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A Counter Example of Invariant Deformation Quantization

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Abstract

In this note, we will give an example of an Hamiltonian Lie algebra action which has no invariant star product.

1 Introduction

Quantization of a Hamiltonian system with symmetries is an important and difficult problem in physics and mathematics. In the deformation quantization formulation [4], this problem can be phrased as follows: given a Hamiltonian Lie group action on a symplectic manifold, does there exist a star product compatible (see Definition 1.3) with the group action?

Since the very early time of deformation quantization, Lichnerowicz has considered this question(see [18] and references therein). Lichnerowicz in [17] showed that each homogeneous space admitting an invariant linear connection admits an invariant Vey star product.

In the literature, there are various definitions of a star product. We fix our star product to be the following one:

Definition 1.1 *Let (M, ω) be a symplectic manifold. A star product on M is an associative product $\star$ on $C^\infty(M)[[\hbar]]$ with the following properties:*

1. *the coefficients $c_k(x)$ of the product*

$$c(x, \hbar) = a(x, \hbar) \star b(x, \hbar) = \sum_{k=0}^{\infty} \hbar^r C^r(a, b),$$

where C^r are locally bidifferential operators.

2. *the leading term $c_0(x)$ is equal to the usual commutative product of functions $a_0(x)b_0(x)$.*

3. *the star product satisfies*

$$[a, b] = a \star b - b \star a = -i\hbar\{a_0, b_0\} + \cdots ,$$

where $\{\ ,\ \}$ means the Poisson bracket of functions and the dots mean higher order terms of $\hbar$.

In this note, we will write a star product as $f \star g = \sum_{r=0}^{\infty} \hbar^r C^r(f,g)$, where C^r is a local bidifferential operator. Next we recall the definition of a Vey star product.

Definition 1.2 *Let ∇ be a symplectic connection on (M,ω). A star product is called a Vey^n-product if the principal symbol of the differential operator C^r is identical to*

$$P^r_{\nabla}(f,g) = \omega^{i_1 j_1} \cdots \omega^{i_r j_r} \nabla_{i_1} \cdots \nabla_{i_r} f \nabla_{j_1} \cdots \nabla_{j_r} g, \quad \textit{for all } f,\ g \in C^{\infty}(M),$$

for all $r \le n$.

At the beginning of this section, when describing the question of quantization with symmetries, we have been very vague by using the word "being compatible". In the literature, there are several related notions of invariant and covariant star products. In this paper, we will focus on the following invariant star product from [2].

Definition 1.3 *For a Hamiltonian Lie group G action on a symplectic manifold (M,ω), a star product is called (geometrically) G invariant* if:*

$$x \cdot (f \star g) = (x \cdot f) \star (x \cdot g), \quad \textit{for all } x \in G,\ f,\ g \in C^{\infty}(M).$$

Looking at the infinitesimal Lie algebra $\mathfrak{g}$ action and $J : \mathfrak{g} \to C^{\infty}(M)$ the dual of the momentum map, we have

$$\{J(X), f \star g\} = \{J(X), f\} \star g + f \star \{J(X), g\},$$

for all X in $\mathfrak{g}$, $f,\ g$ in $C^{\infty}(M)$.

From Definition 1.2 and 1.3, we can easily see ([16]) that if a Vey$^2-$product is G-invariant, then the corresponding symplectic connection is also $G-$invariant. Therefore, Lichnerowicz's result is also necessary for the existence of an invariant Vey^2-product. A G-invariant Vey$^2-$product exists if and only if there is an invariant symplectic connection.

In Fedosov's construction [12] of star products on a symplectic manifold, it is obvious that the existence of an invariant (torsion free) connection implies the existence of an invariant (torsion free) symplectic connection and therefore the existence of an invariant star product. By Palais' theorem, a proper Lie group action on a symplectic manifold M allows a G-invariant metric, and therefore a G-invariant (torsion free) connection, the Levi-Civita connection, on M, and hence a G-invariant star product.

The existence of invariant star products leads to the study of quantum momentum map and reduction theory. Xu in [20] introduced and studied the theory of quantum momentum map. In [11] and [13], Fedosov used his quantization

*In short, we will just say "G invariant" star product in this note.

method to study quantum Marsden-Weinstein reduction of a compact Hamiltonian Lie group action. Bordemann, Herbig, and Waldmann in [5] studied BRST cohomology in the framework of deformation quantization and quantum reduced space.

Recently, there have been many attempts in literature to extend the study of invariant star products and Xu's quantum momentum map to more general types of quantization. In [19], Müller-Bahns and Neumaier considered star products of Wick type; and in [14], Gutt and Rawnsley investigated natural star products. All the known results have suggested that Lichnerowicz's original idea that the existence of an invariant star product is closely related to the existence of an invariant connection is correct.

In the above discussion, we have concentrated on symplectic manifolds. The Poisson version of the question is also worth mentioning. The existence of a star product for a general Poisson manifold was first shown by Kontsevich (and later Tarmarkin with a different method) in [15] using his formality theorem. From Kontsevich's original construction, the conditions needed for the existence of an invariant star product are not very obvious. Dolgushev in [9] gave an alternative construction of the global formality theorem using Fedosov type resolution and Kontsevich's local formality theorem. Dolgushev's construction explicitly shows that the existence of an invariant connection is a sufficient condition for an invariant star product (also an invariant formality theorem). It would be interesting to look at the Poisson version of quantum momentum maps and BRST quotients.

It is also worth mentioning that since [7] and [10], there has been discussion of conformally invariant symbol calculus and star products. These products are different from the star product defined in Definition 1.1 in that they are not defined on all smooth functions but suitable subalgebras. The study of conformally invariant quantization is still at its early stage, and we do not even know whether a conformally invariant quantization always exists. However, we have seen its interesting relations to other areas of mathematics. For example, Cohen, Manin, and Zagier in [7] obtained this type of product by considering deformations of modular forms.

In this note, we will show that there is a Hamiltonian Lie algebra action which has no invariant star product.

In this direction, Arnal, Cortet, Molin, and Pinczon in [2] showed that on some coadjoint orbit $\mathcal{O}$ of a nilpotent Lie algebra $\mathfrak{g}$, there is no $\mathfrak{g}$-invariant Vey$^2-$product by showing that there is no invariant $\mathfrak{g}-$connection.

What we will do is extend their result to any star product. Since we are working in full generality, to show that there is no invariant connection (as in [2]) is not enough any more. We will study properties of general invariant differential operators, which will give us enough information to show the nonexistence of an invariant star product.

Remark 1.4 *This type of counter example is believed to exist among experts we have talked to, but we cannot find any explicit example in the literature. If there is any other example, please let us know.*

Remark 1.5 *On a large class of coadjoint orbits, invariant star products were constructed in [1] and references therein.*

Remark 1.6 *Weaker than invariant star products, people have introduced a notion of "covariant star products" (see [2]). Instead of the keeping the same action, we allow a higher order modification to the group (Lie algebra) action. The existence and uniqueness of covariant star products are related to the lower order Lie algebra (Lie group) cohomology (see [19]). This spring, Kontsevich conjectured that the automorphism group of the Poisson algebra of polynomial functions on $\mathbb{R}^{2n}$ is naturally isomorphic to the automorphism group of the corresponding $2n-$dimensional Weyl algebra. Also this summer in IHP, Gorokhovsky, Nest, and Tsygan showed the author a very interesting construction of their "stacky star product".*

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2 Main result

We look at $(\mathbb{R}^2, dx\wedge dy)$ with the Lie algebra $\mathfrak{g}$ action formed by the Hamiltonian vector fields generated by

$$x^3,\ x^2,\ x,\ y,\ 1.$$

$\mathfrak{g}$ is a 5-dim nilpotent Lie algebra[†]. By the expression of a star product, we can easily see that if a star product is invariant under $\mathfrak{g}$ action, then each C^r of $\star$ has to be $\mathfrak{g}$ invariant, i.e.

$$X(C^r(u,v)) = C^r(X(u),v)+C^r(u,X(v)) \quad \forall X\in\mathfrak{g},\ u,\ v\in C^\infty(M),\ r=1,\ 2,\ 3,\ \cdots.$$

Therefore, in the following, we will first look at properties of differential operators that are invariant under the $\mathfrak{g}$ action. Then we will come back to the existence of an invariant $\star$ product.

[†] We can consider the Lie algebra of the corresponding Hamiltonian vector fields, which has no center.

We write a locally bidifferential operator C in an open set $\mathcal{O}$ containing the origin of $\mathbb{R}^2$ as

$$\sum_{0\le i,j,k,l<\infty} C_{ij;kl}(\partial_x)^i(\partial_y)^j \otimes (\partial_x)^k(\partial_y)^l,$$

where for any $x \in \mathcal{O}$, there are only finite number of i, j, k, l such that $C_{ij,kl} \neq 0$.

Property 2.1 *If C is invariant under the $\mathfrak{g}$ action, then on $\mathcal{O}$, $C_{ij;kl}$ satisfies the following relations:*

1. *$C_{ij;kl}$ are all constants.*
2. *if $i > l$ or $j < k$, then $C_{ij;kl} = 0$;*
3. $$\begin{aligned} C_{ij;kl} &= -C_{i+1,j-1;k-1,l+1}, \quad & for\ j \ge 1,\ k \ge 1; \\ C_{ij;kl} &= -C_{i-1,j+1;k+1,l-1}, \quad & for\ i \ge 1,\ l \ge 1. \end{aligned}$$
4. $$\begin{aligned} C_{ij;kl} &= -C_{i+2,j-1;k-2,l+1}, \quad & for\ j \ge 1,\ k \ge 2; \\ C_{ij;kl} &= -C_{i-2,j+1;k+2,l-1}, \quad & for\ i \ge 2,\ l \ge 1. \end{aligned}$$

Proof. We work on each generator of $\mathfrak{g}$.

1. $1 \in \mathfrak{g}$. This part is trivial. Because the Hamiltonian vector field of 1 is 0, every bidifferential operator is invariant under it.

2. $x \in \mathfrak{g}$. The Hamiltonian vector field generated by x is ∂_y. The invariance of C under ∂_y implies

$$\begin{aligned} &\partial_y(\textstyle\sum_{0\le i,j,k,l<\infty} C_{ij;kl}(\partial_x)^i(\partial_y)^j \otimes (\partial_x)^k(\partial_y)^l) \\ &= \textstyle\sum_{0\le i,j,k,l<\infty} C_{ij;kl}((\partial_x)^i(\partial_y)^{j+1} \otimes (\partial_x)^k(\partial_y)^l + (\partial_x)^i(\partial_y)^j \otimes (\partial_x)^k(\partial_y)^{l+1}). \end{aligned}$$

   We expand the left-hand side of the above equation, and after cancellation, we have

$$\partial_y(C_{ij;kl}) = 0.$$

3. $y \in \mathfrak{g}$. Similar to the case of x, we get

$$\partial_x(C_{ij;kl}) = 0.$$

   From the above, we have that on $\mathcal{O}$, $\partial_x(C_{ij;kl}) = \partial_y(C_{ij;kl}) = 0$, and therefore $C_{ij;kl}$ is a constant.

4. $x^2 \in \mathfrak{g}$. The Hamiltonian vector field generated by x^2 is $2x\partial_y$. The invariance of C under $2x\partial_y$ gives

$$
\begin{aligned}
& 2x\partial_y(C_{ij;kl}(\partial_x)^i(\partial_y)^j \otimes (\partial_x)^k(\partial_y)^l) \\
= & \textstyle\sum_{0\leq i,j,k,l<\infty} C_{ij;kl}((\partial_x)^i(\partial_y)^j(2x\partial_y) \otimes (\partial_x)^k(\partial_y)^l \\
+ & (\partial_x)^i(\partial_y)^j \otimes (\partial_x)^k(\partial_y)^l(2x\partial_y)).
\end{aligned} \tag{1}
$$

Setting $x = 0$ in the above equation, we get

$$
\sum_{0\leq i,j,k,l<\infty} C_{ij;kl}((\partial_x)^{i-1}(\partial_y)^{j+1} \otimes (\partial_x)^k(\partial_y)^l + (\partial_x)^i(\partial_y)^j \otimes (\partial_x)^{k-1}(\partial_y)^{l+1}) = 0, \tag{2}
$$

where the first term exists when $i > 0$, and the second term exists when $k > 0$.

(a) We look at terms of the form $(\partial_x)^i \otimes (\partial_x)^k {\partial_y}^l$. It is easy to find that the first term of Equation (2) does not have this kind of term since its existence requires j to be greater than or equal to 1. From this, we have

$$C_{i0;kl} = 0 \qquad \forall k > 0.$$

(b) Next, we look at terms of the form $(\partial_x)^i(\partial_y)^j \otimes (\partial_x)^k$. Arguments like those above show that

$$C_{ij;k0} = 0 \qquad \forall i > 0.$$

(c) If $j > 0, l > 0$, equation (1) gives

$$\sum_{0\leq i,j,k,l<\infty} C_{i+1,j-1;kl}(\partial_x)^i(\partial_y)^j \otimes (\partial_x)^k(\partial_y)^l + C_{ij;k+1,l-1}(\partial_x)^i(\partial_y)^j \otimes (\partial_x)^k(\partial_y)^l = 0.$$

This shows that

$$C_{i+1,j-1;kl} + C_{ij;k+1,l-1} = 0.$$

Therefore,

i. if $j > 0,\ k > 0$,

$$C_{ij;kl} = -C_{i+1,j-1;k-1,l+1};$$

ii. if $i > 0,\ l > 0$,

$$C_{ij;kl} = -C_{i-1,j+1;k+1,l-1}.$$

According to (a), and iteration using i. of (c), we get that if $j < k$, $C_{ij;kl} = 0$. Similarly, by (b) and ii. of (c), we get that if $i > l$, $C_{ij;kl} = 0$.

5. x^3. The Hamiltonian vector field generated by x^3 is $3x^2\partial_y$.

As in the arguments for x^2, we get

(a) if $k > 1$, $C_{i0;kl} = 0$;
(b) if $i > 1$, $C_{ij;k0} = 0$;
(c) if $j \geq 1$ and $l \geq 1$,

$$C_{i+2,j-1;k,l} + C_{ij;k+2,l-1} = 0.$$

We can rewrite it as the following,

i. if $j \geq 1$ and $k \geq 2$,

$$C_{ij;kl} = -C_{i+2,j-1;k-2,l+1};$$

ii. if $i \geq 2$ and $l \geq 1$,

$$C_{ij;kl} = -C_{i-2,j+1;k+2,l-1}. \quad \square$$

With the above preparation, we prove the following theorem.

Theorem 2.2 *For the Hamiltonian $\mathfrak{g}$ action on $(\mathbb{R}^2,\ dx \wedge dy)$, there is no geometrically $\mathfrak{g}$ invariant $\star$ product.*

Proof. We prove the theorem by contradiction. Assume that there is a $\star$ product of $(\mathbb{R}^2, dx \wedge dy)$ of the form

$$\sum_{r \geq 0} \hbar^r C^r,$$

which is geometrically $\mathfrak{g}$ invariant.

For each $r > 0$, by the assumption of locality, on an open set $\mathcal{O}$ of $\mathbb{R}^2$ containing the origin, we can write

$$C^r = \sum_{0 \leq i,j,k,l < \infty} C^r_{ij;kl}(\partial_x)^i(\partial_y)^j \otimes (\partial_x)^k(\partial_y)^l.$$

According to the associativity of $\star$ for the $\hbar^2$-term and comparing the corresponding coefficients, we have that for any $f,\ g,\ h \in C^\infty(\mathbb{R}^2)$,

(3)
$$C^2(fg,h) + C^1(C^1(f,g),h) + C^2(f,g)h = C^2(f,gh) + C^1(f,C^1(g,h)) + fC^2(g,h).$$

In the following, we will restrict our discussion on the open set $\mathcal{O}$.

1. We look at the coefficient of the term $f_{yy}g_xh_x$.

   - On the left-hand side of equation (3):

     (a) $C^2(fg,h)$. It may possibly contribute the term $C^2_{12;10}$. But according to the conclusion of Proposition 2.1 that if $i > l$, then $C_{ij;kl} = 0$, we have $C^2_{12;10} = 0$. Therefore, $C^2(fg,h)$ has no term of the form $f_{yy}g_xh_x$.

(b) $C^1(C^1(f,g),h)$. There are two C^1. As we have h_x term, the outside C^1 has to be of the form $C^1_{ij;10}$. According the result of Proposition 2.1, if $i>l$, then $C_{ij;kl}=0$, we have that there are only two possibilities for the outside C^1:

$$C^1_{01;10} \quad \textit{and} \quad C^1_{02;10}.$$

If the outside C^1 contributes $C^1_{01;10}$, then as all the $C^1_{ij;kl}$ are constant, the inside one also has to contribute $C^1_{01;10}$. Therefore, there is a contribution of $(C^1_{01;10})^2$.
If the outside C^1 has $C^1_{02;10}$, then the inside C^1 can only contribute $C^1_{00;10}$, but by Proposition 2.1, it has to equal 0, because $j<k$.
So the second term has only one contribution which is $(C_{01;10})^2$.

(c) $C^2(f,g)h$. Because in this term there is no derivative with respect to h, this term cannot contribute anything.

In summary, the left-hand side of the above equation can only contribute $(C_{01;10})^2$ to the coefficient of $f_{yy}g_xh_x$.

- On the right-hand side of equation (3).

(a) $C^2(f,gh)$ can only possibly contribute $C^2_{02;20}$. But according to Proposition 2.1,

$$C^2_{02;20} = -C^2_{21;01}.$$

But from $i>l$, we know $C^2_{21;01}=0$. Therefore, there is no contribution of this term.

(b) $C^1(f,C^1(g,h))$. By comparing the number of derivatives of f, we know that the outside C^1 has to be of the form $C^1_{02;kl}$. As the differential of g and h are all with respect to x, there are three possibilities for the outside C^1:

$$C^1_{02;00}, \quad C^1_{02;10}, \quad \textit{and} \quad C^1_{02;20}.$$

In the following, we will show that all three of them do not have any contribution.

i. $C^2_{02;00}$. Then the inside C^1 has to be of the form $C^1_{10;10}$. This is 0 according to Proposition 2.1.

ii. $C^2_{02;10}$. Then the inside C^1 has to be of the form $C^1_{10;00}$ or $C^1_{00;10}$, which are both 0 because of Proposition 2.1.

iii. $C^2_{02;20}$. From the previous calculation, we know that $C^2_{02;20}=0$.

(c) $fC^2(g,h)$. Because this term has no derivative of f, there is no contribution of this term.

In all, considering both sides of equation (3), there is only one contribution of the term $f_{yy}g_xh_x$, which is $(C^1_{01;01})^2$. Therefore, we have

$$C^1_{01;10}=0.$$

2. We look at the coefficient of $f_{xx}g_yh_y$.

- On the left-hand side of equation (3).
  (a) $C^2(fg,h)$. The only possible contribution is $C^2_{21;01}$. But according to Proposition 2.1, $C^2_{21;01}=0$.
  (b) $C^1(C^1(f,g),h)$. By comparing the derivatives of h, we get that the outside C^1 has to be of the form $C^1_{ij;01}$. As i has to be less than or equal to 1, otherwise this term is 0 according to proposition 2.1, we know that there are four possibilities;

$$C^1_{10;01},\quad C^1_{11;01},\quad C^1_{01;01},\quad \textit{and}\quad C^1_{00;01}.$$

  In the following, we will show that except for $C^1_{10;01}$, the other three cases have no contributions.
    i. $C^1_{10;01}$. In this case, the inside C^1 also has to be of the form $C^1_{10;01}$. The contribution of this term is $(C^1_{10;01})^2$.
    ii. $C^1_{11;01}$. Then the inside C^1 has to be of the form $C^1_{10;00}$, but this has to be 0 because $i>l$. So this term has no contribution.
    iii. $C^1_{01;01}$. Then the inside C^1 has to be of the form $C^1_{20;00}$. This also has to be 0, because $i>l$. This term again has no contribution.
    iv. $C^1_{00;01}$. Then the inside C^1 has to be of the form $C^1_{20;01}$. This is 0 for the same reason as the $C^1_{20;00}$.
  (c) $C^2(f,g)h$. This has no contribution, because there is no derivative on h.
- On the right-hand side of the relation.
  (a) $C^2(f,gh)$. The only possible contribution of $C^2(f,gh)$ is of the form $C^2_{20;02}$. This has to be 0, because $C^2_{20;02}=C^2_{01;21}=0$.
  (b) $C^1(f,C^1(g,h))$. Comparing the part of f, we know that the outside C^1 has to be of the form $C^1_{20,kl}$. As i has to be less than or equal to l, the outside C^2 has to be of the form $C^2_{20;02}$, which is 0.

  In conclusion, total in both sides of equation (3), there is only one contribution $(C^1_{10;01})^2$ for term $f_{xx}g_yh_y$. Therefore $C^1_{10;01}=0$.

We have shown that $C^1_{10;01}$ and $C^1_{01;10}$ are both 0. But on the other hand, from

$$[u,v]=u\star v-v\star u=-i\hbar\{u,v\}+o(\hbar),$$

we have

$$C^1_{10;01}-C^1_{01;10}=-i.$$

If $C^1_{10;01}=C^1_{01;10}=0$, the above equality cannot be true. So we get a contradiction.

Therefore, there is no geometrically $\mathfrak{g}$ invariant star product on $(\mathbb{R}^2, dx\wedge dy)$.
□

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