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Integrating Poisson manifolds via stacks

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Abstract

A symplectic groupoid $G. := (G_1 \rightrightarrows G_0)$ determines a Poisson structure on G_0. In this case, we call $G.$ a symplectic groupoid of the Poisson manifold G_0. However, not every Poisson manifold has such a symplectic groupoid. This keeps us away from some desirable goals: For example, establishing Morita equivalence in the category of all Poisson manifolds. In this paper, we construct symplectic Weinstein groupoids which provide a solution to the above problem (Theorem 1.1). More precisely, we show that a symplectic Weinstein groupoid induces a Poisson structure on its base manifold, and that to every Poisson manifold there is an associated symplectic Weinstein groupoid.

1 Introduction

The notion of a symplectic groupoid (see [10], [23]) was introduced in Weinstein's program of quantization of Poisson manifolds. There is an almost 1-1 correspondence between symplectic groupoids and *integrable* (to be explained below) Poisson manifolds. This correspondence is closely related to the integration problem of Lie algebroids, which we now explain.

Recall that a Lie algebroid over a manifold M is a vector bundle $\pi : A \to M$ with a real Lie bracket structure $[\,,\,]$ on its space of sections $H^0(M, A)$ and a bundle map $\rho : A \to TM$ such that the Leibniz rule

$$[X, fY](x) = f(x)[X, Y](x) + (\rho(X)f)(x)Y(x)$$

holds for all $X, Y \in H^0(M, A)$, $f \in C^\infty(M)$ and $x \in M$. The map ρ is called the anchor. It induces a Lie algebra homomorphism between $H^0(M, A)$ and the space of global vector fields on M.

When M is a point, a Lie algebroid becomes a Lie algebra. A Lie algebra encodes the infinitesimal information of a Lie group. One obtains a Lie algebra from a Lie group by differentiation. One may think the process of obtaining a Lie group from a Lie algebra as a kind of "integration". In analogy, a Lie algebroid can be thought of as an infinitesimal version of a Lie groupoid. One can obtain a Lie algebroid from a Lie groupoid by taking invariant vector fields and restricting them to the identity section. The integrability problem of Lie algebroids asks for a reverse process, namely one that associates to a Lie algebroid A a Lie groupoid whose Lie algebroid is A. This problem, first formulated in [18], has attracted a lot of attention over time. A solution using local groupoids was also given in [18], but the global integrating object, which is important for establishing Morita equivalence of Poisson manifolds, was still missing. An important approach to finding such a global object is the use of path spaces. This idea is not new, see [25] for a nice discussion. We pay particular attention to the recent work [7] of M. Crainic and R. Fernandes and [5] of A. Cattaneo and G. Felder. For a Lie algebroid A, they study the space of A-paths. They are able to give a negative answer to the integrability problem—not every Lie algebroid can be integrated into a Lie groupoid. From the space of A-paths they construct a *topological* groupoid and determine equivalent conditions for this groupoid to be a Lie groupoid that integrates the given Lie algebroid A. So their work shows that every Lie algebroid can be integrated into a topological groupoid, but in general this topological groupoid doesn't have enough information to recover the Lie algebroid we start with. As conjectured by Weinstein, one hopes that there are additional structures on this topological groupoid which allow us to recover the Lie algebroid. In [21], the authors find such structures. The key idea is to enlarge the category one works into the category of differentiable stacks. We introduce the notion of *Weinstein groupoid* which formalizes the additional structures to put on this topological groupoid. By allowing Weinstein groupoids, we answer the integrability problem positively—every Lie algebroid can be integrated into a Weinstein groupoid.

For a Poisson manifold M, there is an associated Lie algebroid $T^*M \to M$. The anchor map $T^*M \to TM$ is given by the contraction with the Poisson bivector, and the Lie bracket is induced by the Poisson bracket $\{,\}$ of M,

$$[df, dg] := d\{f, g\}.$$

If $T^*M \to M$ can be integrated into a Lie groupoid $G_1 \rightrightarrows M$, then there is a natural multiplicative 2-form ω on G_1. The identity

$$m^*\omega = pr_1^*\omega + pr_2^*\omega,$$

holds on the composable pairs $G_1 \times_M G_1$, where m is the multiplication and pr_j are projections onto the j-th components. It turns out [14] that the source map of G_1 is a Poisson map. That is, the symplectic structure of G_1 and the source map recover the Poisson structure on M. In this case, G_1 is called a *symplectic groupoid* of M. Moreover, there is a unique source-simply connected symplectic groupoid of M.

Conversely, to each Poisson manifold M, we want to find a symplectic groupoid over it which induces the Poisson structure of M. As explained above, one may take the Lie groupoid integrating Lie algebroid $T^*M \to M$. Unfortunately, even this special kind of Lie algebroids may not be integrable (see for example [3] and the references therein). However, if we allow Weinstein groupoids, then there is always such a reverse procedure. In this paper, we construct *symplectic Weinstein groupoids* (see Definition 4.4) for every Poisson manifold and prove the analogue of the classical statement above. More precisely,

Theorem 1.1. *For any symplectic Weinstein groupoid $\mathcal{G} \rightrightarrows M$, the base manifold M has a unique Poisson structure such that the source map $\bar{\mathbf{s}}$ is Poisson. In this case, we call $\mathcal{G}$ a symplectic Weinstein groupoid of the Poisson manifold M.*

On the other hand, for any Poisson manifold M, there are two symplectic groupoids $\mathcal{G}(T^*M)$ and $\mathcal{H}(T^*M)$ of M.

We also relate the symplectic Weinstein groupoid to the classical symplectic groupoid in the case when T^*M is integrable.

Theorem 1.2. *A Poisson manifold M is integrable, i.e. M has an associated symplectic groupoid, if and only if $\mathcal{H}(T^*M)$ is representable. In this case, $\mathcal{H}(T^*M)$ is the source-simply connected symplectic groupoid integrating M.*

The paper is organized as follows. In Section 2 we recall the notion and properties of differentiable stacks. In Section 3 we discuss the notion of Weinstein groupoids and results in [21]. In Section 4 we introduce the notion of symplectic Weinstein groupoids and establish a correspondence between symplectic Weinstein groupoids and Poisson manifolds.

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2 Differentiable stacks

In this section we briefly discuss the notion of differentiable stacks. In the past few decades stacks over the category of schemes had be extensively studied in algebraic geometry, especially in connection with moduli problems (see for instance [8], [22], [12], [1]). As known in the early days, stacks can also be defined over other categories such as category of topological spaces or smooth manifolds (see for instance [20], [19], [2],[13]). In this paper we focus on stacks over the category of smooth manifolds, which are called *differentiable stacks*. We refer the readers to [19], [2] and [13] for a more detailed discussion about differentiable stacks.

2.1 Definitions

Let $\mathcal{C}$ be the category of finite dimensional second countable, (but not necessarily Hausdorff) differentiable manifolds[1]. A stack over $\mathcal{C}$ is a category fibred in

[1] In short, these will be called smooth manifolds in this paper.

groupoids satisfying two conditions: "isomorphism is a sheaf" and "descend datum is effective". Both conditions are rather complicated to describe. A precise definition can be found in [2], [13]. See also [9] for an illuminating discussion.

Example 2.1.

1. A manifold M can be viewed as a stack over $\mathcal{C}$. Let $\underline{M}$ denote the following category: The objects are pairs (S, u) where S is a manifold and $u : S \to M$ is a smooth map. A morphism $(S, u) \to (T, v)$ of objects is a smooth map $f : S \to T$ such that $u = v \circ f$. The category $\underline{M}$ is a stack. It contains all the information about the manifold M. In this way the notion of stacks generalizes that of manifolds and we identify manifolds with their associated stacks. A stack over $\mathcal{C}$ is called *representable* if it is of the form $\underline{M}$ for some manifold M.

2. Let G be a Lie group. Recall that the category BG of principal G-bundles is defined as follows: The objects are principal G-bundles $P \to M$ over manifolds. A morphism between two objects $P \to M$ and $P' \to M'$ is a smooth map $M \to M'$ and a G-equivariant map $P \to P'$ that covers $M \to M'$. In fact BG is a stack –the classifying stack of G-bundles.

Morphisms between stacks are functors between their underlying fibred categories. A morphism $f : \mathcal{X} \to \mathcal{Y}$ is a representable submersion if for any morphism $M \to \mathcal{Y}$ from a manifold M, the fiber product $\mathcal{X} \times_{\mathcal{Y}} M$ is representable and the induced morphism $\mathcal{X} \times_{\mathcal{Y}} M \to M$ is a submersion (between manifolds). If in addition $\mathcal{X} \times_{\mathcal{Y}} M \to M$ is surjective, then f is called a *representable surjective submersion*, see [2].

Definition 2.1. A differentiable stack $\mathcal{X}$ is a stack over $\mathcal{C}$ together with a representable surjective submersion $\pi : X \to \mathcal{X}$ from a smooth manifold X. The morphism $\pi : X \to \mathcal{X}$ is called an *atlas* of $\mathcal{X}$. We often abuse notation and call X an atlas of $\mathcal{X}$. Needless to say, atlases are not unique.

Properties of morphisms between differentiable stacks can be defined by considering pullbacks to atlases. In this way one can define what it means for a morphism to be smooth, étale[2], an immersion, etc. A stack $\mathcal{X}$ is said to be *étale* if it has an atlas $\pi : X \to \mathcal{X}$ where π is étale.

2.2 Stacks and Groupoids

Roughly speaking, there is a one-to-one correspondence between differentiable stacks and Lie groupoids. For a differentiable stack $\mathcal{X}$ with an atlas $X_0 \to \mathcal{X}$, we obtain a Lie groupoid $X_1 := X_0 \times_{\mathcal{X}} X_0 \rightrightarrows X_0$ where the two maps are projections.

[2]In the smooth category, being étale means being locally diffeomorphic.

This groupoid is called a groupoid presentation of $\mathcal{X}$. An étale differentiable stack has an étale groupoid presentation. Different atlases give different groupoids. But different groupoid presentations of the same stack are *Morita equivalent* (see [19], [2], [13]). Given a groupoid, one can construct a stack (see [22], [2]). This process is complicated for a general groupoid. We describe only a special case.

Example 2.2. Consider a Lie group G acting on a manifold M. This corresponds to a groupoid $G \times M \rightrightarrows M$ where the two maps are the action and the projection to the second factor. Define a category $[M/G]$ as follows: An object is a principal G-bundle $P \to B$ over a manifold B with a G-equivariant map $P \to M$. A morphism between two objects $B \leftarrow P \to M$ and $B' \leftarrow P' \to M$ is a pair of a map $B \to B'$ and a G-equivariant map $P \to P'$ making all natural diagrams commute. The category $[M/G]$ is in fact a differentiable stack with an atlas $M \to [M/G]$.

Differentiable stacks correspond to Morita equivalence classes of Lie groupoids. 1-morphisms between differentiable stacks correspond to what are called *Hilsum-Skandalis* morphisms (HS morphisms), see [17] and [21] for more details.

Remark 2.1. It is not hard to see that the construction of stacks in the category of smooth manifolds can be extended to the category of Banach manifolds (we use the definition in [11]), yielding the notion of Banach stacks. Main properties discussed here also hold for Banach stacks. Since the category of smooth manifolds is a full subcategory of the category of Banach manifolds, the category of differentiable stacks can be obtained from the category of Banach stacks by restricting the base category.

3 Weinstein groupoids

3.1 The Definition

Definition 3.1 ([21]). A Weinstein groupoid over a manifold M consists of the following data: An étale differentiable stack $\mathcal{G}$, two surjective submersions $\bar{\mathbf{s}}, \bar{\mathbf{t}} : \mathcal{G} \to M$ (source and target), a map $\bar{m} : \mathcal{G} \times_{\bar{\mathbf{s}}, \bar{\mathbf{t}}} \mathcal{G} \to \mathcal{G}$ (multiplication), an injective immersion $\bar{e} : M \to \mathcal{G}$ (identity section), and an isomorphism $\bar{i} : \mathcal{G} \to \mathcal{G}$ (inverse). These maps are required to satisfy identities[3] analogous to those of a groupoid.

Roughly speaking, a Weinstein groupoid is a groupoid in the category of differentiable stacks. Let G be the orbit space of the stack $\mathcal{G}$, which is a topological space. The data of a Weinstein groupoid induce a topological groupoid $G \rightrightarrows M$.

[3] Those identities are required to hold only up to 2-morphisms.

3.2 The Path Spaces

We now explain the use of path spaces in the integrability problem.

Definition 3.2.

1. ([7]) Let $\pi : A \to M$ be a Lie algebroid with anchor $\rho : A \to TM$. A C^1-map $a : I = [0,1] \to A$ is an A-path if the equation
$$\rho(a(t)) = \frac{d}{dt}(\pi \circ a(t))$$
holds.

2. ([21]) Such a map $a : I \to A$ is called an A_0-path if, in addition, both a and $\frac{da}{dt}$ vanish on the boundary.

Denote by PA and P_0A the spaces of A- and A_0-paths respectively. It's known that PA is a Banach manifold (of infinite dimension) and P_0A is a Banach submanifold of PA (see [7], Section 1 and [21], Section 2). We next consider the notion of homotopy.

Definition 3.3 ([7], [21]). Let $a(\epsilon, t)$ be a family of A_0-paths which is C^2 in ϵ. Assume that the base paths $\gamma(\epsilon, t) := \rho \circ a(\epsilon, t)$ have fixed end points. For a connection ∇ on A, consider the equation

$$\partial_t b - \partial_\epsilon a = T_\nabla(a, b), \quad b(\epsilon, 0) = 0. \tag{3.1}$$

Here T_∇ is the torsion of the connection defined by $T_\nabla(\alpha, \beta) = \nabla_{\rho(\beta)}\alpha - \nabla_{\rho(\alpha)}\beta + [\alpha, \beta]$. Two paths $a_0 = a(0, \cdot)$ and $a_1 = a(1, \cdot)$ are homotopic if the solution $b(\epsilon, t)$ satisfies $b(\epsilon, 1) = 0$.

Remark 3.1.

1. A solution $b(\epsilon, t)$ to (3.1) doesn't depend on ∇. Therefore the definition makes sense. Furthermore $b(\cdot, t)$ is an A-path for each fixed t.

2. This definition is analogous to the definition of homotopy of A-paths in [7].

Homotopies of paths generate foliations $\mathcal{F}$ and $\mathcal{F}_0$ on PA and P_0A respectively. The foliation $\mathcal{F}$ restricts to $\mathcal{F}_0$ on P_0A. Now the idea is to consider the *monodromy groupoid* (see [16]) of this foliation: The objects are points in the manifold, and arrows are paths within a leaf (up to homotopies) with fixed end points inside the leaf. Let $Mon(P_0A) \rightrightarrows P_0A$ denote the groupoid associated to the foliation $\mathcal{F}_0$. This groupoid encodes the equivalence relation (i.e. homotopy) of A_0-paths. One could think of $Mon(P_0A)$ as the space of homotopies of A_0-paths. The two maps from $Mon(P_0A)$ to P_0A assign to each homotopy the two paths at the ends.

There are also two maps $P_0A \rightrightarrows M$ which assign to each A_0-path its two end points respectively.

Strictly speaking $Mon(P_0A) \rightrightarrows P_0A$ is not a Lie groupoid since both spaces are infinite dimensional. But it is a smooth groupoid in the category of Banach manifolds. To avoid dealing with infinite dimensional issues, we sometimes consider a variant $\Gamma \rightrightarrows P$ of this groupoid obtained as follows (see for example [21] for details): P is the disjoint union of an open cover of P_0A, and Γ is the disjoint union of slices over this cover that are transversal to the foliation $\mathcal{F}_0$. Then P is a smooth manifold in $\mathcal{C}$, $\Gamma \rightrightarrows P$ is a finite dimensional Lie groupoid. What's even better is that it's an étale groupoid (i.e. the source and target are étale). The two groupoids $Mon(P_0A) \rightrightarrows P_0A$ and $\Gamma \rightrightarrows P$ are in fact Morita equivalent. Also, there are still two maps $P \rightrightarrows M$.

The next step is clear: We want to consider homotopy equivalence classes of paths and declare that points joined by a homotopy class of paths are equivalent. For this we need to take the "quotient" $P_0A/Mon(P_0A)$ and construct a "groupoid" $P_0A/Mon(P_0A) \rightrightarrows M$ where the two maps are endpoint maps. There are at least two ways to do this. We can take the quotient as a topological space (the topological quotient). Then we obtain a *topological* groupoid $P_0A/Mon(P_0A) \rightrightarrows M$ which might not carry any further structure. There is information lost in this process essentially because the topological quotient remembers only orbits of the equivalence relation $Mon(P_0A) \to P_0A \times P_0A$ given by the groupoid $Mon(P_0A) \rightrightarrows P_0A$ but forgets the finer structures of an orbit.

We can also take the quotient as stacks, namely consider the stack associated to the groupoid $Mon(P_0A) \rightrightarrows P_0A$. Given the correspondence between groupoids and stacks, we expect not to lose any information doing this. Denote the stack quotient by $\mathcal{G}(A) := [P_0A/Mon(P_0A)]$. Since $Mon(P_0A) \rightrightarrows P_0A$ and $\Gamma \rightrightarrows P$ are Morita equivalent, the quotient $[P/\Gamma]$ also equals to $\mathcal{G}(A)$. Since $\Gamma \rightrightarrows P$ is étale, $\mathcal{G}$ is an étale stack. From the technical viewpoint, we take $\mathcal{G}(A)$ as an étale groupoid from now on, hence avoid infinite dimensional analysis. Moreover, the two maps to M descend to the quotient, giving two maps $\bar{\mathbf{s}}, \bar{\mathbf{t}} : \mathcal{G} \to M$. There are other maps: By concatenation of paths, we can define a "multiplication" $\bar{m} : \mathcal{G} \times_{\bar{\mathbf{s}},\bar{\mathbf{t}}} \mathcal{G} \to \mathcal{G}$; by reversing the orientation of a path, we can define an "inverse" $\bar{i} : \mathcal{G} \to \mathcal{G}$; by considering constant paths, we can define an "identity section" $\bar{e} : M \to \mathcal{G}$. These maps are defined in detail in [21]. There, we prove that this makes $\mathcal{G} \rightrightarrows M$ into a Weinstein groupoid. A similar procedure using the holonomy groupoid $Hol(P_0A) \rightrightarrows P_0A$ produces another natural Weinstein groupoid $\mathcal{H}(A)$, see [21] for more details.

Theorem 3.1 ([21]).

1. *(Lie's third theorem) To each Weinstein groupoid one can associate a Lie algebroid. For every Lie algebroid A there are two natural Weinstein groupoids $\mathcal{G}(A)$ and $\mathcal{H}(A)$ with Lie algebroid A.*

2. *A Lie algebroid A is integrable in the classical sense if and only if $\mathcal{H}(A)$ is representable, namely it's a Lie groupoid in the category of manifolds. In this case $\mathcal{H}(A)$ is the source-simply connected Lie groupoid*[4] *of A.*

3. *The orbit spaces of $\mathcal{G}(A)$ and $\mathcal{H}(A)$ (which are topological spaces) are both isomorphic to the universal topological groupoid of A constructed in [7].*

4. *Given a Weinstein groupoid $\mathcal{G}$, there is a local groupoid*[5] *G_{loc} whose Lie algebroid is the same as that of $\mathcal{G}$.*

4 Symplectic Weinstein groupoids

In this section we consider the integration problem of Poisson manifolds, namely, the integrability of the Lie algebroid $T^*M \to M$ associated to a Poisson manifold M. We introduce the notion of symplectic and Poisson structures on a differentiable stack and apply Theorem 3.1 to establish a correspondence between Poisson manifolds and what we call symplectic Weinstein groupoids (see Definition 4.4).

4.1 Symplectic and Poisson Structures

Definition 4.1. Let $\mathcal{X}$ be a stack over $\mathcal{C}$. The *sheaf of differential k-forms* of $\mathcal{X}$ is a contravariant functor $\mathcal{F}^k$ from $\mathcal{X}$ to the category of vector spaces. For every $x \in \mathcal{X}$ over $U \in \mathcal{C}$, define $\mathcal{F}^k(x) := \Omega^k(U)$. For every arrow $y \to x$ over $f : V \to U$, there is a map $\mathcal{F}^k(f) : \mathcal{F}^k(x) \to \mathcal{F}^k(y)$ defined by the pullback $f^* : \Omega^k(U) \to \Omega^k(V)$.

The functor $\mathcal{F}^k$ is in fact a sheaf over $\mathcal{X}$, see [2] for the definition of sheaves over stacks and the proof of this fact. A *differential k-form* ω on $\mathcal{X}$ is a map that associates to an element $x \in \mathcal{X}$ over U a section $\omega(x) \in \Omega^k(U)$ such that the following compatibility condition holds: If there is an arrow $y \to x$ over $f : V \to U$, then $\omega(y)$ is the pull back of $\omega(x)$ by f. Notice that according to this definition, the 0-forms on $\mathcal{X}$ are simply the maps from $\mathcal{X}$ to $\mathbb{R}$ (viewed as a stack).

There is a simpler interpretation when the stack is étale:

Lemma 4.1 ([26]). *Let $\mathcal{X}$ be an étale differentiable stack and G an étale groupoid presentation of $\mathcal{X}$. Then there is a 1-1 correspondence between k-forms on $\mathcal{X}$ and G-invariant k-forms on G_0.*

Proof. A G-invariant k-form ω on G_0 defines a differential form on $\mathcal{X}$ as follows: Given a right G-principal bundle $\pi : P \to U$ with moment map $J : P \to G_0$, the pull back form $J^*\omega$ is G-invariant on P. Therefore it induces a k-form $\pi_* J^*\omega$ on

[4] It's called the Weinstein groupoid of A in [7].
[5] It is unique up to isomorphisms near the identity section.

U and this is what P associates to via ω. Notice that we use the fact that π is étale to show that a G-invariant form is a basic form. On the other hand, given any k-form ω on $\mathcal{X}$, consider $\mathbf{t} : G_1 \to G_0$ as a right G-principal bundle with moment map $\mathbf{s} : G_1 \to G_0$. Then $\omega(G_1)$ is a k-form on G_0. Notice that the left multiplication by a bisection $g\cdot : G_1 \to G_1$ is a morphism of G-principal bundles. The compatibility condition of ω implies that $\omega(G_1)$ is G-invariant. □

Definition 4.2 (pullbacks of forms on stacks). Let $\phi : \mathcal{Y} \to \mathcal{X}$ be a map between stacks and ω a form on $\mathcal{X}$. Then $\phi^*\omega$ is a form on $\mathcal{Y}$ defined by associating to $y \in \mathcal{Y}$ the section $\omega(\phi(y))$.

Remark 4.1. We omit here the proof that the above definition is well-defined (see for example [26]). Using Lemma 4.1, the pullbacks of forms on étale differentiable stacks correspond to the ordinary pullbacks on their étale atlases (also see Lemma 4.2 for the proof in the case that ϕ is the identity map).

To simplify the discussion in this paper, we define symplectic, contact, and Poisson structures *only* in the étale case, though in other cases (including the Banach case) similar but more technical definitions could be made. With Lemma 4.1, we can make the following definition:

Definition 4.3. A *symplectic (resp. contact) form on an étale differentiable stack* $\mathcal{X}$ is a G-invariant symplectic (resp. contact) form on G_0, where G is an étale presentation of $\mathcal{X}$.

Remark 4.2. Since the source and target maps $\mathbf{t}$ and $\mathbf{s}$ of G. are étale, a G-invariant form ω is simply a form satisfying $\mathbf{s}^*\omega = \mathbf{t}^*\omega$. By Remark 4.1, a form is symplectic on an étale differentiable stack iff it is symplectic on all étale presentations.

Definition 4.4. A Weinstein groupoid $\mathcal{G} \rightrightarrows M$ is a *symplectic Weinstein groupoid* if there is a symplectic form ω on $\mathcal{G}$ satisfying the following *multiplicative* condition:

$$\bar{m}^*\omega = pr_1^*\omega + pr_2^*\omega,$$

on $\mathcal{G} \times_{\bar{\mathbf{s}},M,\bar{\mathbf{t}}} \mathcal{G}$, where pr_i is the projection onto the i-th factor.

Remark 4.3. When $\mathcal{G}$ is a Lie groupoid, this definition coincides with the definition of symplectic groupoids.

4.2 Integrability

We will show that, after replacing the symplectic groupoid by the symplectic Weinstein groupoid, the correspondence between Poisson manifolds and symplectic groupoids holds for every Poisson manifold.

Lemma 4.2. *Give an étale differentiable stack $\mathcal{X}$ with a symplectic form ω, there is a Poisson bracket $\{,\}$ on the algebra $C^\infty(\mathcal{X})$ of 0-forms (i.e. smooth functions) on $\mathcal{X}$.*

Proof. Take an étale groupoid presentation $G. = (G_1 \rightrightarrows G_0)$ of $\mathcal{X}$ and identify $\mathcal{X}$ with $BG.$. Then $C^\infty(\mathcal{X})$ is the set of $G.$-invariant functions f_G's on G_0, so it is naturally an algebra. Moreover, ω appears as a $G.$-invariant symplectic form ω_G on G_0. Therefore we can define $\{f, g\}_G$, the appearance of $\{f, g\}$ on the presentation $G.$, as $\{f_G, g_G\}_{\omega_G}$ where $\{,\}_{\omega_G}$ is the Poisson bracket defined by ω_G.

We have to show that the above definition is independent of choices of the étale presentations. The groupoid $G.$ is said to be *strongly equivalent* to another groupoid $H.$ if there is a groupoid morphism $\phi : G. \to H.$ and the HS bibundle $E := G_0 \times_{\phi, H_0, \mathbf{t}} H_1$ associated to ϕ is a Morita bibundle. If two groupoids are Morita equivalent, they are both strongly equivalent to a third groupoid (see for example [15]). Hence it suffices to show that if $G.$ is strongly equivalent to $H.$ via ϕ (and E), then they define the same Poisson bracket. Let J_G and J_H be the moment maps from E to G_0 and H_0 respectively. Notice that

$$(G_1 \times_{\bar{\mathbf{t}}\circ\phi, H_0} H_1 \to E) \to (G_1 \xrightarrow{\mathbf{t}} G_0)$$

is a morphism of $G.$-principal bundles (the first principal bundle is the pullback of the second one via J_G). Therefore, we have

$$J_G^* \omega_G = J_G^*(\omega(G_1 \xrightarrow{\mathbf{t}} G_0)) = \omega(G_1 \times_{\mathbf{t}\circ\phi, H_0} H_1 \to E).$$

If we change the presentation from $G.$ to $H.$, the right $G.$-principal bundle $G_1 \times_{\bar{\mathbf{t}}\circ\phi, H_0} H_1 \to E$ transforms via E to an $H.$-principal bundle $(G_1 \times_{H_0} H_1 \times_{G_0} E)/G_1 = G_0 \times_{H_0} H_1 \times_{H_0} H_1$ over E. On the other hand, this principal bundle is also the pullback of $H_1 \xrightarrow{\mathbf{t}} H_0$ via J_H. So

$$J_H^* \omega_H = \omega(G_0 \times_{H_0} H_1 \times_{H_0} H_1 \to E) = \omega(G_1 \times_{\mathbf{t}\circ\phi, H_0} H_1 \to E) = J_G^* \omega_G.$$

Notice that $J_H^* = J_G^* \phi^*$ and J_G is a submersion. We have $\omega_G = \phi^* \omega_H$. Similarly, for functions we also have $f_G = \phi^* f_H$. Therefore, we have

$$\{f_H, g_H\}_{\omega_H} = \{\phi^* f_G, \phi^* g_G\}_{\phi^* \omega_G} = \phi^*(\{f_G, g_G\}_{\omega_G}).$$

So the Poisson bracket on $\mathcal{X}$ is well defined. □

Given two stacks $\mathcal{X}$ and $\mathcal{Y}$ whose smooth functions form Poisson algebras, a morphism $\mathcal{X} \to \mathcal{Y}$ is called *Poisson* if the induced map $C^\infty(\mathcal{Y}) \to C^\infty(\mathcal{X})$ preserves the Poisson brackets.

Now we can prove Theorem 1.1 and 1.2.

Proof of Theorem 1.1. Given a symplectic Weinstein groupoid $\mathcal{G} \rightrightarrows M$, we can associate to it a local symplectic groupoid $G_{loc} \rightrightarrows M$. The method is similar to the proof of Theorem 3.1 (see [21]). We recall the idea: Let $G.$ be an étale presentation of the stack $\mathcal{G}$. Then $\bar{\mathbf{s}}, \bar{\mathbf{t}} : \mathcal{G} \to M$ are represented by G_1-invariant maps $\mathbf{s}, \mathbf{t} : G_0 \to M$. We divide M into pieces M_l and embed them into G_0. Then the local groupoid G_{loc} is obtained by gluing small open neighborhoods $U_l \subset G_0$ of these embedded pieces M_l. Then U_l is a local groupoid over M_l. The multiplicativity of ω on $\mathcal{G}$ implies that the symplectic form $\omega_G|_{U_l}$ is multiplicative. Since the symplectic form ω_G on the étale atlas G_0 is invariant under the G_1-action and the gluing morphisms are also induced by the G_1-action (see [21], Proposition 5.3), the multiplicative symplectic forms on the U_l's also glue together to a multiplicative symplectic form on G_{loc}. Therefore, there is a unique Poisson structure $\{,\}_M$ on M such that the source map $\mathbf{s}_{loc}$ of G_{loc} is Poisson.

Notice that the pullback $\mathbf{s}_{loc}^* f$ of $f \in C^\infty(M)$ is locally the same as $\mathbf{s}^* f$ in $C^\infty(G_0)$. Since Poisson bracket is a local operation on functions and the Poisson bracket on $\mathcal{G}$ is defined via the Poisson bracket on G_0, we conclude that the source map $\mathbf{s} : G_0 \to M$ is Poisson in the union of U_l's. To show $\mathbf{s}$ is globally Poisson on G_0 so that $\bar{\mathbf{s}}$ is a Poisson map, one needs arguments involving Lagrangian subspaces of stacks and we refer to a further paper [4].

For the converse, recall that for any Lie algebroid A, we can associate two Weinstein groupoids $\mathcal{G}(A)$ and $\mathcal{H}(A)$, as discussed in Theorem 3.1. We prove the converse statement for $\mathcal{G}(T^*M)$. The proof for $\mathcal{H}(T^*M)$ is similar. Let ω_c be the canonical symplectic form on T^*M. According to [5], ω_c induces a symplectic form Ω (not necessarily strong[6]) on the path space PT^*M. The restriction of Ω to the A-path space P_aT^*M has kernel exactly the tangent space of the foliation $\mathcal{F}$ and is invariant along the foliation. Consider the étale presentation $\Gamma \rightrightarrows P$ of $\mathcal{G}(T^*M)$ described in Section 3. Since P is the transversal of the foliation $\mathcal{F}$ and Γ is the monodromy groupoid of the restricted foliation of $\mathcal{F}$ on P, the restricted form is a Γ-invariant symplectic form. This form induces a symplectic form ω on $\mathcal{G}(T^*M)$. The multiplicativity of ω follows from the additivity of the integrals after examining the definition of Ω.

It remains to prove that $\bar{\mathbf{s}} : \mathcal{G}(T^*M) \to M$ is Poisson. As shown in the proof of Theorem 1.2 in [21], the local groupoid associated to $\mathcal{G}(T^*M)$ is exactly the symplectic local groupoid associated to M in [6]. An argument analogous to the first part of the proof shows that $\bar{\mathbf{s}}$ is a Poisson map.

Proof of Theorem 1.2. It follows from Theorem 1.1 and 3.1.

[6]A symplectic form defines a map from the tangent space to the cotangent space. A symplectic form is called *weak* if this maps is injective (but not necessarily surjective), and is called *strong* if this map is both injective and surjective.

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