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Dirac structures for generalized Courant and Courant algebroids

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Abstract

We establish some fundamental relations between Dirac subbundles L for the generalized Courant algebroid $(A \oplus A^*, \phi + W)$ over a differentiable manifold M and the associated Dirac subbundles $\tilde{L}$ for the corresponding Courant algebroid $\tilde{A} \oplus \tilde{A}^*$ over $M \times \mathbb{R}$.

1 Introduction

In [1], T. Courant introduces the notion of a *Dirac structure* in order to present a unified framework for the study of symplectic and Poisson structures and foliations. Alan Weinstein and his collaborators develop the theory of these structures and study several problems of Poisson geometry via Dirac structures theory [10], [11]. The notion was exploited by A. Wade ([16]) and recently by the second author and J. Clemente-Gallardo ([14]) in order to interpreter Jacobi manifolds ([9], [2]) by means of Dirac structures. In [14], J.M. Nunes da Costa and J. Clemente-Gallardo approach this problem by introducing the notions of a *generalized Courant algebroid* and of a *Dirac structure for a generalized Courant algebroid* and by proving that the double $(A \oplus A^*, \phi + W)$ of a generalized Lie bialgebroid $((A, \phi), (A^*, W))$ over a differentiable manifold M, notion very close to the Jacobi manifolds ([6]), is a generalized Courant algebroid.

In the present work, being well known that there is an one-to-one correspondence between generalized Lie bialgebroids structures $((A, \phi), (A^*, W))$ over M and Lie bialgebroids structures $(\tilde{A}, \tilde{A}^*)$, $\tilde{A} = A \times \mathbb{R}$, $\tilde{A}^* = A^* \times \mathbb{R}$, over $\tilde{M} = M \times \mathbb{R}$, we establish some basic relations between the Dirac subbundles L for $(A \oplus A^*, \phi + W)$ and the associated Dirac subbundles $\tilde{L} = \{X + e^t\alpha \,/\, X + \alpha \in L\}$ for $\tilde{A} \oplus \tilde{A}^*$. We prove : 1) L is a reducible Dirac structure for $(A \oplus A^*, \phi + W)$ if and only if $\tilde{L}$ is a reducible Dirac structure for $\tilde{A} \oplus \tilde{A}^*$. 2) If $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are the characteristic foliations of M and $\tilde{M}$ defined by L and $\tilde{L}$, respectively, then $\tilde{L}$ induces an homogeneous Poisson structure on $\tilde{M}/\tilde{\mathcal{F}} = M/\mathcal{F} \times \mathbb{R}$ which is the Poissonization of the induced Jacobi structure on $M/\mathcal{F}$ by L.

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Notation : In this paper, M is a C^∞-differential manifold of finite dimension. We denote by $C^\infty(M)$ the space of all real C^∞-differentiable functions on M and by δ the usual de Rham differential operator.

2 Generalized Lie bialgebroids

Let $(A,[\,,\,],a)$ be a Lie algebroid over M ([12]), A^* its dual vector bundle over M, $\bigwedge A^* = \oplus_{k\in\mathbb{Z}} \bigwedge^k A^*$ the graded exterior algebra of A^* whose differential sections are called *A-forms* on M, $d : \Gamma(\bigwedge A^*) \to \Gamma(\bigwedge A^*)$ the exterior derivative of degree 1 and $\phi \in \Gamma(A^*)$ an 1-cocycle in the Lie algebroid cohomology complex with trivial coefficients ([12], [6]), i.e., for all $X, Y \in \Gamma(A)$, $\langle\phi,[X,Y]\rangle = a(X)(\langle\phi,Y\rangle) - a(Y)(\langle\phi,X\rangle)$. We modify the usual representation a of the Lie algebra $(\Gamma(A),[\,,\,])$ on the space $C^\infty(M)$ by defining $a^\phi : \Gamma(A)\times C^\infty(M) \to C^\infty(M)$, $a^\phi(X,f) = a(X)f + \langle\phi,X\rangle f$. The resulting cohomology operator $d^\phi : \Gamma(\bigwedge A^*) \to \Gamma(\bigwedge A^*)$ of the new cohomology complex is called the *ϕ-differential* of A and $d^\phi\eta = d\eta + \phi\wedge\eta$, for all $\eta \in \Gamma(\bigwedge^k A^*)$. d^ϕ allows us to define the *ϕ-Lie derivative by $X \in \Gamma(A)$*, $\mathcal{L}^\phi_X : \Gamma(\bigwedge^k A^*) \to \Gamma(\bigwedge^k A^*)$, as $\mathcal{L}^\phi_X = d^\phi \circ i_X + i_X \circ d^\phi$, where i_X is the contraction by X. Using ϕ we can also modify the Schouten bracket $[\,,\,]$ on $\Gamma(\bigwedge A)$ to the *ϕ-Schouten bracket* $[\,,\,]^\phi$ on $\Gamma(\bigwedge A)$ by setting, for all $P \in \Gamma(\bigwedge^p A)$ and $Q \in \Gamma(\bigwedge^q A)$, $[P,Q]^\phi = [P,Q] + (p-1)P\wedge(i_\phi Q) + (-1)^p(q-1)(i_\phi P)\wedge Q$, where $i_\phi Q$ can be interpreted as the usual contraction of a multivector field with an 1-form. For details, see [12], [6] and [4].

The notion of *generalized Lie bialgebroid* has been introduced by D. Iglesias and J.C. Marrero in [6] and independently by J. Grabowski and G. Marmo in [4] under the name of *Jacobi bialgebroid*, in such a way that a Jacobi manifold ([9]) has a generalized Lie bialgebroid canonically associated and conversely. We recall that a *Jacobi manifold* is a smooth manifold M equipped with a bivector field Λ and a vector field E such that $[\Lambda,\Lambda] = -2E\wedge\Lambda$ and $[E,\Lambda] = 0$, where $[\,,\,]$ denotes the Schouten bracket.

We consider a Lie algebroid $(A,[\,,\,],a)$ over M and an 1-cocycle $\phi \in \Gamma(A^*)$ and we assume that the dual vector bundle $A^* \to M$ admits a Lie algebroid structure $([\,,\,]_*,a_*)$ and that $W \in \Gamma(A)$ is an 1-cocycle in the Lie algebroid cohomology complex with trivial coefficients of $(A^*,[\,,\,]_*,a_*)$. Then, we say that :

Definition 2.1. The pair $((A,\phi),(A^*,W))$ is a *generalized Lie bialgebroid* over M if, for all $X, Y \in \Gamma(A)$ and $P \in \Gamma(\bigwedge^p A)$, the following conditions hold :

$$d^W_*[X,Y] = [d^W_* X, Y]^\phi + [X, d^W_* Y]^\phi \quad\text{and}\quad \mathcal{L}^W_{*\phi}P + \mathcal{L}^\phi_W P = 0\,;$$

d^W_* and $\mathcal{L}^W_*$ are, respectively, the W-differential and the W-Lie derivative of A^*.

Obviously, if $\phi = 0$ and $W = 0$, we recover the notion of *Lie bialgebroid* introduced by K. Mackenzie and P. Xu in [13] and its equivalent definition given by Yv. Kosmann-Schwarzbach in [8].

Given a Lie algebroid $(A, [\,,\,], a)$ over M, we can construct a Lie algebroid structure on $\tilde{A} \to \tilde{M}$, $\tilde{A} = A \times \mathbb{R}$ and $\tilde{M} = M \times \mathbb{R}$. We identify $\Gamma(\tilde{A})$ with the set of the time-dependent sections of $A \to M$, i.e. for any $\tilde{X} \in \Gamma(\tilde{A})$ and $(x,t) \in M \times \mathbb{R}$, t being the canonical coordinate on $\mathbb{R}$, $\tilde{X}(x,t) = \tilde{X}_t(x)$, where $\tilde{X}_t \in \Gamma(A)$, and we take : i) the Lie bracket $[\,,\,]\tilde{}$ on $\Gamma(\tilde{A})$ defined, for any $\tilde{X}, \tilde{Y} \in \Gamma(\tilde{A})$ and $(x,t) \in \tilde{M}$, by $[\tilde{X}, \tilde{Y}]\tilde{}(x,t) = [\tilde{X}_t, \tilde{Y}_t](x)$, ii) the bundle map $\tilde{a} : \tilde{A} \to T\tilde{M}$, $\tilde{a}(\tilde{X})(x,t) = a(\tilde{X}_t)(x)$. Then $(\tilde{A}, [\,,\,]\tilde{}, \tilde{a}) \to \tilde{M}$ is a Lie algebroid. Also, taking an 1-cocycle ϕ of A, we deform $([\,,\,]\tilde{}, \tilde{a})$ in two different ways and we obtain two new Lie algebroid structures on $\tilde{A}$, [6]. Precisely, for any $\tilde{X}, \tilde{Y} \in \Gamma(\tilde{A})$:

$$(2.1)\quad [\tilde{X}, \tilde{Y}]\tilde{}^{\phi} = [\tilde{X}, \tilde{Y}]\tilde{} + i_\phi \tilde{X}_t \partial \tilde{Y}/\partial t - i_\phi \tilde{Y} \partial \tilde{X}/\partial t, \quad \tilde{a}^\phi(\tilde{X}) = \tilde{a}(\tilde{X}) + i_\phi \tilde{X} \partial/\partial t;$$

$$(2.2)\quad \begin{aligned} [\tilde{X}, \tilde{Y}]\hat{}^{\phi} &= e^{-t}([\tilde{X}, \tilde{Y}]\tilde{} + \langle \phi, \tilde{X}_t \rangle (\partial \tilde{Y}/\partial t - \tilde{Y}) - \langle \phi, \tilde{Y}_t \rangle (\partial \tilde{X}/\partial t - \tilde{X})), \\ \hat{a}^\phi(\tilde{X}) &= e^{-t}(\tilde{a}(\tilde{X}) + \langle \phi, \tilde{X}_t \rangle \partial/\partial t). \end{aligned}$$

Theorem 2.1 ([6]). *Let $(A, [\,,\,], a)$ be a Lie algebroid over M and $\phi \in \Gamma(A^*)$ an 1-cocycle. Suppose that A^* has a Lie algebroid structure $([\,,\,]_*, a_*)$ and that $W \in \Gamma(A)$ is an 1-cocycle for this structure. Consider on $\tilde{A} = A \times \mathbb{R}$ and $\tilde{A}^* = A^* \times \mathbb{R}$ the Lie algebroid structures $([\,,\,]\tilde{}^{\phi}, \tilde{a}^\phi)$ and $([\,,\,]\hat{}_*^{W}, \hat{a}_*^W)$, respectively. Then $(\tilde{A}, \tilde{A}^*)$ is a Lie bialgebroid over $\tilde{M} = M \times \mathbb{R}$ if and only if $((A, \phi), (A^*, W))$ is a generalized Lie bialgebroid over M. The induced Poisson structure on $\tilde{M}$ is the Poissonization of the induced Jacobi structure on M.*

Moreover, the image Ima of the anchor map a of $(A, [\,,\,], a) \to M$ is an integrable distribution on M ([3]) which defines a singular foliation $\mathcal{F}_A$ of M, called the *Lie algebroid foliation of M associated with A* ([7]). The relation between the leaves of the Lie algebroid foliation $\mathcal{F}_{\tilde{A}}$ of $M \times \mathbb{R}$ associated with $(\tilde{A}, [\,,\,]\tilde{}^{\phi}, \tilde{a}^\phi)$ (given by (2.1)) and the leaves of the Lie algebroid foliation $\mathcal{F}_A$ of M associated with A was studied in [7] by D. Iglesias and J.C. Marrero. They have proved :

Theorem 2.2 ([7]). *Under the above considerations, suppose that $(x_0, t_0) \in M \times \mathbb{R}$ and that $\tilde{F}$ and F are the leaves of the Lie algebroid foliations $\mathcal{F}_{\tilde{A}}$ and $\mathcal{F}_A$ passing through $(x_0, t_0) \in M \times \mathbb{R}$ and $x_0 \in M$, respectively, and denote by A_{x_0} the fiber of A over x_0. Then : (1) If $\ker(a|_{A_{x_0}}) \nsubseteq \langle \phi(x_0) \rangle^\circ$, $\tilde{F} = F \times \mathbb{R}$. (2) If $\ker(a|_{A_{x_0}}) \subseteq \langle \phi(x_0) \rangle^\circ$ and $\pi_1 : M \times \mathbb{R} \to M$ is the canonical projection onto the first factor, $\pi_1(\tilde{F}) = F$ and $\pi_1|_{\tilde{F}} : \tilde{F} \to F$ is a covering map.*

3 Generalized Courant algebroids

The notion of *generalized Courant algebroid* has been introduced by the second author and J. Clemente-Gallardo in [14] and independently, under the name of *Courant-Jacobi algebroid*, by J. Grabowski and G. Marmo in [5].

Definition 3.1 ([14]). Let $E \to M$ to be a vector bundle over a differentiable manifold M equipped with : (i) a nondegenerate symmetric bilinear form $(\,,\,)$ on the bundle, (ii) a skew-symmetric bilinear bracket $[\,,\,]$ on $\Gamma(E)$, (iii) a bundle map $\rho : E \to TM$ and (iv) an E-1-form θ such that, for any $e_1, e_2 \in \Gamma(E)$, $\langle\theta, [e_1, e_2]\rangle = \rho(e_1)\langle\theta, e_2\rangle - \rho(e_2)\langle\theta, e_1\rangle$. We consider : (a) the bundle map $\rho^\theta : E \to TM \times \mathbb{R}$ defined, for any $e \in E$, by $\rho^\theta(e) = \rho(e) + \langle\theta, e\rangle$, (b) the applications $\mathcal{D}, \mathcal{D}^\theta : C^\infty(M) \to \Gamma(E)$ defined, for any $f \in C^\infty(M)$, respectively, by $\mathcal{D}f = \frac{1}{2}\beta^{-1}\rho^*\delta f$ [2] and $\mathcal{D}^\theta f = \mathcal{D}f + \frac{1}{2}f\beta^{-1}(\theta)$ and (c) for any $e_1, e_2, e_3 \in \Gamma(E)$, the function $T(e_1, e_2, e_3) = \frac{1}{3}([e_1, e_2], e_3) + c.p.$ on the base M. Then, we say that E is a *generalized Courant algebroid* if the following relations are satisfied :

1. $[[e_1, e_2], e_3] + c.p. = \mathcal{D}^\theta T(e_1, e_2, e_3), \quad \forall\, e_1, e_2, e_3 \in \Gamma(E)$;
2. $\rho^\theta([e_1, e_2]) = [\rho^\theta(e_1), \rho^\theta(e_2)]$, [3] $\quad \forall\, e_1, e_2 \in \Gamma(E)$;
3. $[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - (e_1, e_2)\mathcal{D}f, \quad \forall\, e_1, e_2 \in \Gamma(E),\ \forall f \in C^\infty(M)$;
4. $\rho^\theta \circ \mathcal{D}^\theta = 0$, i.e., for any $f, g \in C^\infty(M)$, $(\mathcal{D}^\theta f, \mathcal{D}^\theta g) = 0$;
5. $\rho^\theta(e)(e_1, e_2) = ([e, e_1] + \mathcal{D}^\theta(e, e_1), e_2) + (e_1, [e, e_2] + \mathcal{D}^\theta(e, e_2)),\ \forall\, e, e_1, e_2 \in \Gamma(E)$.

Definition 3.2. A *Dirac structure for a generalized Courant algebroid* (E, θ) over M is a subbundle $L \subset E$ that is maximal isotropic under $(\,,\,)$ and integrable, i.e. $\Gamma(L)$ is closed under $[\,,\,]$.

A Dirac subbundle L of (E, θ) is a Lie algebroid under the restrictions of the bracket $[\,,\,]$ and of the anchor ρ to $\Gamma(L)$. If $\theta \in \Gamma(L^*)$, then it is an 1-cocycle for the Lie algebroid cohomology with trivial coefficients of $(L, [\,,\,]|_L, \rho|_L)$.

The most important example of generalized Courant algebroid is the double $(A \oplus A^*, \phi + W)$ of a generalized Lie bialgebroid $((A, \phi), (A^*, W))$ over M. On $A \oplus A^*$ there exist two natural nondegenerate bilinear forms, one symmetric and another skew-symmetric $(\,,\,)_\pm$: for any $X_1 + \alpha_1, X_2 + \alpha_2 \in A \oplus A^*$, $(X_1 + \alpha_1, X_2 + \alpha_2)_\pm = 1/2(\langle\alpha_1, X_2\rangle \pm \langle\alpha_2, X_1\rangle)$ and on $\Gamma(A \oplus A^*) \cong \Gamma(A) \oplus \Gamma(A^*)$ we introduce the bracket $[\![\,,\,]\!]$: for all $X_1 + \alpha_1, X_2 + \alpha_2 \in \Gamma(A \oplus A^*)$,

$$\begin{aligned}[\![X_1 + \alpha_1, X_2 + \alpha_2]\!] = {}& ([X_1, X_2]^\phi + \mathcal{L}^W_{*\alpha_1} X_2 - \mathcal{L}^W_{*\alpha_2} X_1 - d^W_*(e_1, e_2)_-) + \\ & + ([\alpha_1, \alpha_2]^W_* + \mathcal{L}^\phi_{X_1}\alpha_2 - \mathcal{L}^\phi_{X_2}\alpha_1 + d^\phi(e_1, e_2)_-).\end{aligned}$$

Also, we consider the bundle map $\rho : A \oplus A^* \to TM$ given by $\rho = a + a_*$, i.e., for any $X + \alpha \in E$, $\rho(X + \alpha) = a(X) + a_*(\alpha)$. We have:

Theorem 3.1 ([14]). *If $((A, \phi), (A^*, W))$ is a generalized Lie bialgebroid over M, then $A \oplus A^*$ endowed with $([\![\,,\,]\!], (\,,\,)_+, \rho)$ and $\theta = \phi + W \in \Gamma(E^*)$ is a generalized Courant algebroid over M. The operators $\mathcal{D}$ and $\mathcal{D}^\theta$ are, respectively, $\mathcal{D} = (d_* + d)|_{C^\infty(M)}$ and $\mathcal{D}^\theta = (d^W_* + d^\phi)|_{C^\infty(M)}$.*

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[2] β is the isomorphism from E onto E^* given by the nondegenerate bilinear form $(\,,\,)$.

[3] The bracket on the right-hand side is the Lie bracket defined on $\Gamma(TM \times \mathbb{R})$ by $[(X, f), (Y, g)] = ([X, Y], X \cdot g - Y \cdot f)$.

4 Dirac structures of $((A,\phi),(A^*,W))$ and of $(\tilde{A},\tilde{A}^*)$

Let $((A,[\,,\,],a,\phi),(A^*,[\,,\,]_*,a_*,W))$ be a generalized Lie bialgebroid over M and $(A\oplus A^*,[\![\,,\,]\!],(\,,\,)_+,a+a_*,\phi+W)$ the associated generalized Courant algebroid.

Definition 4.1. We say that a Dirac subbundle L of $A\oplus A^*$ is *reducible* if the image $a(D)$ of its *characteristic subbundle* $D=L\cap A$ by a defines a simple foliation $\mathcal{F}$ of M. By the term "simple foliation" we mean that $\mathcal{F}$ is a regular foliation such that the space $M/\mathcal{F}$ is a nice manifold and the canonical projection $M\to M/\mathcal{F}$ is a submersion.

Definition 4.2. Let L be a Dirac subbundle of $A\oplus A^*$. A function $f\in C^\infty(M)$ is called *L-admissible* if there exists $Y_f\in\Gamma(A)$ such that $Y_f+d^\phi f\in\Gamma(L)$. We denote by $C^\infty_L(M,\mathbb{R})$ the set of all L-admissible functions of $C^\infty(M)$.

Let $((\tilde{A},[\,,\,]\tilde{}^{\phi},\tilde{a}^\phi),(\tilde{A}^*,[\,,\,]\hat{}^{W}_*,\hat{a}^W_*))$ be the Lie bialgebroid over $\tilde{M}$ defined by $((A,[\,,\,],a,\phi),(A^*,[\,,\,]_*,a_*,W))$ as in Theorem 2.1. Then, $\tilde{A}\oplus\tilde{A}^*$ endowed with : (i) the two nondegenerate bilinear forms $(\,,\,)_\pm$ on $\tilde{A}\oplus\tilde{A}^*$: for all $\tilde{X}_1+\tilde{\alpha}_1,\tilde{X}_2+\tilde{\alpha}_2\in\tilde{A}\oplus\tilde{A}^*$, $(\tilde{X}_1+\tilde{\alpha}_1,\tilde{X}_2+\tilde{\alpha}_2)_\pm=1/2(\langle\tilde{\alpha}_1,\tilde{X}_2\rangle\pm\langle\tilde{\alpha}_2,\tilde{X}_1\rangle)$, (ii) the bracket $[\![\,,\,]\!]\tilde{}$ on $\Gamma(\tilde{A}\oplus\tilde{A}^*)$: for all $\tilde{X}_1+\tilde{\alpha}_1,\tilde{X}_2+\tilde{\alpha}_2\in\Gamma(\tilde{A}\oplus\tilde{A}^*)$,

$$\begin{aligned}[\![\tilde{X}_1+\tilde{\alpha}_1,\tilde{X}_2+\tilde{\alpha}_2]\!]\tilde{} \;&=\; \big([\tilde{X}_1,\tilde{X}_2]\tilde{}^{\phi}+\hat{\mathcal{L}}^W_{\tilde{\alpha}_1}\tilde{X}_2-\hat{\mathcal{L}}^W_{\tilde{\alpha}_2}\tilde{X}_1-\hat{d}^W_*((\tilde{e}_1,\tilde{e}_2)_-)\big)+\\ &\quad\big([\tilde{\alpha}_1,\tilde{\alpha}_2]\hat{}^{W}_*+\tilde{\mathcal{L}}^\phi_{\tilde{X}_1}\tilde{\alpha}_2-\tilde{\mathcal{L}}^\phi_{\tilde{X}_2}\tilde{\alpha}_1+\tilde{d}^\phi((\tilde{e}_1,\tilde{e}_2)_-)\big),\end{aligned}$$

(for any $\tilde{f}\in C^\infty(\tilde{M})$, $\tilde{d}^\phi\tilde{f}=\tilde{d}\tilde{f}+\frac{\partial\tilde{f}}{\partial t}\phi$ and $\hat{d}^W_*\tilde{f}=e^{-t}(\tilde{d}\tilde{f}+\frac{\partial\tilde{f}}{\partial t}\phi)$, [6]), (iii) the bundle map $\tilde{\rho}:\tilde{A}\oplus\tilde{A}^*\to T\tilde{M}$, $\tilde{\rho}=\tilde{a}^\phi+\hat{a}^W_*$, is a Courant algebroid over $\tilde{M}$ ([10]).

Let $\mathbf{E}:\Gamma(A\oplus A^*)\to\Gamma(\tilde{A}\oplus\tilde{A}^*)$ be the embedding of $\Gamma(A\oplus A^*)$ into $\Gamma(\tilde{A}\oplus\tilde{A}^*)$ defined, for any $X+\alpha\in\Gamma(A\oplus A^*)$, by

$$\mathbf{E}(X+\alpha)=X+e^t\alpha,$$

where X and α are regarded as time-independent sections of $\tilde{A}$ and $\tilde{A}^*$, respectively. If L is a subbundle of $A\oplus A^*$, we write $\tilde{L}=\mathbf{E}(L)$ in order to denote the vector subbundle $\tilde{L}$ of $\tilde{A}\oplus\tilde{A}^*$ whose space of global cross sections is the image by $\mathbf{E}$ of the space of global cross sections of L, i.e. $\Gamma(\tilde{L})=\mathbf{E}(\Gamma(L))$.

Proposition 4.1. *Let L be a vector subbundle of $A\oplus A^*$ and $\tilde{L}=\mathbf{E}(L)$. Then, L is a Dirac structure for the generalized Courant algebroid $(A\oplus A^*,\phi+W)$ if and only if $\tilde{L}$ is a Dirac structure for the Courant algebroid $\tilde{A}\oplus\tilde{A}^*$.*

Proof. It is easy to check that $\tilde{L}$ is a maximally isotropic subbundle of $(\tilde{A}\oplus\tilde{A}^*,(\,,\,)_+)$ if and only if L is a maximally isotropic subbundle of $(A\oplus A^*,(\,,\,)_+)$. Moreover, by a straightforward calculation we get that

$$[\![\mathbf{E}(X_1+\alpha_1),\mathbf{E}(X_2+\alpha_2)]\!]\tilde{}=\mathbf{E}([\![X_1+\alpha_1,X_2+\alpha_2]\!]),\quad\forall\,X_1+\alpha_1,X_2+\alpha_2\in\Gamma(L),$$

i.e. $\Gamma(\tilde{L})$ is closed under $[\![\,,\,]\!]\tilde{}$ if and only if $\Gamma(L)$ is closed under $[\![\,,\,]\!]$. $\square$

Proposition 4.2. *Let L be a Dirac structure of $(A \oplus A^*, \phi + W)$ and $\tilde{L} = \mathbf{E}(L)$ the associated Dirac structure of $\tilde{A} \oplus \tilde{A}^*$. Then $\tilde{f} \in C^\infty(\tilde{M})$ is a $\tilde{L}$-admissible function if and only if $\tilde{f} = e^t f$ and $f \in C^\infty_L(M)$.*

Proof. Let $\tilde{f} \in C^\infty_{\tilde{L}}(\tilde{M})$, i.e. there exists $Y \in \Gamma(A) : Y + \tilde{d}^\phi \tilde{f} \in \Gamma(\tilde{L})$. But, $Y + \tilde{d}^\phi \tilde{f} \in \Gamma(\tilde{L})$ implies that there exists $\xi \in \Gamma(A^*) : Y + \xi \in \Gamma(L)$ and $Y + \tilde{d}^\phi \tilde{f} = \mathbf{E}(Y + \xi)$, thus $\tilde{d}^\phi \tilde{f} = e^t \xi$. From *Theorem of normal forms for Lie algebroids* ([3]) we have that, if the rank of $a(D)$, $D = L \cap A$, at a point $q \in M$ is k, then we can construct on a neighborhood U of q in M a system of local coordinates $(x_1, \ldots, x_k, \ldots, x_n)$ $(n = \dim M)$ and a basis of sections $(X_1, \ldots, X_k, \ldots X_r)$ of $\Gamma(A)$ (r is the dimension of the fibres of $A \to M$), with $(X_1, \ldots, X_k)$ sections of $\Gamma(D)$, such that $a(X_i) = \frac{\partial}{\partial x_i}$, for every $i = 1, \ldots, k$. Let $(\alpha_1, \ldots, \alpha_k, \ldots, \alpha_r)$ be the basis of $\Gamma(A^*)$, dual of $(X_1, \ldots, X_k, \ldots X_r)$. Since $\phi, \xi \in \Gamma(A^*)$, there exist $\phi_i, \xi_i \in C^\infty(U)$, $i = 1, \ldots, r$, such that $\phi = \sum_{i=1}^r \phi_i \alpha_i$ and $\xi = \sum_{i=1}^r \xi_i \alpha_i$. So, for any $i = 1, \ldots, r$,

$$\tilde{d}^\phi \tilde{f} = e^t \xi \Rightarrow \langle \tilde{d}\tilde{f} + (\partial \tilde{f}/\partial t)\phi, X_i \rangle = \langle e^t \xi, X_i \rangle \Leftrightarrow \langle \tilde{d}\tilde{f}, X_i \rangle + (\partial \tilde{f}/\partial t)\phi_i = e^t \xi_i. \tag{4.1}$$

But, for $i = 1, \ldots, k$, $\langle \tilde{d}\tilde{f}, X_i \rangle = \langle \delta \tilde{f}, \tilde{a}(X_i) \rangle = \langle \delta \tilde{f}, a(X_i) \rangle = \langle \delta \tilde{f}, \frac{\partial}{\partial x_i} \rangle = \frac{\partial \tilde{f}}{\partial x_i}$. Hence, the last equation of (4.1) can be written, for any $i = 1, \ldots, k$, as

$$\partial \tilde{f}/\partial x_i + (\partial \tilde{f}/\partial t)\phi_i = e^t \xi_i. \tag{4.2}$$

By resolving the characteristic system $\frac{\delta x_i}{1} = \frac{\delta t}{\phi_i} = \frac{\delta \tilde{f}}{e^t \xi_i}$ of (4.2), we obtain that $\tilde{f}$ must be, at least locally, of the form $\tilde{f} = e^t f$ with $f \in C^\infty(U)$. Taking into account Definition 4.2 and that $\tilde{L} = \mathbf{E}(L)$, we get $\tilde{f} = e^t f \in C^\infty_{\tilde{L}}(\tilde{M}) \Leftrightarrow f \in C^\infty_L(M)$. □

Proposition 4.3. *Let L be a Dirac subbundle for $(A \oplus A^*, \phi + W)$ and $\tilde{L} = \mathbf{E}(L)$ the associated Dirac subbundle of $\tilde{A} \oplus \tilde{A}^*$. Then, L is reducible if and only if $\tilde{L}$ is reducible.*

Proof. Let $D = L \cap A$ and $\tilde{D} = \tilde{L} \cap \tilde{A}$ be the characteristic subbundles of L and $\tilde{L}$, respectively, $\mathcal{F}$ and $\tilde{\mathcal{F}}$ the foliations of M and $\tilde{M}$, respectively, defined by $a(D)$ and $\tilde{a}^\phi(\tilde{D})$, respectively. Obviously, $\tilde{D} \cong D$ and $\tilde{a}^\phi(\tilde{D}) = \{\tilde{a}^\phi(X) \,/\, X \in D\} = \{a(X) + \langle \phi, X \rangle \partial/\partial t \,/\, X \in D\}$. Let (x_0, t_0) be a point of $\tilde{M} = M \times \mathbb{R}$, $\tilde{F}$ and F the leaves of $\tilde{\mathcal{F}}$ and $\mathcal{F}$ passing through $(x_0, t_0) \in \tilde{M}$ and $x_0 \in M$, respectively, and D_{x_0} the fibre of D over x_0. By Theorem 2.2, we have : (i) if $ker(a|_{D_{x_0}}) \nsubseteq \langle \phi(x_0) \rangle^\circ$, then $\tilde{F} = F \times \mathbb{R}$, so $\dim \tilde{F} = \dim F + 1$ and the vector field $\partial/\partial t$ is tangent to $\tilde{F}$; (ii) if $ker(a|_{D_{x_0}}) \subseteq \langle \phi(x_0) \rangle^\circ$ and $\pi_1 : M \times \mathbb{R} \to M$ is the canonical projection, then $\pi_1(\tilde{F}) = F$ and $\pi_1|_{\tilde{F}} : \tilde{F} \to F$ is a covering map, thus $\dim \tilde{F} = \dim F$ and the vector field $\partial/\partial t$ is not tangent to $\tilde{F}$. Since every $\tilde{L}$-admissible function $\tilde{f}$ is of type $\tilde{f} = e^t f$, $f \in C^\infty_L(M)$, (Proposition 4.2) and also it is constant along the leaves of $\tilde{\mathcal{F}}$ ([1],[11]), it is not possible the leaves $\tilde{F}$ of $\tilde{\mathcal{F}}$ to be of type $\tilde{F} = F \times \mathbb{R}$ (because, in this case, $\partial/\partial t$ is tangent to $\tilde{F}$ and $\tilde{f} = e^t f$ is not constant along

$\partial/\partial t$). Thus, for any leaf $\tilde{F}$ of $\tilde{\mathcal{F}}$ and for the corresponding leaf F of $\mathcal{F}$, we have $\pi_1(\tilde{F}) = F$ and $\pi_1|_{\tilde{F}} : \tilde{F} \to F$ is a covering map. Hence, we get : (1) Every leaf $\tilde{F}$ of $\tilde{\mathcal{F}}$ is of the same dimension as the corresponding leaf F of $\mathcal{F}$, so $\mathcal{F}$ is a regular foliation of M if and only if $\tilde{\mathcal{F}}$ is a regular foliation of $\tilde{M}$. (2) $\tilde{\mathcal{F}} \cong \mathcal{F}$, so $\tilde{M}/\tilde{\mathcal{F}} \cong (M \times \mathbb{R})/\mathcal{F} \cong (M/\mathcal{F}) \times \mathbb{R}$; thus, $M/\mathcal{F}$ is a nice manifold if and only if $\tilde{M}/\tilde{\mathcal{F}}$ is a nice manifold and the projection $M \to M/\mathcal{F}$ is a submersion if and only if the projection $M \times \mathbb{R} = \tilde{M} \to \tilde{M}/\tilde{\mathcal{F}} \cong (M/\mathcal{F}) \times \mathbb{R}$ is a submersion. Consequently, L is a reducible Dirac subbundle for $A \oplus A^*$ if and only if $\tilde{L} = \mathbf{E}(L)$ is a reducible Dirac subbundle for $\tilde{A} \oplus \tilde{A}^*$. □

Let L be a Dirac structure of $(A \oplus A^*, \phi + W)$ and $\tilde{L}$ the associated Dirac structure of $\tilde{A} \oplus \tilde{A}^*$. On $C^\infty_L(M)$ we define the bracket $\{\,,\,\}_L$ by setting, for all $f, g \in C^\infty_L(M)$, $\{f, g\}_L := \rho^\theta(e_f)g$, where $e_f = Y_f + d^\phi f \in \Gamma(L)$. Also, on $C^\infty_{\tilde{L}}(\tilde{M})$ we define the bracket $\{\,,\,\}_{\tilde{L}}$ by setting, for all $\tilde{f}, \tilde{g} \in C^\infty_{\tilde{L}}(\tilde{M})$, $\tilde{f} = e^t f$, $\tilde{g} = e^t g$ with $f, g \in C^\infty_L(M)$, $\{\tilde{f}, \tilde{g}\}_{\tilde{L}} := \tilde{\rho}(\tilde{e}_{\tilde{f}})\tilde{g}$, where $\tilde{e}_{\tilde{f}} = Y_f + \tilde{d}^\phi \tilde{f} \in \Gamma(\tilde{L})$. By a straightforward calculation we get :

$$\{\tilde{f}, \tilde{g}\}_{\tilde{L}} = \{e^t f, e^t g\}_{\tilde{L}} = e^t \{f, g\}_L. \tag{4.3}$$

Theorem 4.1 ([15]). *1) If $1 \in C^\infty_L(M)$[4], then $(C^\infty_L(M), \{\,,\,\}_L)$ is a Jacobi algebra. 2) If L is a reducible Dirac subbundle of $(A \oplus A^*, \phi + W)$ and $1 \in C^\infty_L(M)$, then L induces a Jacobi structure on $M/\mathcal{F}$ defined by the Jacobi bracket $\{\,,\,\}_L$.*

Theorem 4.2. *1) If $1 \in C^\infty_L(M)$, then $(C^\infty_{\tilde{L}}(\tilde{M}), \{\,,\,\}_{\tilde{L}})$ is an homogeneous Poisson algebra with respect $\partial/\partial t$[5]. 2) If L is a reducible Dirac subbundle of $(A \oplus A^*, \phi + W)$ and $1 \in C^\infty_L(M)$, then $\tilde{L}$ induces an homogeneous Poisson structure on $\tilde{M}/\tilde{\mathcal{F}}$ defined by the homogeneous Poisson bracket $\{\,,\,\}_{\tilde{L}}$. 3) $\tilde{M}/\tilde{\mathcal{F}} = (M/\mathcal{F}) \times \mathbb{R}$ and the induced homogeneous Poisson structure on $\tilde{M}/\tilde{\mathcal{F}}$ by $\tilde{L}$ is the Poissonization of the induced Jacobi structure on $M/\mathcal{F}$ by L.*

Proof. 1) It is checked by taking account (4.3) and the fact that, if $1 \in C^\infty_L(M)$, then $(C^\infty_L(M), \{\,,\,\}_L)$ is a Jacobi algebra. 2) By applying the results of [11] to the reducible Dirac subbundle $\tilde{L}$ and the homogeneous Poisson algebra $(C^\infty_{\tilde{L}}(\tilde{M}), \{\,,\,\}_{\tilde{L}})$. 3) We have $\tilde{\mathcal{F}} = \mathcal{F} \times \{0\}$ ([15]), thus $\tilde{M}/\tilde{\mathcal{F}} = (M/\mathcal{F}) \times \mathbb{R}$, and by (4.3) we conclude the announced result. □

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[4] We have ([15]) : 1 is an L-admissible function if and only if, for any $Y \in \Gamma(D)$, $\langle \phi, Y \rangle = 0$.

[5] In the sense of Dazord-Lichnerowicz-Marle ([2]) terminology.

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