Travaux mathématiques, Volume 16 (2005), 9–40, 

# (Bi)Modules, morphisms, and reduction of star-products: the symplectic case, foliations, and obstructions

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## Contents

# Introduction

Poisson geometry is naturally linked to the theory of deformation quantization invented by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer [2]: here the algebra of quantum observables in quantum theory is described by an associative formal deformation (a so-called star-product $*$) of the commutative associative algebra $\mathcal{C}^\infty(M,\mathbb{K})$ of smooth functions on a Poisson manifold $(M,P)$ such that its first order commutator is proportional to the Poisson bracket. The algebra part of this theory is finished by now: there is a general existence theorem (DeWilde-Lecomte 1983 [20] for the symplectic and Kontsevitch 1997 [38] for the general Poisson case) and a classification of formal isomorphy or equivalence classes (Deligne [18]; Nest-Tsygan [44], [45]; Bertelsson-Cahen-Gutt [3] for the symplectic (see also [33] for a review) and Kontsevitch (1997) [38] for the Poisson case).

The subject of this talk is to give an introduction to three algebraic question in deformation quantization, namely what are the *algebra morphisms*, the *modules*, and what are *commutants of modules*? I shall show that to each of these topics there is a geometric situation in 'the classical limit', i.e. *Poisson maps*, *coisotropic maps*, and *phase space reduction* whose quantization problem is most interesting and finds applications in quantum physics such as *quantization of symmetries and integrable systems*, *quantization of first class constraints*, and *quantization of the reduced Poisson algebra*. I shall present some results on these topics which are listed in the following table and also a recursive and total obstruction analysis. As opposed to the abovementioned algebra part there are still many open questions in that field. Of course the list of names in the result column is not complete, and I apologize in advance for any omission.

| Algebraic object | Geometric object in classical limit | Quantization problem | Results |
|---|---|---|---|
| **Algebra**: $\mathcal{A} := \big(\mathcal{C}^\infty(M,\mathbb{K})[[\nu]], *\big)$ associative deformation of commutative algebra | Poisson manifold $(M,P)$ | Quantization of Poisson structures | DeWilde-Lecomte 1983: symplectic case Kontsevitch 1997: general Poisson case |
| **Morphism** of algebras $\Phi : \mathcal{A} \to \mathcal{B}$ $\mathcal{B} = \big(\mathcal{C}^\infty(M',\mathbb{K})[[\nu]], *'\big)$ | Poisson map $\phi$ $(M',P') \to (M,P)$ | Quantization of Poisson maps | open, even in the symplectic-to-symplectic case, partial results for vanishing Atiyah-Molino class: MB, 2004, [10] |
| $\mathcal{A}$-**Module** $\rho : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ $\mathcal{M} = \mathcal{C}^\infty(C,\mathbb{K})[[\nu]]$ | coisotropic map $i$ $C \to (M,P)$ | Quantization of coisotropic submanifolds | open, partial results by [10], [8], and Cattaneo-Felder 2003 |
| **Commutant** $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M},\mathcal{M})$ $D : \mathcal{M} \to \mathcal{M}$ $D\rho(f) = \rho(f)D \ \forall\ f \in \mathcal{A}$ | phase space reduction $M \xleftarrow{i} C \xrightarrow{\pi} M_{\mathrm{red}}$ | Quantization of phase space reduction | Fedosov 1996, for general coisotropic in symplectic: MB 2004, [10] |

Most of the material of this talk comes from my detailed preprint [10] (covering some of my research during the last three years) to which I refer for most of the proofs, and some additional things are contained in the preprint [8]. I apologize for the incomplete reference list and refer the reader to the more extensive list in [10].

**Acknowledgement**

I should like to thank the organizers of the Poisson 2004 workshop, Karine Molitor-Braun, Norbert Poncin, and Martin Schlichenmaier for the invitation and the excellent organisation and *accueil* in Luxembourg. I also like to thank the referee for his comments.

# 1 Motivation: Simple symbol calculus

One of the motivations for deformation quantization comes from the standard symbol calculus used in differential operator theory and in quantum mechanics where it is called *canonical quantization*: Let $A := \mathbb{C}[q,p]$ be the associative commutative algebra of complex polynomials in two variables, and $D$ the algebra of differential operators on the real line with complex polynomial coefficients, i.e.

$$D := \left\{ \sum_{k,l=0}^{N} b_{kl} q^k \frac{\partial^l}{\partial q^l} \;\middle|\; N \in \mathbb{N}, b_{kl} \in \mathbb{C} \right\}.$$

For the nonzero real number $\hbar$ (which is Planck's constant in physics) define the linear bijection

$$\rho_s : A \to D : \sum_{k,l=0}^{N} a_{kl} q^k p^l \mapsto \sum_{k,l=0}^{N} a_{kl} q^k \left( \frac{\hbar}{i} \frac{\partial}{\partial q} \right)^l. \tag{1.1}$$

In particular, one has for $\varphi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ the position and momentum operators in quantum mechanics:

$$\rho_s(q) := Q \quad \text{with} \quad (Q\varphi)(q) = q\varphi(q) \quad \text{and} \quad \rho_s(p) =: P := \frac{\hbar}{i} \frac{\partial}{\partial q}.$$

The bijection (1.1) can be rewritten in the following way: let $f \in A$ and $\varphi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$, then

$$\rho_s(f)(\varphi)(q) = \sum_{r=0}^{\infty} \frac{(\hbar/i)^r}{r!} \frac{\partial^r f}{\partial p^r}(q,0) \frac{\partial^r \varphi}{\partial q^r}(q).$$

The multiplication of differential operators in $D$ is again a differential operator in $D$, whence for $f, g \in A$:

$$\begin{aligned} \rho_s(f)\rho_s(g) &= \rho_s(f *_s g) \quad \text{with} \\ f *_s g &:= \sum_{r=0}^{\infty} \frac{(\hbar/i)^r}{r!} \frac{\partial^r f}{\partial p^r} \frac{\partial^r g}{\partial q^r}. \end{aligned} \tag{1.2}$$

Hence the pair $(A, *_s)$ is an associative noncommutative algebra with unit 1. Expansion in $\hbar$ yields

$$f *_s g = fg + \frac{\hbar}{i} \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} + \cdots .$$

The formula (1.2) does not converge when $f$ and $g$ are just elements of $\mathcal{C}^\infty(\mathbb{R}^2,\mathbb{C})$. But in case $\hbar$ is seen as a formal parameter, the multiplication $*_s$ makes sense on $\mathcal{A} := A[[\hbar]]$ as seen as an algebra over the power series ring $\mathbb{C}[[\hbar]]$. Moreover, by construction the $\mathbb{C}[[\hbar]]$-module $\mathcal{M} := \mathcal{C}^\infty(\mathbb{R},\mathbb{C})[[\hbar]]$ becomes an $\mathcal{A}$-module via the natural extension of the map $\rho_s$ from $\mathbb{C}$ to $\mathbb{C}[[\hbar]]$-modules.

# 2 Recall of deformation quantization

## 2.1 Formal associative deformation of algebras

The example of the previous section can be generalized in the following way (see Gerstenhaber's work 1963, see [27], [28]: let $(A, \mathsf{C}_0)$ be an associative algebra with unit 1 over a commutative ring $\mathbf{k}$ and let $\nu$ be a formal parameter. We shall sometimes use the convention $\nu = \frac{i\hbar}{2}$. Consider the $\mathbf{k}[[\nu]]$-module of all formal power series

$$\mathcal{A} = A[[\nu]] := \left\{ \sum_{r=0}^{\infty} \nu^r a_r \;\middle|\; a_r \in A \;\forall\; r \in \mathbb{N} \right\}$$

equipped with a multiplication $\mathsf{C} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ subject to the following conditions (for arbitrary $f, g, h \in A$):

$$\mathsf{C} \;:=\; \sum_{r=0}^{\infty} \nu^r \mathsf{C}_r \quad \text{with} \quad \mathsf{C}_r \in Hom_{\mathbf{k}}(A \otimes A, A) \quad \text{and} \tag{2.1}$$

$$0 \;=\; \sum_{s=0}^{r} \Big( \mathsf{C}_s\big(\mathsf{C}_{r-s}(f,g),h\big) - \mathsf{C}_s\big(f,\mathsf{C}_{r-s}(g,h)\big)\Big) \quad \text{and} \tag{2.2}$$

$$0 \;=\; \mathsf{C}_r(1,f) = \mathsf{C}_r(f,1) \quad \forall r \geq 1. \tag{2.3}$$

Then $(\mathcal{A}, \mathsf{C})$ is an associative algebra over the commutative ring $\mathbf{k}[[\nu]]$, a *deformation* of $(A, \mathsf{C}_0)$.

The associativity condition (2.2) at order $r+1$ of $\nu$ yields

$$\begin{aligned}
0 = (\mathsf{C} \circ_G \mathsf{C})_{r+1}(f,g,h) \;&:=\; \mathsf{C}\big(\mathsf{C}(f,g),h\big)_{r+1} - \mathsf{C}\big(f,\mathsf{C}(g,h)\big)_{r+1} \\
&=\; -f\mathsf{C}_{r+1}(g,h) + \mathsf{C}_{r+1}(fg,h) - \mathsf{C}_{r+1}(f,gh) + \mathsf{C}_{r+1}(f,g)h \\
&\qquad + \sum_{s=1}^{r} \Big( \mathsf{C}_s\big(\mathsf{C}_{r+1-s}(f,g),h\big) - \mathsf{C}_s\big(f,\mathsf{C}_{r+1-s}(g,h)\big)\Big) \\
&=:\; -(\mathsf{b}\mathsf{C}_{r+1})(f,g,h) \\
&\qquad + \hat{\mathsf{D}}_{r+1}(f,g,h)
\end{aligned} \tag{2.4}$$

where $\mathsf{b}$ denotes the Hochschild coboundary operator. On the other hand, the obvious identity

$$\begin{aligned} 0 \;=\; & \mathsf{C}\Big(\mathsf{C}\big(\mathsf{C}(f,g),h\big),p\Big) - \mathsf{C}\Big(\mathsf{C}\big(f,\mathsf{C}(g,h)\big),p\Big) \\ & +\mathsf{C}\Big(f,\mathsf{C}\big(\mathsf{C}(g,h),p\big)\Big) - \mathsf{C}\Big(f,\mathsf{C}\big(g,\mathsf{C}(h,p)\big)\Big) \\ & -\mathsf{C}\Big(\mathsf{C}\big(\mathsf{C}(f,g),h\big),p\Big) + \mathsf{C}\Big(\mathsf{C}(f,g),\mathsf{C}(h,p)\Big) \\ & +\mathsf{C}\Big(\mathsf{C}\big(f,\mathsf{C}(g,h)\big),p\Big) - \mathsf{C}\Big(f,\mathsf{C}\big(\mathsf{C}(g,h),p\big)\Big) \\ & -\mathsf{C}\Big(\mathsf{C}(f,g),\mathsf{C}(h,p)\Big) + \mathsf{C}\Big(f,\mathsf{C}\big(g,\mathsf{C}(h,p)\big)\Big) \end{aligned}$$

–which is true for any $\mathbf{k}[[\nu]]$-bilinear map $\mathsf{C}$– yields the following equation in case $\mathsf{C}$ is associative up to order $r$, i.e. $(\mathsf{C}\circ_G\mathsf{C})_k = 0$ for all $0\leq k\leq r$:

$$\mathsf{b}\hat{\mathsf{D}}_{r+1} = 0. \tag{2.5}$$

If one wants to construct the formal series $\mathsf{C} = \sum_{r=0}^{\infty}\nu^r\mathsf{C}_r$ order by order in a recursive manner, the two preceding equations (2.4) and (2.5) show that the *recursive obstructions to associativity* from order $r$ to $r+1$ lie in the *third Hochschild cohomology group* of the underlying algebra $A$,

$$HH^3(A,A).$$

Two formal associative deformations $\mathsf{C} = \sum_{r=0}^{\infty}\nu^r\mathsf{C}_r$ and $\mathsf{C}' = \sum_{r=0}^{\infty}\nu^r\mathsf{C}'_r$ of the same underlying assoaciative algebra $(A,\mathsf{C}_0 = \mathsf{C}'_0)$ over $\mathbf{k}$ are said to be *equivalent* iff they are formally isomorphic in the following sense: there is a series $\mathsf{S} = \sum_{r=0}^{\infty}\nu^r\mathsf{S}_r$ of linear maps $\mathsf{S}_r : A\to A$, called an *equivalence transformation* such that

$$\mathsf{S}_0 = id_A \tag{2.6}$$

$$\mathsf{S}_r(1) = 0 \quad \forall\, r\geq 1 \tag{2.7}$$

$$\mathsf{S}\big(\mathsf{C}(f,g)\big) = \mathsf{C}'\big(\mathsf{S}(f),\mathsf{S}(g)\big) \quad \forall\, f,g\in A \tag{2.8}$$

We shall write $\mathsf{S}(\mathsf{C})(f,g)$ for $\mathsf{S}\Big(\mathsf{C}\big(\mathsf{S}^{-1}(f),\mathsf{S}^{-1}(g)\big)\Big)$. Note that the first condition (2.6) implies that $\mathsf{S}$ is invertible whence the stated equivalence is a true equivalence relation.

The obstructions to recursively construct an equivalence transformation between to given deformations are also described by Hochschild cohomology: if $\mathsf{C}_s = \mathsf{C}'_s$ for all $0\leq s\leq r$ then by associativity at order $r+1$ it follows that

$$\mathsf{b}(\mathsf{C}'_{r+1} - \mathsf{C}_{r+1}) = 0.$$

On the other hand, an equivalence transformation $\mathsf{S}$ with $\mathsf{S}_s = 0$ for all $1 \leq s \leq r$ changes $\mathsf{C}_{r+1}$ into

$$\big(\mathsf{S}(\mathsf{C})\big)_{r+1} = \mathsf{C}_{r+1} - \mathsf{b}\mathsf{S}_{r+1}.$$

It follows that the recursive obstructions for this construction lie in the second Hochschild cohomology group of $A$:

$$HH^2(A, A).$$

## 2.2 Classical Limit: Poisson algebras and manifolds

If $(A, \mathsf{C}_0)$ is *commutative* and $\mathsf{C} = \sum_{r=0}^{\infty} \nu^r \mathsf{C}_r$ is a formal associative deformation then it turns out that

$$\{f, g\} := \mathsf{C}_1(f, g) - \mathsf{C}_1(g, f) \quad \forall\, f, g \in A$$

defines a *Poisson bracket on* $A$, i.e. a Lie bracket which satisfies the *Leibniz rule*:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g \quad \forall\, f, g, h \in A.$$

Indeed, the Jacobi identity for the bracket $\{\ ,\ \}$ follows from total antisymmetrization of the associativity condition of $\mathsf{C}$ at order 2. In order to obtain the Leibniz identity one takes the associativity condition of $\mathsf{C}$ at order 1,

$$0 = -f\mathsf{C}_1(g, h) + \mathsf{C}_1(fg, h) - \mathsf{C}_1(f, gh) + \mathsf{C}_1(f, g)h,$$

and adds to it the same identity with $f$ and $h$ interchanged, which gives

$$0 = -f\{g, h\} + \{fg, h\} - \{f, gh\} + \{f, g\}h$$

(showing that $\mathsf{b}\{\ ,\ \} = 0$). Adding to this identity the one with $g$ and $h$ interchanged and subtracting the one with $f$ and $g$ interchanged yields (twice) the Leibniz rule.

Let $M$ be a differentiable manifold, let $\mathbf{k} = \mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let $A$ be the commutative associative unital algebra $A = \mathcal{C}^\infty(M, \mathbb{K})$ equipped with the pointwise multiplication $\mathsf{C}_0$. A *Poisson structure* $P$ is a bivector field, i.e. a section in $\Gamma^\infty(M, \Lambda^2 TM)$ such that its associated Poisson bracket

$$\{f, g\} := P(df, dg) \tag{2.9}$$

satisfies the Jacobi identity

$$0 \overset{!}{=} [P, P]_S(df, dg, dh) := -2\Big(\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}\Big) \ \forall\, f, g, h \in A.$$

Clearly $(A, C_0, \{\ ,\ \})$ is a Poisson algebra (since the exterior derivative satisfies the Leibniz rule). Conversely, every algebraically given Poisson bracket on $A$ is of the above form (2.9): indeed the map

$$f \mapsto X_g(f) := \{f, g\} \tag{2.10}$$

is a derivation of $A$, hence a vector field (the so-called *Hamiltonian vector field associated to* $g$), which depends in a derivative manner on $g$; so the map $g \mapsto X_g$ descends to an $A$-module morphism of the Kähler differentials of $A$, i.e. $\Gamma^\infty(M, T^*M)$, into the derivations of $A$, hence the vector fields $\Gamma^\infty(M, TM)$. It follows that $\{\ ,\ \}$ is in $Hom_A\big(\Gamma^\infty(M, T^*M), \Gamma^\infty(M, TM)\big)$ which is isomorphic to $\Gamma^\infty(M, TM \otimes TM)$.

- Example: symplectic manifold: $(M, \omega)$ where $\omega$ is a nondegenerate closed 2-form on $M$; for instance $M = \mathbb{R}^{2n}$, $\omega = \sum_{k=1}^n dq^k \wedge dp_k$ and $P = \sum_{k=1}^n \frac{\partial}{\partial q^k} \wedge \frac{\partial}{\partial p_k}$.
- Example: dual $\mathfrak{g}^*$ of a finite-dimensional Lie algebra $(\mathfrak{g}, [\ ,\ ])$:

$$P_\alpha := \alpha([\ ,\ ])$$

## 2.3 Quantization Problem: Deformation quantization on Poisson manifolds

The inverse problem of deforming a given Poisson algebra had been formulated by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer in 1978, see [2]:

**Definition 2.1** *Let $(M, P)$ be a given Poisson manifold. A* star-product $* = \sum_{r=0}^\infty \nu^r \mathsf{C}_r$ *is a formal associative deformation of the Poisson algebra $A$ given by $\big(\mathcal{C}^\infty(M, \mathbb{K}), \mathsf{C}_0, \{\ ,\ \}\big)$ where $\mathsf{C}_0$ is the pointwise multiplication in $A$ such that conditions (2.1), (2.2) and (2.3) are satisfied and in addition*

$$\begin{aligned} \mathsf{C}_1(f,g) - \mathsf{C}_1(g,f) &= 2P(df, dg) = 2\{f, g\} \\ \mathsf{C}_\mathsf{r} & \quad \text{bidifferential operators} \quad \forall r \in \mathbb{N}. \end{aligned}$$

Here a *multidifferential operator* of rank $p$ is defined by a $p$-multilinear (w.r.t. $\mathbb{K}$) map

$$\mathsf{D} \ : \ \mathcal{C}^\infty(M, \mathbb{K}) \times \cdots \times \mathcal{C}^\infty(M, \mathbb{K}) \to \mathcal{C}^\infty(M, \mathbb{K})$$

such that there is $N \in \mathbb{N}$ for which in any chart $\big(U, (x^1, \ldots, x^n)\big)$

$$\mathsf{D}(f_1, \ldots, f_p)|_U = \sum_{|I_1|, \ldots, |I_p| \leq N} \mathsf{D}^{I_1 \cdots I_p} \frac{\partial^{|I_1|} f_1}{\partial x^{I_1}} \cdots \frac{\partial^{|I_p|} f_p}{\partial x^{I_p}} \tag{2.11}$$

where $I_1, \ldots, I_p$ are multi-indices in $\mathbb{N}^n$, for $I = (i_1, \ldots, i_n)$ the symbol $|I|$ means $i_1 + \cdots + i_n$, $\frac{\partial^{|I|}}{\partial x^I}$ is short for $\frac{\partial^{i_1 + \cdots + i_k}}{\partial x^{1 i_1} \cdots \partial x^{n i_n}}$, and $\mathsf{D}^{I_1 \cdots I_p}$ is a $\mathcal{C}^\infty$-function on $U$.

Often this condition is generalized to *locality*, i.e. the maps $\mathsf{D}$ are support nonincreasing: by Peetre's theorem such maps are locally $p$-differential, but the orders of the partial derivatives may not be bounded on noncompact manifolds.

Still another generalization is given by continuity with respect ot the standard Fréchet topology. These requirements simplify computations and are also motivated by asymptotic limits (stationary phase expansions) of pseudodifferential operator calculus.

The notion of *equivalence* of formal associative deformations is transferred to star-products: here the series $\mathsf{S} = id_A + \sum_{r=1}^{\infty} \nu^r S_r$ is such that the conditions (2.6), (2.7), and (2.8) are satisfied and each $S_r$ is a differential operator $\mathcal{C}^\infty(M, \mathbb{K}) \to \mathcal{C}^\infty(M, \mathbb{K})$ vanishing on the constants for all $r \geq 1$.

The *Hochschild cohomology* for multidifferential or local or continuous cochains of $A = \mathcal{C}^\infty(A, \mathbb{K})$ is well-known by the $\mathcal{C}^\infty$-version of *Hochschild-Kostant-Rosenberg-Theorem* [35], namely

$$HH^p_{\mathrm{diff}}\big(\mathcal{C}^\infty(M, \mathbb{K}), \mathcal{C}^\infty(M, \mathbb{K})\big) \cong \Gamma^\infty(M, \Lambda^p TM).$$

One can find a proof for $\mathcal{C}^\infty(M, \mathbb{K})$ in Cahen-DeWilde-Gutt 1980 [13], DeWilde-Lecomte 1983 [19] and for the continuous case in Connes 1982 [17], Pflaum 1998 [49], and Nadaud 1999 [43]. In particular $HH^3_{\mathrm{diff}}\big(\mathcal{C}^\infty(M, \mathbb{K}), \mathcal{C}^\infty(M, \mathbb{K})\big)$ is non zero for manifolds of dimension greater or equal than 3, so there is no simple recursion construction for star-products.

The *Existence of star-products* has been shown by DeWilde-Lecomte 1983 [20], Fedosov 1985 (see [22]), and Omori-Maeda-Yoshioka 1990 [47] in the symplectic case and by Kontsevitch 1997 [38] in the general Poisson case. Moreover the *Classification of equivalence classes* of star-products has been done by Deligne 1995 [18], Nest-Tsygan 1995 [44] [45], and Bertelson-Cahen-Gutt 1997 [3] in the symplectic case where the result is

$$\{[*] \mid * \ \mathrm{star-product\ on}\ (M, \omega)\} \cong \frac{[\omega]_{dR}}{\nu} + H^2_{dR}(M, \mathbb{K})[[\nu]],$$

and the class $[*]$ is called the *Deligne class of* $*$, see also [33] for an excellent review. For general Poisson manifolds Kontsevitch (1997) [38] classified the equivalence classes by formal diffeomorphy classes of formal Poisson structures.

For later use we mention the *Gutt star-product* of the Poisson manifold $\mathfrak{g}^*$ where $(\mathfrak{g}, [\ ,\ ])$ is a finite-dimensional real Lie algebra, see [32]: for $\xi \in \mathfrak{g}$ let $e_\xi$ denote the exponential function $x \mapsto e^{\langle x, \xi \rangle}$, and for $\xi, \eta \in \mathfrak{g}$ let $H(\xi, \eta)$ be the Baker-Campbell-Hausdorff series $\frac{1}{\nu} log(e^{\nu\xi} e^{\nu\eta})$ (where the formal exponential functions $e^{\nu\xi}$ are computed in $U\mathfrak{g}[[\nu]]$ where $U\mathfrak{g}$ is the universal envelopping algebra of $\mathfrak{g}$). The Gutt star-product is given by

$$e_\xi * e_\eta := e_{H(\xi, \eta)},$$

and reflects the multiplication in $U\mathfrak{g}$ by applying $*$ to polynomials and setting $\nu = 1$.

Let me mention two obvious constructions which I shall need later on: for two given star-products $*$ on $M$, $*'$ on $M'$, one may form their *tensor product* $* \otimes *'$

on $M \times M'$ by

$$* \otimes *' := \sum_{r=0}^{\infty} \nu^r \sum_{s=0}^{r} (pr_1^* \mathsf{C}_s) \otimes (pr_2^* \mathsf{C}'_{r-s})$$

where $pr_1 : M \times M' \to M$ and $pr_2 : M \times M' \to M'$ denote the canonical projections. The Deligne class of $* \otimes *'$ is computed to be (see e.g. [10])

$$[* \otimes *'] = pr_1^*[*] + pr_2^*[*'].$$

The *opposite star-product* of a star-product $*$ is defined in the equally obvious manner

$$f *^{\mathrm{opp}} g := g * f$$

and defines a star-product for the Poisson manifold $(M, -P)$. Its Deligne class is given by (see [46])

$$[*^{\mathrm{opp}}] = -[*].$$

# 3 Morphisms

## 3.1 Morphisms of deformed algebras

Let $(A, \mathsf{C}_0)$ and $(B, \mathsf{C}'_0)$ two associative algebras with unit over the commutative ring $\mathbf{k}$. Consider two formal associative deformations ($\mathcal{A} = A[[\nu]], * = \mathsf{C} = \sum_{r=0}^{\infty} \nu^r \mathsf{C}_r$) and ($\mathcal{B} = B[[\nu]], *' = \mathsf{C}' = \sum_{r=0}^{\infty} \nu^r \mathsf{C}'_r$).

It is natural to look at *algebra morphisms* $\mathcal{A} \to \mathcal{B}$, i.e. $\mathbf{k}[[\nu]]$-linear maps $\Phi = \sum_{r=0}^{\infty} \nu^r \Phi_r$ with

$$\Phi(f * g) = \big(\Phi(f)\big) *' \big(\Phi(g)\big) \quad \text{and} \quad \Phi(1) = 1.$$

This identity at order $r$ yields

$$(3.1) \quad 0 \overset{!}{=} \mathsf{D}_r(f,g) := \sum_{s+t=r} \Phi_t\big(\mathsf{C}_s(f,g)\big) - \sum_{s+t+u=r} \mathsf{C}'_s\big(\Phi_t(f), \Phi_u(g)\big) \quad \forall\, f, g \in A.$$

For $r = 0$ this immediately implies that $\Phi_0 : A \to B$ is a morphism of associative algebras with unit. It follows that $B$ becomes an $A - A$-bimodule via $f\phi g := \Phi_0(f)\phi\Phi_0(g)$ for all $f, g \in A, ; \phi \in B$ where we write simple multiplication for $\mathsf{C}_0$ and $\mathsf{C}'_0$.

In order to perform a recursive construction of $\Phi$: suppose that there are $\mathbf{k}$-linear maps $\Phi_1, \ldots, \Phi_r : A \to B$ such that the identity (3.1) holds up to order $r$. Now there is the following general identity for

$$\mathsf{D}(f,g) := \sum_{r=0^{\infty}} \nu^r \mathsf{D}_r(f,g) = \Phi\big(\mathsf{C}(f,g)\big) - \mathsf{C}'\big(\Phi(f), \Phi(g)\big),$$

namely

$$0 = \mathsf{C}'\big(\Phi(f), \mathsf{D}(g,h)\big) - \mathsf{D}\big(\mathsf{C}(f,g),h\big) + \mathsf{D}\big(f,\mathsf{C}(g,h)\big) - \mathsf{C}'\big(\mathsf{D}(f,g),\Phi(h)\big) \tag{3.2}$$

where only the associativity of $\mathsf{C}$ and $\mathsf{C}'$ (and not (3.1)) is needed. At order $r+1$ (3.1) reads

$$0 \stackrel{!}{=} \mathsf{D}_{r+1} =: \mathsf{b}\Phi_{r+1} + \hat{\mathsf{D}}_{r+1}$$

where $\hat{\mathsf{D}}_{r+1}$ contains only $\Phi_s$ up to order $r$. Supposing $0 = \mathsf{D}_0 = \cdots = \mathsf{D}_r$ identity (3.2) at order $r+1$ gives

$$0 = \mathsf{b}\mathsf{D}_{r+1} = \mathsf{b}\hat{\mathsf{D}}_{r+1}$$

whence the recursive obstructions lie in the following second Hochschild cohomology group

$$HH^2(A,B).$$

More information can be found in [29] and the references cited therein.

## 3.2 Classical limit: Poisson maps

Let $(M,P)$ and $(M',P')$ two Poisson manifolds, and $A = \mathcal{C}^\infty(M,\mathbb{K})$, $B = \mathcal{C}^\infty(M',\mathbb{K})$. Given two star-products $*$ on $M$ and $*'$ on $M'$ suppose there is a *star-product morphism*, i.e. an algebra morphism $\Phi : \mathcal{A} = A[[\nu]] \to \mathcal{B} = B[[\nu]]$.

By Milnor's exercise (see e.g. [37]) the algebra morphism $\Phi_0$ has to be the pull-back $\phi^*$ with respect to a unique smooth map $\phi : M' \to M$, i.e. $\Phi_0(f) = f \circ \phi$. Moreover, considering the morphism identity (3.1) at order 1, antisymmetrized in $f,g$, we get

$$\Phi_0(\{f,g\}) = \{\Phi_0(f), \Phi_0(g)\}'$$

(where $\{f,g\}$ (resp. $\{f,g\}'$) denotes the Poisson bracket w.r.t. $P$ (resp. $P'$)), hence $\phi$ is a *Poisson map*

$$T\phi \otimes T\phi(P') = P \circ \phi.$$

Examples of Poisson maps:

- *Projections on first factor*: let $(M_1,P_1)$ and $(M_2,P_2)$ two Poisson manifolds and $(M_1 \times M_2, P_1 + P_2)$ their product then

$$\begin{array}{ccc} M_1 & \times & M_2 \\ & pr_1 \downarrow & \\ & M_1 & \end{array}$$

  is obviously a Poisson map.

- *Momentum maps*: let $(\mathfrak{g}, [\ ,\ ])$ be a finite-dimensional real Lie algebra and $J$ be a

$$\mathcal{C}^\infty\text{−map } J : M \to \mathfrak{g}^* \quad \text{with} \quad \{\langle J, \xi\rangle, \langle J, \eta\rangle\} = \langle J, [\xi, \eta]\rangle$$

  $\forall\ \xi, \eta \in \mathfrak{g}$. Then $J$ is a Poisson map.

  For $\mathfrak{g}$ abelian, $2 \dim \mathfrak{g} = \dim P$ and $J$ regular on an open subset whose complement is of measure zero, one speaks of a *completely integrable system.*

- *Symplectic to symplectic*: let $(M', \omega')$ and $(M, \omega)$ be two symplectic manifolds and $\phi : M' \to M$ a Poisson map. There is the following –at first glance surprisingly simple– result: it can be shown that

$$\begin{array}{c} M' \\ \phi\ \downarrow \\ M \end{array} \quad \text{is locally like} \quad \begin{array}{ccc} M & \times & M_2 \\ pr_1\ \downarrow & & \\ M & & \end{array},$$

  i.e. $\phi$ is a *submersion* and $M'$ has two *transverse symplectic foliations*: this means that the tangent bundle of $M'$ decomposes into the direct sum

$$TM' = \mathrm{Ker}T\phi \oplus (\mathrm{Ker}T\phi)^{\omega'}$$

  of two *integrable symplectic subbundles*, see e.g. [10] for a proof.

## 3.3 Differentiability of morphisms: differential operators along maps

One may wonder whether the $\mathbb{K}$-linear maps $\Phi_r : \mathcal{C}^\infty(M, \mathbb{K}) \to \mathcal{C}^\infty(M', \mathbb{K})$ of a star-product morphism are constrained to some more geometric subset: since $\Phi_0$ is the pull-back w.r.t a smooth map $\phi : M' \to M$ it is reasonable to ask whether the $\Phi_r$ of higher order are "differential along $\phi$". To be more precise, one defines *multidifferential operators along maps* in the following natural way: to a diagram of manifolds and smooth maps

$$\begin{array}{ccc} M_1 & & \\ & \nwarrow \phi_1 & \\ \vdots & \vdots & M \\ & \swarrow \phi_p & \\ M_p & & \end{array}$$

one associates a space of certain multilinear maps, called multidifferential along $\phi_1, \ldots, \phi_p$,

$$\mathbf{D}^p\big(\mathcal{C}^\infty(M_1, \mathbb{K}), \ldots, \mathcal{C}^\infty(M_p, \mathbb{K}); \mathcal{C}^\infty(M, \mathbb{K})\big),$$

locally given by the obvious generalization of (2.11)

$$(3.3) \qquad \mathsf{D}(f_1,\ldots,f_p)|_U \;=\; \sum_{|I_1|,\ldots,|I_p|\leq N} \mathsf{D}^{I_1\cdots I_p}\;\frac{\partial^{|I_1|} f_1}{\partial x_{(1)}^{I_1}}\circ\phi_1\cdots\frac{\partial^{|I_p|} f_p}{\partial x_{(p)}^{I_p}}\circ\phi_p$$

in local charts $(U,x),(U_1,x_{(1)}),\ldots,(U_p,x_{(p)})$ of $M,M_1,\ldots,M_p$ with $\mathsf{D}^{I_1\cdots I_p}\in\mathcal{C}^\infty(U,\mathbb{K})$.
The generalization to sections of vector bundles is possible, see [10]. It would be interesting to study the operadic nature of these objects.

Shortly after my preprint [10] appeared on the web in which I had conjectured that star-product morphisms are differential, S.Gutt and J.Rawnsley informed me that this was indeed the case:

**Theorem 3.1 (Gutt-Rawnsley 2004)** *: Every $*$-product morphism $\Phi=\sum_{r=0}^\infty\nu^r\Phi_r$ is a series of differential operators along the Poisson map $\phi$ (where $\Phi_0=\phi^*$).*

For their proof they computed a variant of the $\mathcal{C}^\infty$ Hochschild-Kostant-Rosenberg Theorem, namely for any smooth map $\phi: M'\to M$:

$$HH^p_{\mathrm{diff},\phi}\big(\mathcal{C}^\infty(M,\mathbb{K}),\mathcal{C}^\infty(M',\mathbb{K})\big)\cong\Gamma^\infty(M',\phi^*\Lambda^pTM).$$

Among other things this had also been shown for the particular case of the canonical embedding $\phi$ of $\mathbb{R}^{n-l}$ into $\mathbb{R}^n$ in [8].

## 3.4 Quantization problem of Poisson maps

The natural inverse problem is the following: given a Poisson map

$$\phi: M'\to M$$

between two Poisson manifolds $(M',P')$ and $(M,P)$, are there star-products $*$ on $M$ and $*'$ on $M'$ and a star-product homomorphism

$$\Phi:\mathcal{A}\to\mathcal{A}' \text{ with } \Phi_0=\phi^* \;?$$

$\Phi$ will then be called a *quantization of the Poisson map $\phi$.*

## 3.5 Results

### 3.5.1 (Quantum) Momentum maps

The problem of quantizing a momentum map $J: M\to\mathfrak{g}^*$ as a Poisson map can slightly be modified in the following way: on $\mathfrak{g}^*$ the Gutt star-product is fixed and the quantized morphism $\Phi$ is only applied to the polynomial functions on $\mathfrak{g}^*$, that is to the $\nu$-adic completion $U_\nu\mathfrak{g}$ of the universal envelopping algebra of the Lie algebra $\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{K}[[\nu]]$ over $\mathbb{K}[[\nu]]$ equipped with the Lie bracket $\nu[\ ,\ ]$. This modified quantization –where $\Phi$ may in general no longer be applied to $\mathcal{C}^\infty(\mathfrak{g}^*,\mathbb{K})$– is equivalent to the concept of a *quantum momentum map*

$$\mathbb{J}: M \to \mathfrak{g}^*[[\nu]]$$

(Lu 1993 [39], Xu 1998 [56]) where $\mathbb{J}$ is $\Phi$ restricted to $\mathfrak{g}$ (hence $J=\mathbb{J}_0$) and satisfies

$$\langle \mathbb{J},\xi\rangle * \langle \mathbb{J},\eta\rangle - \langle \mathbb{J},\eta\rangle * \langle \mathbb{J},\xi\rangle \;=\; 2\nu\langle \mathbb{J},[\xi,\eta]\rangle \quad \forall\ \xi,\eta\in\mathfrak{g}.$$

for a given star-product $*$ on the Poisson manifold $(M,P)$. For $\mathbb{J}=J$ the star-product $*$ is also called *covariant* by Arnal, Cortet, Molin,Pinczon (1983) [1].
An important particular case of this is a *quantized integrable system* when $\mathfrak{g}$ is abelian and $J$ defines an integrable system.

A large class of covariant star-products is provided by Fedosov's

**Theorem 3.2 (Fedosov 1996)** *Let $(M,\omega)$ be a symplectic manifold, $J: M\to\mathfrak{g}^*$ a momentum map, and $\nabla$ a connection in the tangent bundle $TM$ such that*

$$0=\left(L_{X_{\langle J,\xi\rangle}}\nabla\right)_X Y := [X_{\langle J,\xi\rangle},\nabla_X Y]-\nabla_{[X_{\langle J,\xi\rangle},X]}Y-\nabla_X[X_{\langle J,\xi\rangle},Y].$$

*for all $\xi\in\mathfrak{g}$ and vector fields $X,Y$ on $M$, i.e. $\nabla$ is invariant by the Hamiltonian Lie algebra action induced by $\mathfrak{g}$.*
*Then there exists a strongly invariant star-product $*$ on $M$, i.e. for which*

$$\langle \mathbb{J},\xi\rangle * g - g * \langle \mathbb{J},\xi\rangle = 2\nu\{\langle \mathbb{J},\xi\rangle, g\} \quad \forall\ \xi\in\mathfrak{g}\ \ \forall\ g\in\mathcal{C}^\infty(M,\mathbb{K}).$$

*In particular ($g=\langle J,\eta\rangle$) the star-product $*$ is covariant.*

See [24] for a proof. Many examples are provided by momentum maps associated to a proper Hamiltonian Lie group action (since the action preserves a Riemannian metric thanks to a classical theorem by R.Palais, [48]).

### 3.5.2 Symplectic to symplectic

In spite of the simple geometric nature of a Poisson map $\phi: M'\to M$ between two symplectic manifolds $(M',\omega')$ and $(M,\omega)$, the full answer of the quantizability of $\phi$ is not yet known. A partial result has been obtained in [10]:

**Theorem 3.3 (M.B. 2004)** *If*

- *the Atiyah-Molino class $\kappa_{AM}(M',(KerT\phi)^{\omega'})$ of the symplectic integrable subbundle $(KerT\phi)^{\omega'}$ vanishes and*
- *the Deligne classes satisfy $[*'](\nu)=\frac{[\omega']}{\nu}+\phi^*\left([*](\nu)-\frac{[\omega]}{\nu}\right)$*

*then the Poisson map $\phi : M \to M'$ is quantizable. Moreover, in that case the star-products can be modified by equivalence transformations such that $\phi^*$ is already a star-product morphism.*

See section 6 for a short definition of the Atiyah-Molino class.
The proof uses a Fedosovian representation of both star-products where the Atiyah-Molino class arises as a natural obstruction, see also section 7.1.2 for a discussion of total obstructions up to order three.

# 4 (Bi)Modules

## 4.1 (Bi)Modules of deformed algebras

Let $(A, \mathsf{C}_0)$ be an associative algebra with unit over the commutative ring $\mathbf{k}$. Consider a formal associative deformation $(\mathcal{A} = A[[\nu]], * = \mathsf{C} = \sum_{r=0}^{\infty} \nu^r \mathsf{C}_r)$.

It is natural to look at (left) *modules* or *representations* of $\mathcal{A}$, i.e. at a $\mathbf{k}$-module $\mathcal{M}_0$ such that for $\mathcal{M} := \mathcal{M}_0[[\nu]]$ there is a

$$\mathbf{k}[[\nu]]\text{−bilinear map } \rho \ : \ \begin{array}{ccccc} \mathcal{A} & \times & \mathcal{M} & \to & \mathcal{M} \\ (f & , & \varphi) & \mapsto & \rho(f)(\varphi) \end{array}$$

with the representation identity

$$\rho(f)\rho(g)(\varphi) = \rho(f * g)(\varphi) \quad \text{and} \quad \rho(1) = id_{\mathcal{M}} \tag{4.1}$$

for all $f, g \in A$ and $\varphi \in \mathcal{M}_0$. A right $\mathcal{A}$-module is defined as a left $\mathcal{A}^{\mathrm{opp}}$-module. For another associative $\mathbf{k}$-algebra $(B, \mathsf{C}'_0)$ with associative deformation $(\mathcal{B} = B[[\nu]], *' = \mathsf{C}' = \sum_{r=0}^{\infty} \nu^r \mathsf{C}'_r)$ a $\mathcal{A} - \mathcal{B}$ *bimodule* is a left $\mathcal{A} \otimes_{\mathbf{k}[[\nu]]} \mathcal{B}^{\mathrm{opp}}$-module.

In order to see some recursive constructions let

$$\mathsf{D}(f,g) := \sum_{r=0}^{\infty} \nu^r \mathsf{D}_r(f,g) := \rho(f)\rho(g) - \rho\big(\mathsf{C}(f,g)\big) \in \ Hom_{\mathbf{k}[[\nu]]}(\mathcal{M}, \mathcal{M}) \tag{4.2}$$

for all $f, g \in A$, hence (4.1) holds iff $\mathsf{D} = 0$. At order $r = 0$ the representation identity entails that $\rho_0$ is a representation of $A$ on $\mathcal{M}_0$. Moreover, the space $Hom_{\mathbf{k}}(\mathcal{M}_0, \mathcal{M}_0)$ becomes an $A - A$ bimodule via $(fKg)(\phi) := \rho_0(f)\big(K(\rho_0(g)\phi)\big)$ for all $f, g \in A$, $K \in Hom_{\mathbf{k}}(\mathcal{M}_0, \mathcal{M}_0)$, and $\varphi \in \mathcal{M}_0$. At order $r+1$ the representation identity gives

$$0 \overset{!}{=} \mathsf{D}_{r+1} =: \mathsf{b}\rho_{r+1} + \hat{\mathsf{D}}_{r+1}.$$

where $\hat{\mathsf{D}}_{r+1}$ only contains $\rho_s$ up to order $r$. There is the following general identity for $\mathsf{D}$:

$$0 = \rho(f)\mathsf{D}(g,h) - \mathsf{D}\big(\mathsf{C}(f,g),h\big) + \mathsf{D}\big(f,\mathsf{C}(g,h)\big) - \mathsf{D}(f,g)\rho(h) \quad \forall\ f,g,h \in A$$

where only the associativity of $\mathsf{C}$ and the multiplication in $Hom_{\mathbf{k}[[\nu]]}(\mathcal{M},\mathcal{M})$ is needed. Supposing the representation identity true up to order $r$, i.e. $0 = \mathsf{D}_0 = \cdots = \mathsf{D}_r$ we can deduce the following equation from the preceding identity:

$$0 = \mathsf{b}\mathsf{D}_{r+1} = \mathsf{b}\hat{\mathsf{D}}_{r+1},$$

hence the recursive obstructions to this construction lie in the second Hochschild cohomology group

$$HH^2\big(A, Hom_{\mathbf{k}}(\mathcal{M}_0,\mathcal{M}_0)\big).$$

## 4.2 Classical Limit: vanishing ideal and coisotropic maps

Let $(M,P)$ be a Poisson manifold, let $A = \mathcal{C}^\infty(M,\mathbb{K})$, and let $* = \sum_{r=0}^\infty \mathsf{C}_r$ be a star-product on $M$. A possible framework for modules and representations in deformation quantization –but certainly not the only one– is the following: let $E \xrightarrow{\tau} C$ a $\mathbb{K}$-vector bundle over a manifold $C$ (where the most important case will be the trivial bundle $E = C\times\mathbb{K}$), let $\mathcal{M}_0 := \Gamma^\infty(C,E)$ and $\mathcal{M} := \mathcal{M}_0[[\nu]]$. We demand that a *star-product representation* $\rho$ to be an algebra morphism of $(\mathcal{A},*)$ into the associative unital algebra of all formal series of differential operators on $\mathcal{M}_0$, i.e. $\mathbf{D}(\mathcal{M}_0,\mathcal{M}_0)[[\nu]] =: \mathbf{D}(\mathcal{M},\mathcal{M})$.

In what follows, we shall restrict on the case $E = C\times\mathbb{K}$. The representation identity (4.1) at $r=0$ shows that $\rho_0$ is an algebra morphism of the associative commutative algebra $A = \mathcal{C}^\infty(M,\mathbb{K})$ into $\mathbf{D}\big(\mathcal{C}^\infty(C,\mathbb{K}),\mathcal{C}^\infty(C,\mathbb{K})\big)$. Since for real-valued $f$ the function $1+f^2$ is invertible, and invertible differential operators are of order zero (we do not consider inverses as pseudodifferential operators!), hence there is an algebra morphism $\Phi_0 : \mathcal{C}^\infty(M,\mathbb{K}) \to \mathcal{C}^\infty(C,\mathbb{K})$ such that

$$\rho_0(f)(\varphi) =: \Phi_0(f)\varphi \quad \forall\, f\in A \ \ \forall\, \varphi \in \mathcal{M}_0 = \mathcal{C}^\infty(C,\mathbb{K})$$

whence (thanks to Milnor's exercise) there exists a $\mathcal{C}^\infty$-map

$$i : C \to M \ \text{ with } \ \rho(f)(\varphi) = (i^*f)\varphi.$$

A very important notion will be the *vanishing ideal of* $i$:

$$I := \{g \in \mathcal{C}^\infty(M,\mathbb{K}) \mid i^*g = 0\} \ \text{ which is an ideal of } \mathcal{C}^\infty(M,\mathbb{K}).$$

Upon antisymmetrizing the representation identity (4.1) at $r=1$ for $g,g'\in I$ we get

$$\begin{aligned} 0 &= \rho_1(g)\rho_0(g') + \rho_0(g)\rho_1(g') - \rho_1(gg') - \rho_0\big(\mathsf{C}_1(g,g')\big) - (g\leftrightarrow g') \\ &= 0 + 0 - 0 - 2\rho_0(\{g,g'\}) \end{aligned}$$

whence

$$\begin{gathered} I \text{ is a Poisson subalgebra of } \mathcal{C}^\infty(M,\mathbb{K}) \\ \Leftrightarrow: \\ i \text{ is a } \textit{coisotropic map.} \end{gathered}$$

These maps are important in Poisson geometry as the following examples of coisotropic maps show:

- $i$ is surjective or has dense image.
- $i$ is a Poisson map.
- $i$ is an embedding on a *coisotropic or 1st class submanifold* of $M$ which can alternatively be defined in the following way:

$$\begin{array}{ll} \text{Poisson :} & \forall\, c \in C,\ \forall\, \alpha \in T_cC^{\text{ann}} \subset T_cM^* : P_c(\alpha,\ \ ) \in T_cC \\ \text{symplectic :} & \qquad\qquad \forall\, c \in C :\ T_cC^{\omega} \subset T_cC \end{array}$$

  Note that coisotropic submanifolds come with a (singular) integrable distribution $c \mapsto P_c(T_cC^{\text{ann}},\ \ ) \subset T_cC$ giving rise to a (singular) *foliation* $\mathcal{F}$ on $C$: the quotient space $\pi : C \to C/\mathcal{F}$ is known to be the *reduced phase space*. For symplectic manifolds $M$ the distribution is an integrable subbundle $E = TC^{\omega}$ and the foliation is regular. If in this case the quotient $C/\mathcal{F} =: M_{\text{red}}$ is a manifold such that $\pi$ is a submersion, then $M_{\text{red}}$ is known to carry a canonical symplectic structure $\omega_{\text{red}}$ defined by $\pi^*\omega_{\text{red}} = i^*\omega$.
- $i : M' \to M \times \overline{M'} : m' \mapsto \big(\phi(m'), m'\big)$ i.e. a *graph of a Poisson map* (A.Weinstein [55]), where $\overline{M'}$ indicates the Poisson manifold $(M', -P')$.

## 4.3 Differentiability of representations

In the same way as for morphisms, one may ask whether a star-product representation $\rho$ is also differential in its first argument:

**Theorem 4.1 (M.B. 2004)** *Every $*$ star-product representation on $\mathcal{C}^\infty(C, \mathbb{K})$, $\rho = \sum_{r=0}^{\infty} \nu^r \rho_r$, is a series of bidifferential operators along $i$ and $id_C$ according to the diagram*

$$\begin{array}{ccc} M & & \\ & \nwarrow i & \\ & & C \\ & \swarrow id_C & \\ C & & \end{array}$$

The idea of the proof is similar to the proof of the fact that all derivations of $A$ are vector fields: by induction the case $\rho_0$ being clear.

- Use the representation identity at $r+1$:

$$\rho_{r+1}(fg)(\psi) = \rho_{r+1}(f)\big((i^*g)\psi\big) + (i^*f)\rho_{r+1}(g)(\psi) - \hat{\mathsf{D}}_{r+1}(f, g, \psi)$$

- Embed $\iota : M \to \mathbb{R}^N$ by Whitney's theorem (Pflaum's trick).

- Use Hadamard's trick

$$\hat{h}(x) = \hat{h}(y) + \sum_{j=1}^{N} \int_0^1 \frac{\partial \hat{h}}{\partial x^j}(tx + (1-t)y)dt\ (x^j - y^j)$$

on a prolongation $\hat{h}$ of a function $h$ on $M$ to $\mathbb{R}^N$, restrict this formula to $M$, and express $\rho_{r+1}(h)(\psi)$ upon using $\rho_{r+1}(1) = 0$ at $x = y$ in terms of $\hat{\mathsf{D}}_{r+1}(f, g, \psi)$ which by induction is tridifferential along $i, i, id_C$.

## 4.4 Quantization problem of coisotropic submanifolds

The following inverse problem is interesting: given a coisotropic submanifold of $M$

$$i : C \to M$$

and a star-product $*$ on $M$, is there a star-product representation on $\mathcal{M} := \mathcal{C}^\infty(C, \mathbb{K})[[\nu]]$

$$\rho : \mathcal{A} \to \mathbf{D}(\mathcal{M}, \mathcal{M}) \ \text{ with } \ \rho_0(f)(\phi) = (i^* f)\phi \ ?$$

$*$ is called *representable* in that case.

## 4.5 Deformation of the vanishing ideal and adapted star-products

Let $i : C \to M$ be a coisotropic submanifold (with vanishing ideal $I$) of the Poisson manifold $(M, P)$, and let $*$ be a star-product on $M$. The following simplification has turned out to be useful:

*Adapted star-products* are star-products $*$ such that

$$I[[\nu]] \text{ is a left ideal of } \mathcal{A}$$

These star-products are immediately representable since

$$\mathcal{C}^\infty(C, \mathbb{K})[[\nu]] \cong \mathcal{A}/I[[\nu]].$$

For example the star-product (1.2) is adapted. Moreover, *every representable star-product is equivalent to an adapted star-product*: in a tubular neighbourhood of $C$ in $M$ one shows that the map $f \mapsto \rho(f)1$ can be written as $i^*(\mathsf{S}f)$ where $\mathsf{S} = id + \sum_{r=1}^{\infty} \nu^r \mathsf{S}_r$ is a formal series of differential operators such that $\mathsf{S}_r 1 = 0$ for all $r \geq 1$. Then for the equivalent star-product $*' = \mathsf{S}(*)$ the map $\rho'(f) := \rho(\mathsf{S}^{-1} f)$ will be a star-product representation such that

$$I[[\nu]] = \{g \in \mathcal{A} \mid \rho'(g)1 = 0\}$$

and the right hand side obviously is a left ideal of $(\mathcal{A}, *')$.

## 4.6 Results

### 4.6.1 Partial results

Finitely generated *projective modules* of $A = \mathcal{C}^\infty(M, \mathbb{K})$ are in bijection with $\Gamma^\infty(M, E)$ where $E$ is a $\mathbb{K}$-vector bundle over $M$ thanks to the Serre-Swan Theorem. By explicitly deforming the projection valued map on $M$ defining $E$ as a subbundle of a trivial bundle B.Fedosov (1985, 1996, see e.g. [24]) has shown that these modules are always quantizable. This analysis has been carried out further by Bursztyn and Waldmann who investigated Morita equivalence bimodules, see e.g. [11], [12].

Waldmann and myself (1998) [4] transferred the notion of the *Gel'fand-Naimark-Segal construction* in $C^*$-algebra theory to deformation quantization: here the modules are constructed by means of a positive linear functional on $\mathcal{A}$ ('positivity' refers to the nonarchimedian asymptotic ring ordering in $\mathbb{R}[[\nu]]$). We saw that some of the relevant representations in quantum mechanics (Schrödinger and Wick representation and projectable WKB) could be obtained that way. Waldmann has continued this work [51], [52].

The aforementioned *symbol calculus* of differential operators (cf. Section 1) is a particular case of representations of star-products on the *cotangent bundle* $T^*Q$ of any manifold $Q$ which has been studied in a series of papers by Pflaum (1998) [50], M.B., N.Neumaier, S.Waldmann [5], [6], [9].

### 4.6.2 Results in the symplectic case

See subsection 5.2 for the notion of symplectic reduction:

**Theorem 4.2 (M.B. 2003)** *: Let $i : C \to M$ be a closed coisotropic submanifold of a symplectic manifold $(M, \omega)$ such that the reduced phase space $\pi : C \to M_{\mathrm{red}}$ exists. Let $*$ be a star-product on $M$. Then if for the Deligne class $[*]$ of $*$ it is true that $i^*[*]$ is basic, i.e. there is a class $\beta$ on $M_{\mathrm{red}}$ with $i^*[*] = \pi^*\beta$ then the star-product $*$ is representable.*

The idea of the proof is to note that $i \times \pi : C \to M \times \overline{M_{\mathrm{red}}}$ embeds $C$ as a *Lagrangian* (i.e. minimal coisotropic) submanifold. According to a classical theorem by Alan Weinstein [53], a tubular neighbourhood of $C$ in $M \times \overline{M_{\mathrm{red}}}$ is symplectomorphic to an open neighbourhood of the zero section of $T^*C$. The theorem follows by symbol calculus on $C$. See also the reduction theorem in subsection 5.4.3.

A more general result is true:

**Theorem 4.3 (M.B. 2003)** *: Let $i : C \to M$ be a closed coisotropic submanifold of a symplectic manifold $(M, \omega)$ and let $E = TC^\omega$ be the integrable subbundle of $TC$. Let $*$ be a star-product on $M$ having Deligne class $[*]$. Suppose that*

- *The Atiyah-Molino class $\kappa_{AM}(C, E)$ of $E$ vanishes.*

- $i^*[*] = 0$.

*Then the star-product $*$ is representable.*

The idea of the proof in [10] is to reduce this situation to the morphism quantization theorem 3.3:

- Construct a **fibration of locally reduced phase spaces**, i.e. a Poisson map $\phi : E^C \to$ tubular neighbourhood of $C$ in $M$ where the symplectic manifold $E^C$ generalizes the aforementioned situation $M \times \overline{M_{\rm red}} \to M$ and the fibres are local quotients in a foliation chart.

- $C$ embeds into $E^C$ as *Lagrangian submanifold.*

- Represent $\mathcal{C}^\infty(E^C, \mathbb{K})[[\nu]]$ on $C$,

- quantize the Poisson map $\phi$ (here the Atiyah-Molino class becomes relevant) to get an algebra morphism of $\mathcal{C}^\infty(M, \mathbb{K})[[\nu]]$ into $\mathcal{C}^\infty(E^C, \mathbb{K})[[\nu]]$, and represent.

### 4.6.3 Adapted multidifferential operators and formality

Let $C$ be a fixed closed submanifold with vanishing ideal $I$ of a differentiable manifold $M$. Motivated by adapted star-products we have introduced in [8] the notion of an *adapted multidifferential operator* $\mathsf{D}$ in $\mathbf{D}^p\big(\mathcal{C}^\infty(M, \mathbb{K}), \ldots, \mathcal{C}^\infty(M, \mathbb{K}); \mathcal{C}^\infty(M, \mathbb{K})\big)$ by demanding

$$\mathsf{D}(f_1, \ldots, f_{p-1}, g) \in I \quad \forall\ f_1, \ldots, f_{p-1} \in \mathcal{C}^\infty(M, \mathbb{K}),\ \forall\ g \in I.$$

Let $\mathfrak{G}_I^p$ be the space of adapted $p$-multidifferential operators on $M$ and $\mathfrak{G}_I := \bigoplus_{p\in\mathbb{Z}} \mathfrak{G}_I^p$. Then $\mathfrak{G}_I$

- contains the bidifferential operators of an adapted star-product,

- is closed by composition in the $i$th argument,

- and thus a subcomplex of the Hochschild complex.

- There is the Hochschild-Kostant-Rosenberg type Theorem

$$\mathfrak{g}_I^p := H\mathfrak{G}_I^p \cong \{T \in \Gamma^\infty(M, \Lambda^p TM \mid T(dg_1, \ldots, dg_p) \in I\ \forall\ g_1, \ldots, g_p \in I\}.$$

Note also that $HH^p_{\rm diff}\big(A, \mathbf{D}(\mathcal{M}_0, \mathcal{M}_0)\big) \cong \Gamma^\infty(C, \Lambda^p(TM/C))$. For certain particular situations in $M = \mathbb{R}^n$ and $C = \mathbb{R}^{n-l}$ one can show a $G_\infty$-formality statement. A different aproach to the same problem which generalizes the Kontsevitch graphs has been done by A.Cattaneo, G.Felder, see [15], [16].

#### 4.6.4 Glößners's representability theorem in codimension one

**Theorem 4.4 (P.Glößner 1998)** *Let $i : C \to M$ be a closed coisotropic submanifold of codimension* 1 *of a Poisson manifold* $(M, P)$*.*
*Then every star-product on $M$ is representable.*

See [30]. This theorem can be deduced from the absence of recursive obstructions in codimension 1, see subsection 7.2.1.

# 5 Commutants of modules and quantum reduction

## 5.1 Commutants of modules of deformed algebras

Let $(A, \mathsf{C}_0)$ be an associative unital algebra over the commutative ring $\mathbf{k}$, let $* = \mathsf{C} = \sum_{r=0}^{\infty} \nu^r \mathsf{C}_r$ an associative formal deformation of $A$ and $\mathcal{A} = A[[\nu]]$. Let $\mathcal{M}_0$ be an $A$-module, and $\mathcal{M} = \mathcal{M}_0[[\nu]]$ be an $\mathcal{A}$-module where the representation is denoted by $\rho$. We specialize to the situation where $\mathcal{M}_0$ is generated by one element $b$ as an $A$-module. By induction it is clear that $\mathcal{M}$ is also generated by $b$ as an $\mathcal{A}$-module.

It is interesting to study the algebra $\mathcal{A}_r := \mathbf{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$ of all homomorphisms of $\mathcal{M}$, i.e. $\mathbf{k}$-linear maps $\mathsf{D} : \mathcal{M} \to \mathcal{M}$ such that

$$\mathsf{D}\big(\rho(f)(\varphi)\big) = \rho(f)\big(\mathsf{D}(\varphi)\big) \quad \forall\, f \in \mathcal{A} \;\; \forall\, \varphi \in \mathcal{M} \tag{5.1}$$

The algebra $\mathcal{A}_r$ is also known as the *commutant of the $\mathcal{A}$-module $\mathcal{M}$*. It is classical that $\mathcal{M}$ becomes a *$\mathcal{A}$-$\mathcal{A}_r^{\mathrm{opp}}$-bimodule* via $(f, \varphi, \mathsf{D}) \mapsto \rho(f)\big(\mathsf{D}\varphi\big)$ for all $f \in \mathcal{A}$, $\mathsf{D} \in \mathcal{A}_r$, and $\varphi \in \mathcal{M}$.

The algebra $\mathcal{A}_r$ can be related to a certain subquotient of $\mathcal{A}$ as follows: let $\mathcal{I}$ be the left ideal in $\mathcal{A}$ defined by

$$\mathcal{I} := \{g \in \mathcal{A} \mid \rho(g)b = 0\}.$$

Let $\varphi \in \mathcal{M}$. There is $f \in \mathcal{A}$, unique up to elements in $\mathcal{I}$, with $\rho(f)b = \varphi$. Let $\mathsf{D} \in \mathcal{A}_r$ and $h \in \mathcal{A}$ (also unique up to elements in $\mathcal{I}$) with $\rho(h)b = \mathsf{D}b$. Then we have

$$\mathsf{D}\varphi = \mathsf{D}\rho(f)b = \rho(f)\mathsf{D}b = \rho(f)\rho(h)b = \rho(f * h)b.$$

Hence $\mathsf{D}$ is determined by $h$ which has the additional property that $g * h \in \mathcal{I}$ for all $g \in \mathcal{I}$. Define the *idealizer of $\mathcal{I}$* by

$$\mathcal{N}(\mathcal{I}) := \{h \in \mathcal{A} \mid g * h \in \mathcal{I} \;\forall\, g \in \mathcal{I}\}. \tag{5.2}$$

It follows that $\mathcal{N}(\mathcal{I})$ is a subalgebra of $\mathcal{A}$, that $\mathcal{I}$ is a two-sided ideal of $\mathcal{N}(\mathcal{I})$, and that

$$\mathcal{N}(\mathcal{I})/\mathcal{I} \to \mathcal{A}_r^{\mathrm{opp}} : \bar{h} := h + \mathcal{I} \mapsto \mathsf{D}_{\bar{h}} : (\rho(f)b \mapsto \rho(f * h)b$$

is a well-defined isomorphism of associative algebras.

## 5.2 Classical limit in the symplectic case: symplectic reduction

Let $(M, \omega)$ be a symplectic manifold, $A = \mathcal{C}^\infty(M, \mathbb{K})$, and let $*$ be a star-product on $M$. Let $i : C \to M$ be a closed coisotropic submanifold and $\mathcal{M} = \mathcal{C}^\infty(C, \mathbb{K})[[\nu]]$. Suppose that there is a star-product representation $\rho$ of $*$ on $\mathcal{M}$. Computing zeroth and first order of (5.2) we get the following classical limits:

$$\begin{array}{rcl} \mathcal{I} & \stackrel{\nu\to 0}{\longrightarrow} & I \\ \mathcal{N}(\mathcal{I}) & \stackrel{\nu\to 0}{\longrightarrow} & N(I) := \{f \in \mathcal{C}^\infty(M, \mathbb{K}) \mid \{f, g\} \in I \ \forall \ g \in I\} \end{array}$$

Note that in these 'limits' only the conditions up to first order of the zeroth components of $f, g$ are considered; there may be additional ones in higher orders: in general it is NOT true that $\mathcal{N}(\mathcal{I})$ is isomorphic to $N(I)[[\nu]]$ as $\mathbb{K}[[\nu]]$-module. Since $C$ is coisotropic, $I$ is a Poisson subalgebra of $A$ (and a commutative ideal), and it follows that $N(I)$, its Lie normalizer, is a Poisson subalgebra with unit of $A$ and $I$ is a Poisson ideal of $N(I)$. Hence the quotient $N(I)/I$ is a Poisson algebra which can be seen as the classical limit

$$\mathcal{N}(\mathcal{I})/\mathcal{I} \stackrel{\nu\to 0}{\longrightarrow} N(I)/I.$$

The *geometric interpretation* is a well-known construction called *phase space reduction*: the coisotropic submanifold $C$ is foliated along the integrable subbundle $TC^\omega$. Since the differentials of elements of $I$ vanish along $C$ it can be shown that the Hamiltonian vector fields of $g \in I$ are along the leaves of the foliation $\mathcal{F}$. Hence $N(I)$ is a space of functions constant along the leaves and thus project down to the quotient space (or *reduced phase space*) $M_{\mathrm{red}} := C/\mathcal{F}$. Hence the Poisson algebra $N(I)/I$ models a function space on $M_{\mathrm{red}}$. In case $M_{\mathrm{red}}$ is a smooth manifold and the canonical projection $C \stackrel{\pi}{\to} M_{\mathrm{red}}$ is a smooth submersion it is known that $M_{\mathrm{red}}$ carries a canonical symplectic structure $\omega_{\mathrm{red}}$ defined by

$$i^*\omega =: \pi^*\omega_{\mathrm{red}}.$$

In that case there is the following isomorphism

$$N(I)/I \cong \mathcal{C}^\infty(M_{\mathrm{red}}, \mathbb{K}) \qquad \text{as Poisson algebras.}$$

This point of view is one of the features of the *classical BRST approach to constraints*, incorporating the ideas of Dirac 1964 and Batalin-Fradkin-Vilkovisky 1983/85, see e.g. Henneaux-Teitelboim 1988 , Stasheff 1988 [34] [26], Kimura 1992 [36].

A.Weinstein and J.Hua Lu (1993) [39] emphasized the *bimodule structure* of the geometric situation (‘*coisotropic creed*’)

$$\begin{array}{ccccccc} & & M & & & & \mathcal{C}^\infty(M,\mathbb{K}) \\ & \overset{i}{\nearrow} & & & & \overset{i^*}{\swarrow} & \\ C & & & \text{corresponds to} & \mathcal{C}^\infty(C,\mathbb{K}) & & \\ & \underset{\pi}{\searrow} & & & & \underset{\pi^*}{\nwarrow} & \\ & & M_{\text{red}} & & & & \mathcal{C}^\infty(M_{\text{red}},\mathbb{K}) \end{array}$$

hence $\mathcal{C}^\infty(C,\mathbb{K})$ is a $\mathcal{C}^\infty(M,\mathbb{K})$-$\mathcal{C}^\infty(M_{\text{red}},\mathbb{K})$-bimodule via $f\varphi h = (i^*f)(\pi^*h)\varphi$. Note that both $i$ and $\pi$ are coisotropic maps.

Example: Marden-Weinstein reduction (1974) [40]: let $J : M \to \mathfrak{g}^*$ be a momentum map (coming from symplectic $G$-action), 0 regular value of $J$:

$$C := J^{-1}(0) \text{ is coisotropic}, \quad M_{\text{red}} = C/G.$$

## 5.3 Quantization Problem: quantum reduction

The inverse problem is interesting: in the above geometric situation $M \overset{i}{\leftarrow} C \overset{\pi}{\rightarrow} M_{\text{red}}$ what would a reasonable quantum reduction be?

- *Coisotropic creed*: are there star-products $*$ on $M$ and $*_r$ on $M_{\text{red}}$ such that

$$\mathcal{M} := \mathcal{C}^\infty(C,\mathbb{K})[[\nu]] \text{ becomes a } * - *_r - \textbf{bimodule} \text{ ?}$$

- *Classical BRST*: is there a star-product $*$ on $M$ such that

$$\begin{gathered} I[[\nu]] \text{ is a } *-\text{subalgebra of } \mathcal{A}, \\ N(I)[[\nu]] \text{ is a } *-\text{subalgebra of } \mathcal{A}, \\ I[[\nu]] \text{ is a } *-\text{ideal of } N(I)[[\nu]] \text{ ?} \end{gathered}$$

  This implies that

$$N(I)[[\nu]]/I[[\nu]] \cong \mathcal{C}^\infty(M_{\text{red}},\mathbb{K})[[\nu]] \text{ has a star} - \text{product}.$$

  The star-product $*$ is called *projectable* in that case. Note that $I[[\nu]]$ is not demanded to be a one-sided ideal of $\mathcal{A}$ as in the first approach.

## 5.4 Results

### 5.4.1 Partial results

A deformation quantization version of the BRST construction in the situation of the Marsden-Weinstein reduction has been given by M.B, H.-C. Herbig, S.Waldmann (2000) [7].

### 5.4.2 Subalgebra deformations of the vanishing ideal

The difference of the two aforementioned approaches becomes smaller in the symplectic case, see [10]

**Theorem 5.1 (M.B. 2003)** *: For every* symplectic *manifold $M$ and connected closed coisotropic submanifold $C$ we have: if there is a star-product $*$ on $M$ such that $I[[\nu]]$ is a* subalgebra *of $\mathcal{A}$, then it is* either a left or a right ideal *of $\mathcal{A}$.*

### 5.4.3 Quantum reduction in the symplectic case

For smooth reduced phase spaces, the quantization of the reduction procedure can in some sense be classified, see [10]:

**Theorem 5.2** *(M.B. 2003): Let $i : C \to M$ be a connected coisotropic submanifold of a symplectic manifold $(M, \omega)$ such that the reduced phase space $\pi : C \to M_{\mathrm{red}}$ exists. Let $*$ be a star-product on $M$. Then the following conditions are equivalent:*

1. *$i^*[*]$ is basic, i.e. there is a class $\beta$ on $M_{\mathrm{red}}$ with $i^*[*] = \pi^*\beta$.*
2. *$*$ is reducible, i.e. equivalent to a projectable star-product.*
3. *There is a star-product $*_r$ on $M_{\mathrm{red}}$ such that $\mathcal{C}^\infty(C, \mathbb{K})[[\nu]]$ becomes a $* - *_r$ or a $*_r - *$-bimodule.*

*If one of these conditions is satisfied then*

- *$*$ is representable.*
- *$i^*[*] = \pi^*[*_r]$.*
- *$\big(\mathcal{C}^\infty(M_{\mathrm{red}}, \mathbb{K})[[\nu]], *_r\big)$ is the commutant of the algebra $\big(\mathcal{C}^\infty(M, \mathbb{K})[[\nu]], *\big)$ acting on $\mathcal{C}^\infty(C, \mathbb{K})[[\nu]]$.*
- *The isomorphism classes of $* - *_r$-bimodule-structures on $\mathcal{C}^\infty(C, \mathbb{K})[[\nu]]$ are in bijection to the following deRham cohomology groups*

$$\nu H^{1\prime}_{dR}(C, \mathbb{K}) \oplus \nu^2 H^1_{dR}(C, \mathbb{K})[[\nu]]$$

*where the $'$ means quotienting by $2\pi i$ times the integer classes for $\mathbb{K} = \mathbb{C}$.*

Idea of proof:

1. Note that (A.Weinstein)
$$C \to M \times \overline{M_{\mathrm{red}}}$$
is a Lagrangian submanifold.

2. An open neighbourhood of $C$ in $M \times \overline{M_{\rm red}}$ looks like an open neighbourhood of $T^*C$ (A.Weinstein).

3. The symbol calculus on $C$ by means of a star-product is possible iff the restriction of the Deligne class vanishes (M.B., N.Neumaier, M.Pflaum, S.Waldmann, 2003, [9]).

4. Thereby one gets a representation of the tensor product $* \otimes *_r^{\rm opp}$ on $C$ which gives the bimodule structure.

Note that there is equivalence of Coisotropic Creed and Classical BRST in this case.

# 6 The Atiyah-Molino-Class of a foliation

In order to understand the nature of the total obstructions I am going to present in the next section I shall give some overview of the Atiyah-Molino class of a foliation (P.Molino 1971 [41], [42]):

Let $E$ be an integrable subbundle of the tangent bundle of a manifold $C$. Frobenius' theorem guarantees the existence of a foliation of $C$ along $E$. Consider the quotient bundle $Q := TC/E$, then one has $\Gamma^\infty(C,Q) = \Gamma^\infty(C,TC)/\Gamma^\infty(C,E)$. Since $\Gamma^\infty(C,E)$ is a Lie subalgebra of the Lie algebra $\Gamma^\infty(C,TC)$ of all vector fields, there is a natural representation of the 'vertical fields', $\Gamma^\infty(C,E)$ on $\Gamma^\infty(C,Q)$ by means of

$$\nabla_V^{\rm Bott}\overline{X} := \overline{[V,X]}$$

for all $V \in \Gamma^\infty(C,E), X \in \Gamma^\infty(C,TC)$ and $X \mapsto \overline{X}$ denoting the canonical projection $\Gamma^\infty(C,TC) \to \Gamma^\infty(C,Q)$. This representation is called the *Bott connection* since it satisfies the Koszul axioms of a partial connection along $E$ in the vector bundle $Q$. The representation identity implies the flatness of the Bott connection. The spaces of smooth sections of tensor, symmetric or Grassmann products of $Q$ or its dual are thus modules of the Lie algebra $\Gamma^\infty(C,E)$ and the space $\Gamma^\infty(C, \Lambda E^* \otimes \mathsf{F})$ (where $\mathsf{F}$ is one of these bundles constructed out of $Q$) becomes a cocomplex by the natural vertical version $d_v$ of the exterior derivative.

It is always possible to extend the Bott connection in a nonunique manner to a connection $\nabla$ on $C$ in $Q$, i.e. for which $\nabla_V\overline{X} = \nabla_V^{\rm Bott}\overline{X}$ for all vertical vector fields $V$. The curvature tensor $R$ of this connection has the property that $R(V,W)\overline{X} = 0$ for all vertical $V, W$ and arbitrary vector fields $X$ due to the flatness of the Bott connection. Hence the curvature tensor descends to a section $r_{AM}$ in $\Gamma^\infty(C, \Lambda^1 E^* \otimes Q \otimes Q^* \otimes Q^*)$ defined by $r_{AM}(V)(\overline{X}, \overline{Y}) := R(V,X)\overline{Y}$. By the Bianchi identity it follows that $d_v r_{AM} = 0$. The class of this 1-cocyle, the so-called *Atiyah-Molino class* $c_{AM}(C,E)$ turns out to be independent of the connection one has chosen to extend the Bott connection, and it is thus a differential topological invariant of the foliation. An example of a foliation with nonzero Atiyah-Molino class is provided by the *Reeb foliation of the three-sphere*.

Molino has also shown that the vanishing of the Atiyah-Molino class is equivalent to the existence of a *locally projectable connection* in $TC$, i.e. a connection for which the locally defined projections on the local leaf spaces (in a foliation chart) are affine maps with respect to a local connection on the local leaf space. In the Fedosov construction this fact is crucial.

For coisotropic submanifolds one can find a variant of the Atiyah-Molino class, $\kappa_{AM}(C, E)$ where only connections are considered which preserve the symplectic structure on $Q$, see [10].

# 7 Obstructions and foliations

## 7.1 Morphisms

### 7.1.1 Reduction to bimodule problem

Since the graph of a Poisson map is a coisotropic submanifold of the cartesian product of the two Poisson manifolds concerned, the problem of quantizing a Poisson map can be reduced to a *bimodule representation problem on their graphs*, see [10].

### 7.1.2 Total obstructions in the symplectic case

By a rather technical *Fedosov analysis* of a Poisson map $\phi : M \to M'$ between symplectic manifolds we get the

**Theorem 7.1 (M.B.2003)** *Let $*$ be a star-product on $M$ and $*'$ be a star-product on $M'$. Let $E := (KerT\phi)^\omega$ Let $\alpha_0$ be a representative of $[*]_0 - \phi^*[*']_0$. Then $*$ is adapted up to order 3 iff*

1. $0 = p_v\big([*]_0 - \phi^*[*']_0\big)$ *(vertical restriction of class of $\alpha_0$ vanishes)*
2. $\alpha_0$ *can be chosen in such a way that*

$$\frac{1}{12}\overline{P}^{(3)}\big(\kappa_{AM}(M,E), \kappa_{AM}(M,E)\big) + \frac{1}{2}\overline{P}^{(1)}([\alpha_0]_{(1,1)}, [\alpha_0]_{(1,1)}) - p_v\big([*]_1 - \phi^*[*']\big) = 0$$

*where $\overline{P}^{(p)} \in \Gamma^\infty(C, Q^{\otimes 2p})$ are $p$ factors of the transverse Poisson structure, $[\alpha_0]_{(1,1)}$ denotes the class of $\alpha_0$ seen as form in $\Gamma^\infty\big(M, \Lambda^1 M \otimes (TM/E)\big)$ and $\kappa_{AM}(M,E)$ is the Atiyah-Molino class of the presymplectic manifold $(M, \phi^*\omega')$.*

See [10] for a proof.

## 7.2 (Bi)modules

### 7.2.1 Recursive obstructions

Recursive obstructions (M.B 2002/04; A.Cattaneo, G.Felder 2003 [15]) answered the question: if a star-product $*$ is representable on a coisotropic submanifold $C$ of a Poisson manifold $(M, P)$ up to order $r$, what are necessary and sufficient conditions to continue up to order $r+1$ if one can only modify the orders $r$ and $r+1$?

- The good cocomplex is given by $\Gamma^\infty\big(C, \Lambda^\bullet(TM|_C/TC)\big)$ with
$$\text{differential} \;\; d_P : \bar{A} \mapsto \overline{[P, A]}.$$
- The obstructions lie in
$$H^2_P(C, \mathbb{K}) \;\; \big(\cong \;\; H^2_{\mathrm{vert.dR}}(C, \mathbb{K}) \;\; \text{if } M \text{ is symplectic}\big).$$
- In the particular case where $C$ *is of codimension* 1 (P.Glößner 1998) $H^2_P(C, \mathbb{K})$ always vanishes, hence *every star-product is representable.*

### 7.2.2 Symplectic case

Total obstructions to modules up to order three (M.B.2003) [10]: take the *general* symplectic star-product $*$ up to order three (Lichnerowicz 1980; see e.g. DeWilde, Lecomte 1988 [21]; Fedosov 1996 [24]) and check whether it is adapted:

- Restrict to a tubular neighbourhood $U$ of $C$;
- Use the *Gotay/Weinstein-Theorem* [31] $U \cong V \subset E^* := (TC^\omega)^*$;
- Use an *adapted symplectic covariant derivative* $\tilde{\nabla}$ constructed out of a presymplectic $E$-preserving connection $\nabla$ on $C$, and
- Perform some computations of higher order loop type....

**Theorem 7.2 (M.B. 2002)** *Let $\alpha_0, \alpha_1$ be representatives of $[*]_0, [*]_1$. Then $*$ is adapted up to order 3 iff*

1. $0 = p_v i^*[\alpha_0]$ *(vertical restriction of class of $\alpha_0$ vanishes)*
2. *$\alpha_0$ can be chosen in such a way that*
$$\frac{1}{12}\overline{P}^{(3)}\big(\kappa_{AM}(C,E), \kappa_{AM}(C,E)\big) + \frac{1}{2}\overline{P}^{(1)}([i^*\alpha_0]_{(1,1)}, [i^*\alpha_0]_{(1,1)}) - p_v i^*[*]_1 = 0$$

*where $\kappa_{AM}(C, E)$ is the Atiyah-Molino class of the presymplectic manifold $C$.*

See [10] for a proof. It is hard to see whether these condition can always be satisfied which I doubt although I do not know of any concrete example.

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