathcal{A}$. π is an element $\wedge^2\mathcal{A}^*$, however we choose and fix a splitting to the projection $\mathrm{T}(M) \to \mathcal{A}^*$, and consider $\mathcal{A}^*$ as a direct summand of $\mathrm{T}(M)$, obtaining a 2-vector field which extends π. This is possible since we are assuming $\mathcal{A}$ is of constant rank. We denote this extended 2-vector field by the same letter π, since we hope this will not cause any confusion. Then $\mathcal{L}$ is given by

$$\mathcal{L} = \{(X, \alpha) \ \mid \ \alpha \in \mathcal{A}, \tilde{\pi}(\alpha) = X|_{\mathcal{A}}\} = \{(\tilde{\pi}(\alpha) + \bar{X}, \alpha) \ \mid \ \alpha \in \mathcal{A}, \bar{X} \in \mathcal{A}^\circ\}. \tag{3.1}$$

For $e_1 = (X, \alpha), e_2 = (Y, \beta)$ and $e_3 = (Z, \gamma)$ in $\mathcal{L}$, we look for the condition on e_3 under which $T(e_1, e_2, e_3) = 0$ holds for all $e_1, e_2 \in \mathcal{L}$. We can write $e_1 = (\tilde{\pi}(\alpha) + \bar{X}, \alpha), e_2 = (\tilde{\pi}(\beta) + \bar{Y}, \beta)$ and $e_3 = (\tilde{\pi}(\gamma) + \bar{Z}, \gamma)$, respectively, where $\bar{X}, \bar{Y}, \bar{Z} \in \mathcal{A}^\circ$. With these notations, we have

$$\begin{aligned} [\![(X, \alpha), (Y, \beta)]\!] = \Big(& [\tilde{\pi}(\alpha),\tilde{\pi}(\beta)] + [\tilde{\pi}(\alpha), \bar{Y}] + [\bar{X}, \tilde{\pi}(\beta)] + [\bar{X}, \bar{Y}], \\ & L_{\tilde{\pi}(\alpha)}\beta + L_{\tilde{\pi}(\beta)}\alpha - d(\pi(\alpha, \beta)) + L_{\bar{X}}\beta - L_{\bar{Y}}\alpha \Big). \end{aligned}$$

Writing $\{\alpha, \beta\}_\pi$ for $L_{\tilde{\pi}(\alpha)}\beta - L_{\tilde{\pi}(\beta)}\alpha - d(\pi(\alpha, \beta))$ and making the pairing $\langle \cdot, \cdot \rangle_+$ of the above element and $(Z, \gamma) = (\tilde{\pi}(\gamma) + \bar{Z}, \gamma)$, we obtain

$$\begin{aligned} & [\tilde{\pi}(\alpha), \tilde{\pi}(\beta)](\gamma) + [\tilde{\pi}(\alpha), \bar{Y}](\gamma) + [\bar{X}, \tilde{\pi}(\beta)](\gamma) + [\bar{X}, \bar{Y}](\gamma) \\ & \quad + \pi(\gamma, \{\alpha, \beta\}_\pi) + \bar{Z}(\{\alpha, \beta\}_\pi) + \pi(\gamma, L_{\bar{X}}\beta - L_{\bar{Y}}\alpha) + \bar{Z}(L_{\bar{X}}\beta - L_{\bar{Y}}\alpha), \end{aligned} \tag{3.2}$$

which is nothing but $T(e_1, e_2, e_3)$.

If we choose $\bar{X} = \bar{Y} = 0$, then (3.2) gives

$$[\tilde{\pi}(\alpha), \tilde{\pi}(\beta)](\gamma) + \pi(\gamma, \{\alpha, \beta\}_\pi) + \bar{Z}(\{\alpha, \beta\}_\pi) = 0, \tag{3.3}$$

for $\alpha, \beta \in \mathcal{A}$. We put $\bar{Y} = 0$ and $\alpha = 0$ in (3.2), we obtain

$$[\bar{X}, \tilde{\pi}(\beta)](\gamma) + \pi(\gamma, L_{\bar{X}}\beta) + \bar{Z}(L_{\bar{X}}\beta) = 0, \quad \bar{X} \in \mathcal{A}^\circ, \beta \in \mathcal{A}. \tag{3.4}$$

If we put $\alpha = \beta = 0$ into (3.2), we get $\gamma([\bar{X}, \bar{Y}]) = 0$ ($\bar{X}, \bar{Y} \in \mathcal{A}^\circ$). It is easy to see that this is equivalent to

$$L_{\bar{X}}\gamma \in \mathcal{A}, \quad \bar{X} \in \mathcal{A}^\circ. \tag{3.5}$$

Conversely, it can also be seen that if $(Z, \gamma) = (\tilde{\pi}(\gamma) + \bar{Z}, \gamma)$ satisfies conditions (3.3), (3.4) and (3.5) then (3.2) vanishes identically.

In the following, we will simplify the conditions (3.3) and (3.4). First, we note (3.4) is equivalent to the following:

$$(L_{\bar{X}}\pi)(\beta, \gamma) + \pi(L_{\bar{X}}\beta, \gamma) + \pi(\gamma, L_{\bar{X}}\beta) + L_{\bar{X}}(\bar{Z}(\beta)) - [\bar{X}, \bar{Z}](\beta) = 0.$$

Since $\bar{Z}(\beta) = 0$, this means

$$(L_{\bar{X}}\pi)(\gamma) + L_{\bar{X}}\bar{Z} = 0, \quad \text{on } \mathcal{A}. \tag{3.6}$$

To simplify the condition (3.3), we use the following

Lemma 3.1. *For $\bar{Z} \in \mathcal{A}^\circ$ and $\alpha, \beta \in \mathcal{A}$, we have*

$$[\bar{Z}, \pi](\alpha, \beta) = \bar{Z}(\{\alpha, \beta\}_\pi) \ .$$

Proof. By the definition of $\{\alpha, \beta\}_\pi$, we have

$$\begin{aligned} \bar{Z}(\{\alpha, \beta\}_\pi) =& (L_{\tilde{\pi}(\alpha)}\beta)(\bar{Z}) - (L_{\tilde{\pi}(\beta)}\alpha)(\bar{Z}) - L_{\bar{Z}}(\pi(\alpha, \beta)) \\ =& L_{\tilde{\pi}(\alpha)}(\beta(\bar{Z})) - \beta(L_{\tilde{\pi}(\alpha)}\bar{Z}) - L_{\tilde{\pi}(\beta)}(\alpha(\bar{Z})) \\ & + \alpha(L_{\tilde{\pi}(\beta)}\bar{Z}) - L_{\bar{Z}}(\pi(\alpha, \beta)) \end{aligned}$$

(since $\alpha(\bar{Z}) = \beta(\bar{Z}) = 0$)

$$\begin{aligned} =& - \alpha(L_{\bar{Z}}(\tilde{\pi}(\beta))) - \tilde{\pi}(\alpha)(L_{\bar{Z}}\beta) \\ =& - \alpha([\bar{Z}, \tilde{\pi}])(\beta)) = [\bar{Z}, \pi](\alpha, \beta) \ . \end{aligned}$$

□

Lemma 3.2. *The condition* (3.3) *for* $(Z, \gamma) = (\tilde{\pi}(\gamma) + \bar{Z}, \gamma)$ *can be replaced by the next equality:*

$$[\tilde{\pi}(\gamma), \tilde{\pi}(\beta)] + (L_{\bar{Z}}\tilde{\pi})(\beta) - \tilde{\pi}(\{\gamma, \beta\}_\pi) = 0, \quad \beta \in \mathcal{A},$$

or equivalently by

$$\frac{1}{2}[\pi, \pi](\gamma) + L_{\bar{Z}}\pi = 0 \quad \textit{on} \quad \mathcal{A} \ .$$

Proof. By Lemma 3.1, (3.3) can be replaced by

$$[\tilde{\pi}(\alpha), \tilde{\pi}(\beta)](\gamma) + \pi(\gamma, \{\alpha, \beta\}_\pi) + [\bar{Z}, \pi](\alpha, \beta) = 0 \quad \alpha, \beta \in \mathcal{A}. \tag{3.7}$$

Using the general formula for a 2-vector field (see [7], [9])

$$[\tilde{\pi}(\alpha), \tilde{\pi}(\beta)] = \tilde{\pi}(\{\alpha, \beta\}_\pi) + \frac{1}{2}[\pi, \pi](\alpha, \beta)\ , \tag{3.8}$$

we thus rewrite (3.7) as

$$\frac{1}{2}[\pi, \pi](\alpha, \beta, \gamma) + (L_{\bar{Z}}\pi)(\alpha, \beta) = 0 \quad \text{i.e.,} \quad \frac{1}{2}[\pi, \pi](\gamma) + L_{\bar{Z}}\pi = 0\ .$$

□

From the above lemmas, we can summarize the conditions on $\mathcal{L}_0$ as follows.

Proposition 3.1. *Let $\mathcal{A}$ and $\mathcal{A}^\circ$ be as before and π a skew symmetric bilinear form on $\mathcal{A}$. Let*

$$\mathcal{L} = \{(X, \alpha) \mid \alpha \in \mathcal{A}, \pi(\alpha) = X|_{\mathcal{A}}\} = \{(\tilde{\pi}(\alpha) + \bar{X}, \alpha) \mid \alpha \in \mathcal{A}, \bar{X} \in \mathcal{A}^\circ\}$$

be an almost Dirac structure defined by π. We put

$$\mathcal{L}_0 = \{e = (Z, \gamma) = (\tilde{\pi}(\gamma) + \bar{Z}, \gamma) \in \mathcal{L} \mid T(e_1, e_2, e) = 0,\ e_1, e_2 \in \mathcal{L}\}.$$

Then $(Z, \gamma) = (\tilde{\pi}(\gamma) + \bar{Z}, \gamma) \in \mathcal{L}$ belongs to $\mathcal{L}_0$ if and only if the following conditions (C1)*,* (C2) *and* (C3) *are satisfied:*

(C1) $L_{\bar{X}}\gamma \in \mathcal{A}$, *for all* $\bar{X} \in \mathcal{A}^\circ$,

(C2) $(L_{\bar{X}}\pi)(\gamma) + L_{\bar{X}}\bar{Z} = 0$ *on* $\mathcal{A}$, *for all* $\bar{X} \in \mathcal{A}^\circ$,

(C3) $\frac{1}{2}[\pi, \pi](\gamma) + [\bar{Z}, \pi] = 0$ *on* $\mathcal{A}$.

Example 3.1. Let $\mathcal{A}$ be an arbitrary Pfaffian system. We consider the case when $\pi \equiv 0$. Then

$$\mathcal{L} = \{(X, \alpha) \mid \alpha \in \mathcal{A}, X \in \mathcal{A}^\circ\}\ .$$

(C1) means $L_X\gamma \in \mathcal{A}$ for any $X \in \mathcal{A}^\circ$, and (C2) mean $[X, Z] \in \mathcal{A}^\circ$ for any $X \in \mathcal{A}^\circ$. Clearly, (C3) is vacuous in this case. Thus $\mathcal{L}_0 = \mathrm{Char}(\mathcal{A}) \times \mathcal{A}_1$, where $\mathrm{Char}(\mathcal{A})$ is the Cauchy characteristic of $\mathcal{A}$ and $\mathcal{A}_1$ is the first derived (Pfaffian) system of $\mathcal{A}$, respectively. In particular, if $\mathcal{A}$ is completely integrable and hence $\mathcal{A}$ is the tangent bundle of a foliation $\mathcal{F}$, $\mathcal{L}_0$ is just the product $T\mathcal{F} \times (T\mathcal{F})^\circ$. The bracket in $\mathcal{L}_0$ is given by

$$[\![(X, \alpha), (Y, \beta)]\!] = ([X, Y], L_X\beta - L_Y\alpha).$$

Example 3.2 ([7]). We consider the case when $\mathcal{A} = \mathrm{T}^*(M)$ and $\pi : \mathrm{T}^*(M) \to \mathrm{T}(M)$ is an arbitrary 2-vector field. Since $\mathcal{A}^\circ = \{\mathbf{0}\}$, the conditions (C1) and (C2) are trivial. (C3) implies $[\pi, \pi](\alpha, \gamma, \cdot) = 0$ for any $\alpha \in \mathrm{T}^*(M)$. Thus, $\mathcal{L}_0 = \{(\tilde{\pi}(\gamma), \gamma) \mid \gamma \in \ker[\pi, \pi]\}$ and $\ker[\pi, \pi]$ is a Lie algebroid with respect to $\{\cdot, \cdot\}_\pi$. This is our previous result in [7].

4 Description by 2-forms

In this section, we describe an almost Dirac structure by a '2-form' on $\mathcal{E} = \rho_{\mathcal{L}}(\mathcal{L}) \subset \mathrm{T}(M)$ and find the conditions which characterize $\mathcal{L}_0$. To justify the computation, we assume $\mathcal{E}$ is of constant rank again.

Let $\omega : \mathcal{E} \to \mathcal{E}^*$ be a skew symmetric bundle homomorphism as before. The almost Dirac structure is given by

$$\mathcal{L} = \{(X, \alpha) \in \mathrm{T}(M) \oplus \mathrm{T}^*(M) \ \mid \ i_X\omega = \alpha|_{\mathcal{E}},\ X \in \mathcal{E},\ \alpha \in \mathrm{T}^*(M)\}.$$

The bracket on $\Gamma(\mathcal{L})$ is given by

$$[\![(X_1, \alpha_1), (X_2, \alpha_2)]\!] = ([X_1, X_2]\ ,\ L_{X_1}\alpha_2 - L_{X_2}\alpha_1 + d\,(\omega(X_1, X_2)))\,.$$

Let $e_1 = (X, \alpha), e_2 = (Y, \beta), e_3 = (Z, \gamma)$ be three elements in $\Gamma(\mathcal{L})$. We look for the conditions on $e_3 = (Z, \gamma)$, so that $T(e_1, e_2, e_3) = 0$ holds for all $e_1, e_2 \in \Gamma(\mathcal{L})$. We choose a section s of the natural projection $i^* : \mathrm{T}^*(M) \to \mathcal{E}^*$ and consider the map $s \circ \omega : \mathcal{E} \to \mathrm{T}^*(M)$. Extending $s \circ \omega$ to a map from $\mathrm{T}(M)$ to $\mathrm{T}^*(M)$, we obtain a 2-form $\tilde{\omega} \in \wedge^2(\mathrm{T}^*(M))$ satisfying $\tilde{\omega}(e_1, e_2) = \omega(e_1, e_2)$, for $e_1, e_2 \in \mathcal{E}$. We write an element (X, α) in $\mathcal{L}$ as $(X, i_X\tilde{\omega} + \bar{\alpha})$, where $\bar{\alpha} \in \mathcal{E}^\circ$ (= the annihilators of $\mathcal{E}$). We compute $T(e_1, e_2, e_3)$ using the formula in Proposition 2.2:

$$\begin{aligned} &T((X, \alpha), (Y, \beta), (Z, \gamma)) \\ =&\alpha([Y, Z]) + \beta([Z, X]) + \gamma([X, Y]) + L_X(\beta(Z)) + L_Y(\gamma(X)) + L_Z(\alpha(Y)) \\ =&- d\alpha(Y, Z) - d\beta(Z, X) - d\gamma(X, Y) + L_Y(\alpha(Z)) + L_Z(\beta(X)) + L_X(\gamma(Y)). \end{aligned}$$

Making use of

$$\begin{aligned} d\alpha =& di_X\tilde{\omega} + d\bar{\alpha} = L_X\tilde{\omega} - i_Xd\tilde{\omega} + d\bar{\alpha}, \\ d\alpha(Y, Z) =& (L_X\tilde{\omega})(Y, Z) - (d\tilde{\omega})(X, Y, Z) + (d\bar{\alpha})(Y, Z), \\ L_Y(\alpha(Z)) =& L_Y(\tilde{\omega}(X, Z) + \bar{\alpha}(Z)) = L_Y(\tilde{\omega}(X, Z)), \end{aligned}$$

we see the above $T(e_1, e_2, e_3)$ is equal to

$$d\tilde{\omega}(X, Y, Z) + \bar{\alpha}([Y, Z]) + \bar{\beta}([Z, X]) + \bar{\gamma}([X, Y]).$$

From this, we obtain the following conditions (4.1) and (4.2) on $e_3 = (Z, \gamma)$ which assure $T((X, \alpha), (Y, \beta), (Z, \gamma)) = 0$ for all $(X, \alpha), (Y, \beta) \in \mathcal{L}$.

$$d\tilde{\omega}(X, Y, Z) + \bar{\gamma}([X, Y]) = 0 \quad \text{for } X, Y \in \mathcal{E}, \tag{4.1}$$

$$\bar{\beta}([Z, X]) = 0 \quad \text{for } X \in \mathcal{E},\ \bar{\beta} \in \mathcal{E}^\circ. \tag{4.2}$$

Now, (4.1) is equivalent to that $(d\tilde{\omega})(Z) - d\bar{\gamma} = 0$ on $\mathcal{E}$ and from $\gamma = i_Z\tilde{\omega} + \bar{\gamma}$, this is equivalent to $L_Z\tilde{\omega} - d\gamma = 0$ (on $\mathcal{E}$). Similarly, (4.2) is equivalent to that $L_Z\mathcal{E} \subset \mathcal{E}$. Thus $\mathcal{L}_0$ is given by the following:

$$\mathcal{L}_0 = \{(Z, \gamma) \in \mathcal{L} \ \mid \ L_Z\mathcal{E} \subset \mathcal{E},\ L_Z\tilde{\omega} - d\gamma = 0 \text{ on } \mathcal{E}\}.$$

We note that $L_Z\omega$ is well-defined since the right-hand side of

$$i_X(L_Z\tilde{\omega}) = L_Z(i_X\tilde{\omega}) - i_{[Z,X]}\tilde{\omega}$$

is independent of the choice of $\tilde{\omega}$. The bracket (in $\mathcal{L}$) is given by

$$[\![(Z,\gamma),(W,\delta)]\!] = ([Z,W], L_Z\delta - L_W\gamma + d(\gamma(W))) \ .$$

Since $L_{[Z,W]}\mathcal{E} = L_Z(L_W\mathcal{E}) - L_W(L_Z\mathcal{E}) \subset \mathcal{E}$ and

$$\begin{aligned} L_{[Z,W]}\tilde{\omega} =& L_Z(L_W\tilde{\omega}) - L_W(L_Z\tilde{\omega}) = L_Z(d\delta) - L_W(d\gamma) \\ =& d(L_Z\delta - L_W\gamma + d(\gamma(W))), \text{ on } \mathcal{E}, \end{aligned}$$

that $[\![(Z,\gamma),(W,\delta)]\!] \in \mathcal{L}_0$ is verified.

Example 4.1. Consider the case where $\mathcal{E} = \mathrm{T}(M)$ and ω is an arbitrary 2-form. Then

$$\mathcal{L} = \{(X,\alpha) \in \mathrm{T}(M) \oplus \mathrm{T}^*(M) \ \mid \ i_X\omega = \alpha\} = \{(X, i_X\omega) \ \mid \ X \in \mathrm{T}(M)\}.$$

It is easy to see $\mathcal{L}_0 = \{(Z, i_Z\omega) \ \mid \ Z \in \ker d\omega\}$. In particular, if ω is closed, $\mathcal{L}_0$ is a Dirac structure given by the presymplectic structure on M.

Example 4.2. Let $\mathcal{E}$ be a contact distribution with its contact 1-form θ, and let $\omega = d\theta$. The only vector field in $\mathcal{E}$ satisfying $L_Z\mathcal{E} \subset \mathcal{E}$ is the zero vector field. Thus $\Gamma(\mathcal{L}_0) = \{(0, f\theta) \mid f \in C^\infty(M)\}$ with trivial bracket. Similar situations occur with distributions whose Cauchy characteristic is trivial, since the condition $L_Z\mathcal{E} \subset \mathcal{E}$ means that Z is contained in the Cauchy characteristic of $\mathcal{E}$. With such distributions, it is appropriate to consider the ϕ-deformed bracket ([3],[4]).

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