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Quasi-Poisson structures as Dirac structures

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Abstract

We show that quasi-Poisson structures can be identified with Dirac structures in suitable Courant algebroids. This provides a geometric way to construct Lie algebroids associated with quasi-Poisson spaces.

1 Introduction

In this note we use the theory of Courant algebroids to give a geometrical construction of the Lie algebroids associated with quasi-Poisson spaces considered in [5, 6]. Our main observation is that, just as ordinary Poisson structures, quasi-Poisson structures [1] can be described as Dirac structures, but in a different Courant algebroid.

Let M be a manifold, and let $\mathfrak{X}^k(M)$ denote the space of k-multivector fields on M. For a bivector field $\pi \in \mathfrak{X}^2(M)$, consider the bundle map

$$\pi^\sharp : T^*M \to TM, \quad \beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta), \tag{1.1}$$

and the bracket on $\Gamma(T^*M) = \Omega^1(M)$ given by

$$[\alpha, \beta]_\pi := \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d\pi(\alpha, \beta). \tag{1.2}$$

Let $TM \oplus T^*M$ be equipped with its original Courant bracket [7]. In Poisson geometry, we have the following well-known result:

Proposition 1.1 *The following conditions are equivalent:*

i) The bivector field π defines a Poisson structure on M;

ii) T^*M is a Lie algebroid with anchor (1.1) *and bracket* (1.2)*;*

iii) $L_\pi := \mathrm{graph}(\pi^\sharp) \subset TM \oplus T^*M$ is a Dirac structure.

The equivalence between *i*) and *iii*) is one of the motivating examples for the theory of Dirac structures [7, 8]; on the other hand, whenever L_π is a Dirac subbundle of $TM \oplus T^*M$, it inherits a Lie algebroid structure, and the equivalence of *ii*) and *iii*) follows from the natural identification

$$T^*M \xrightarrow{\sim} L_\pi, \quad \alpha \mapsto (\pi^\sharp(\alpha), \alpha). \tag{1.3}$$

This note concerns the analogous description of *quasi*-Poisson structures in terms of Lie algebroids and Dirac structures. If $\mathfrak{g}$ is a Lie quasi-bialgebra, it is shown in [5, 6] that a quasi-Poisson $\mathfrak{g}$-action on M is equivalent to a certain Lie algebroid structure on $\mathfrak{g} \oplus T^*M$ (see Thm. 2.1 in Section 2). This is the analog in quasi-Poisson geometry of the equivalence of *i*) and *ii*) above. The proof of this result in [6] is purely algebraic, based on the construction of a degree-one differential on $\Gamma(\wedge(\mathfrak{g}^* \oplus TM))$. Our main result (Thm. 4.1) provides the analog of *iii*): any quasi-Poisson $\mathfrak{g}$-structure on M can be identified with a Dirac structure

$$L \subset \mathfrak{d} \oplus (TM \oplus T^*M), \tag{1.4}$$

where now the Courant algebroid in question is the direct sum of $TM \oplus T^*M$ and the Drinfeld double $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$. Moreover, this Dirac structure naturally induces the Lie algebroid structure on $\mathfrak{g} \oplus T^*M$ through an identification analogous to (1.3). This completes the picture of the quasi-Poisson counterpart of the equivalences in Proposition 1.1.

The description of quasi-Poisson spaces in terms of Lie algebroids has several interesting consequences. It shows, in particular, that *any* quasi-Poisson $\mathfrak{g}$-space carries a singular foliation (the "orbits" of the Lie algebroid). In the hamiltonian context, these foliations have been studied in [1, 2] in order to relate quasi-Poisson geometry to the momentum map theory of [3]. More generally, the Lie algebroids of quasi-Poisson spaces are essential to unravel the connection between the theory of D/G-valued momentum maps [1] and Dirac geometry, see [5, 6].

The paper is organized as follows: In Section 2 we recall Lie quasi-bialgebras, quasi-Poisson spaces and their associated Lie algebroids; Section 3 recalls Courant algebroids and Lie quasi-bialgebroids; In Section 4 we describe quasi-Poisson spaces in terms of Dirac structures and prove our main result (Thm. 4.1). In Section 5, we point out various interesting aspects of the Lie algebroids of quasi-Poisson spaces from this new geometric point of view.

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2 Lie quasi-bialgebras and quasi-Poisson spaces

In this section we recall some definitions in quasi-Poisson geometry.

A **Lie quasi-bialgebra** [9] is a triple $(\mathfrak{g}, F, \chi)$, where $\mathfrak{g}$ is a (finite-dimensional, real) Lie algebra, $F \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g} \wedge \mathfrak{g})$, and $\chi \in \wedge^3\mathfrak{g}$, satisfying compatibility conditions which are equivalent to the requirement that $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ is a Lie algebra with respect to the bracket:

$$[(u,0),(v,0)]_{\mathfrak{d}} = ([u,v]_{\mathfrak{g}}, 0), \tag{2.1}$$

$$[(v,0),(0,\mu)]_{\mathfrak{d}} = (-\mathrm{ad}^*_{\mu}v, \mathrm{ad}^*_{v}\mu), \tag{2.2}$$

$$[(0,\mu)(0,\nu)]_{\mathfrak{d}} = (\chi(\mu,\nu), F^*(\mu,\nu)), \tag{2.3}$$

for $u, v \in \mathfrak{g}$ and $\mu, \nu \in \mathfrak{g}^*$. The Lie algebra $(\mathfrak{d}, [\cdot,\cdot]_{\mathfrak{d}})$ is called the **Drinfeld double** [4] of the Lie quasi-bialgebra $(\mathfrak{g}, F, \chi)$.

Given a Lie quasi-bialgebra $(\mathfrak{g}, F, \chi)$, a **quasi-Poisson $\mathfrak{g}$-space**[1] [1] is a smooth manifold M equipped with a bivector field $\pi \in \mathfrak{X}^2(M)$ and a $\mathfrak{g}$-action $\rho_M : \mathfrak{g} \to \mathfrak{X}^1(M)$ so that

$$\frac{1}{2}[\pi,\pi] = \rho_M(\chi), \tag{2.4}$$

$$\mathcal{L}_{\rho_M(v)}\pi = -\rho_M(F(v)), \quad \text{for all } v \in \mathfrak{g}. \tag{2.5}$$

In (2.4), (2.5), we keep the notation $\rho_M : \wedge^\bullet\mathfrak{g} \to \mathfrak{X}^\bullet(M)$ for the induced map of exterior algebras.

We saw that the integrability condition of a Poisson bivector field is equivalent to the Jacobi identity of (1.2), and the axioms of a Lie quasi-bialgebra are equivalent to the Jacobi identity of $[\cdot,\cdot]_{\mathfrak{d}}$. Analogously, it is shown in [6] that the compatibility conditions (2.4), (2.5) defining a quasi-Poisson action are equivalent to the Jacobi identity of a certain bracket on $\Gamma(\mathfrak{g} \oplus T^*M) = C^\infty(M, \mathfrak{g}) \oplus \Omega^1(M)$. More precisely, we have [6]:

Theorem 2.1 *Let $(\mathfrak{g}, F, \chi)$ be a Lie quasi-bialgebra, let M be a smooth manifold equipped with a bivector field π, and let $\rho_M : \mathfrak{g} \to \mathfrak{X}^1(M)$ be an $\mathbb{R}$-linear map. Then the following are equivalent:*

1. *$\rho_M : \mathfrak{g} \to \mathfrak{X}^1(M)$ preserves brackets and makes (M, π) into a quasi-Poisson $\mathfrak{g}$-space;*
2. *$(A, r, [\cdot,\cdot]_A)$ is a Lie algebroid, where $A = \mathfrak{g} \oplus T^*M$, $r : \mathfrak{g} \oplus T^*M \to TM$ is the bundle map*

$$r(u,\alpha) = \rho_M(u) + \pi^\sharp(\alpha), \tag{2.6}$$

[1] We restrict our attention to Lie quasi-bialgebras and their infinitesimal actions; the reader is referred to [1, 11] for their global versions.

and the bracket $[\cdot,\cdot]_A$ *on* $C^\infty(M,\mathfrak{g})\oplus\Omega^1(M)$ *is given by*

$$[(u,0),(v,0)]_A = ([u,v]_{\mathfrak{g}},0), \tag{2.7}$$

$$[(v,0),(0,\alpha)]_A = (-i_{\rho_M^*(\alpha)}(F(v)),\mathcal{L}_{\rho_M(v)}\alpha), \tag{2.8}$$

$$[(0,\alpha)(0,\beta)]_A = (i_{\rho_M^*(\alpha\wedge\beta)}\chi,[\alpha,\beta]_\pi), \tag{2.9}$$

for $\alpha,\beta\in\Omega^1(M)$, *and* $u,v\in\mathfrak{g}$, *considered as constant sections in* $C^\infty(M,\mathfrak{g})$ *(the bracket is extended to general elements by the Leibniz rule).*

A direct corollary of this result is that the generalized distribution defined by $\rho_M(u)+\pi^\sharp(\alpha)\subseteq TM$, $u\in\mathfrak{g}$, $\alpha\in T^*M$, is integrable.

Theorem 2.1 is the counterpart for quasi-Poisson spaces of the equivalence of *i*) and *ii*) in Proposition 1.1. The remainder of this note is devoted to showing that this Lie algebroid structure on $\mathfrak{g}\oplus T^*M$ is inherited from a Dirac structure.

3 Courant algebroids and Lie quasi-bialgebroids

A **Courant algebroid** [12] over a manifold M is a vector bundle $E\to M$ equipped with a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on the bundle, a bundle map $\rho: E\to TM$ and a bilinear bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(E)$ such that for all $e,e_1,e_2,e_3\in\Gamma(E)$, $f\in C^\infty(M)$ the following is satisfied:

1. $[\![e_1,[\![e_2,e_3]\!]]\!]=[\![[\![e_1,e_2]\!],e_3]\!]+[\![e_2,[\![e_1,e_3]\!]]\!]$;
2. $[\![e,e]\!]=\frac{1}{2}\mathcal{D}\langle e,e\rangle$;
3. $\mathcal{L}_{\rho(e)}\langle e_1,e_2\rangle=\langle[\![e,e_1]\!],e_2\rangle+\langle e_1,[\![e,e_2]\!]\rangle$;
4. $\rho([\![e_1,e_2]\!])=[\rho(e_1),\rho(e_2)]$;
5. $[\![e_1,fe_2]\!]=f[\![e_1,e_2]\!]+(\mathcal{L}_{\rho(e_1)}f)e_2$,

where $\mathcal{D}: C^\infty(M)\to\Gamma(E)$ is defined by $\langle\mathcal{D}f,e\rangle=\mathcal{L}_{\rho(e)}f$. We chose to use non-skew-symmetric brackets as in [18].

A subbundle $L\subset E$ is called a **Dirac structure** (or a **Dirac subbundle**) if it is maximal isotropic with respect to $\langle\cdot,\cdot\rangle$ and if $\Gamma(L)$ is closed under $[\![\cdot,\cdot]\!]$. The latter requirement is referred to as the *integrability condition*.

The following two standard examples will play a central role in this note.

Example 3.1 A Courant algebroid over a point is just a Lie algebra $\mathfrak{d}$ equipped with an ad-invariant nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$ (condition 3.). In this case, a Dirac structure is a Lie subalgebra $\mathfrak{g}\subset\mathfrak{d}$ which is a maximal isotropic subspace.

Example 3.2 The vector bundle $TM \oplus T^*M$ over M equipped with the symmetric pairing $\langle (X,\alpha),(Y,\beta)\rangle := \beta(X) + \alpha(Y)$ and bracket on $\mathfrak{X}^1(M) \oplus \Omega^1(M)$ given by

$$[\![(X,\alpha),(Y,\beta)]\!]_M := ([X,Y], \mathcal{L}_X\beta - i_Y d\alpha) \tag{3.1}$$

is the (non-skew-symmetric version [12, 18] of the) original Courant algebroid of [7].

Important examples of maximal isotropic subbundles are graphs of bundle maps $\omega^\sharp : TM \to T^*M$ (resp. $\pi^\sharp : T^*M \to TM$) associated with 2-forms $\omega \in \Omega^2(M)$ (resp. bivector fields $\pi \in \mathfrak{X}^2(M)$); in this case, the integrability condition amounts to $d\omega = 0$ (resp. $[\pi,\pi] = 0$, where $[\cdot,\cdot]$ is the Schouten bracket).

More general Courant brackets on $TM \oplus T^*M$ are considered in [20].

We restrict our attention to Courant algebroids $E \to M$ that can be written as $E = L \oplus K$, where L is a Dirac structure and K is a complementary isotropic subbundle of L (not necessarily satisfying the integrability condition). We identify K with L^* using $\langle \cdot,\cdot \rangle$ so that $E = L \oplus L^*$ is now equipped with the symmetric form

$$\langle (l_1,\xi_1),(l_2,\xi_2)\rangle = \xi_2(l_1) + \xi_1(l_2), \quad l_1,l_2 \in \Gamma(L),\ \xi_1,\xi_2 \in \Gamma(L^*).$$

The natural projections are denoted by $\mathrm{pr}_L : E \to L$ and $\mathrm{pr}_{L^*} : E \to L^*$.

If $[\cdot,\cdot]_L$ is the restriction of $[\![\cdot,\cdot]\!]$ to $\Gamma(L)$, then $(L, [\cdot,\cdot]_L, \rho|_L)$ is a Lie algebroid. The associated coboundary operator is denoted by

$$d_L : \Gamma(\wedge^\bullet L^*) \to \Gamma(\wedge^{\bullet+1} L^*),$$

and the Schouten-type bracket on $\Gamma(\wedge L)$ is denoted by

$$[\cdot,\cdot]_L : \Gamma(\wedge^k L) \times \Gamma(\wedge^m L) \to \Gamma(\wedge^{k+m-1} L).$$

For each $l \in \Gamma(L)$, we denote the corresponding Lie derivative operator on $\Gamma(\wedge L^*)$ by $\mathcal{L}_l$, see e.g. [16, Sec. 2]. Dually, we may define a bracket $[\cdot,\cdot]_{L^*}$ on $\Gamma(L^*)$ by

$$[\xi_1,\xi_2]_{L^*} := \mathrm{pr}_{L^*}([\![\xi_1,\xi_2]\!]), \quad \xi_1,\xi_2 \in \Gamma(L^*). \tag{3.2}$$

The bracket (3.2) and the map $\rho|_{L^*} : L^* \to TM$ then induce, as before, a derivation d_{L^*} of degree $+1$ on $\Gamma(\wedge L)$ and a bracket $[\cdot,\cdot]_{L^*}$ of degree -1 on $\Gamma(\wedge L^*)$, but now d_{L^*} is just a "quasi" differential (it may not square to zero) and $[\cdot,\cdot]_{L^*}$ is just a "quasi" Gerstenhaber bracket, see [19]. We keep the notation $\mathcal{L}_\xi$ for the Lie derivative operator on $\Gamma(\wedge L)$ associated with $\xi \in \Gamma(L^*)$.

It follows from condition 3. in the definition of $[\![\cdot,\cdot]\!]$ that, for $l \in \Gamma(L)$ and $\xi \in \Gamma(L^*)$, we have

$$[\![(l,0),(0,\xi)]\!] = (-i_\xi d_{L^*} l, \mathcal{L}_l \xi).$$

Hence, for $l_1, l_2 \in \Gamma(L)$ and $\xi_1, \xi_2 \in \Gamma(L^*)$, the bracket $[\![\cdot,\cdot]\!]$ on $E = L \oplus L^*$ has the form

$$[\![(l_1,\xi_1),(l_2,\xi_2)]\!] = ([l_1,l_2]_L - i_\beta d_{L^*} l_1 + \mathcal{L}_{\xi_1} l_2 + \Phi(\xi_1,\xi_2), [\xi_1,\xi_2]_{L^*} + \mathcal{L}_{l_1}\xi_2 - i_{l_2} d_L \xi_1), \tag{3.3}$$

where $\Phi : \Gamma(\wedge^2 L^*) \to \Gamma(L)$ is given by

$$\Phi(\xi_1,\xi_2) = \mathrm{pr}_L([\![(0,\xi_1),(0,\xi_2)]\!]), \quad \xi_1,\xi_2 \in \Gamma(L^*). \tag{3.4}$$

(We often view Φ as an element in $\Gamma(\wedge^3 L)$.)

Example 3.3 We saw in Example 3.1 that Courant algebroids over a point are Lie algebras $(\mathfrak{d}, [\![\cdot,\cdot]\!])$ equipped with an ad-invariant nondegenerate symmetric form. If one can write $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{k}$, where $\mathfrak{g} \subset \mathfrak{d}$ is a maximal isotropic Lie subalgebra (i.e., a Dirac structure) and $\mathfrak{k}$ is an isotropic complement, then $(\mathfrak{d},\mathfrak{g},\mathfrak{k})$ is called a **Manin quasi-triple**. These are essentially the same as Lie quasi-bialgebra structures on $\mathfrak{g}$, see e.g. [1]:

On one hand, if $(\mathfrak{g}, F, \chi)$ is a Lie quasi-bialgebra and $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ is its Drinfeld double, then it is easy to check that $(\mathfrak{d},\mathfrak{g},\mathfrak{g}^*)$ is a Manin quasi-triple. Conversely, let $(\mathfrak{d},\mathfrak{g},\mathfrak{k})$ be a Manin quasi-triple, and let us identify $\mathfrak{k}$ with $\mathfrak{g}^*$. If we define $F \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}\wedge\mathfrak{g})$ as the dual of the bracket $[\cdot,\cdot]_{\mathfrak{g}^*} \in \mathrm{Hom}(\mathfrak{g}^*\wedge\mathfrak{g}^*, \mathfrak{g}^*)$ as in (3.2), and if we set $\chi = \Phi \in \wedge^3\mathfrak{g}$ as in (3.4), then writing the Lie bracket $[\![\cdot,\cdot]\!]$ on $\mathfrak{d}$ as in (3.3), one can check that it coincides with (2.1),(2.2) and (2.3). Hence $(\mathfrak{g}, F, \chi)$ is a Lie quasi-bialgebra.

Following Example 3.3, a **Lie quasi-bialgebroid** [18, 19] is defined as a Lie algebroid $(L, [\cdot,\cdot]_L, \rho_L)$ together with a bundle map $\rho_{L^*} : L^* \to TM$, an element $\Phi \in \Gamma(\wedge^3 L)$, and a skew-symmetric bracket $[\cdot,\cdot]_{L^*}$ on $\Gamma(L^*)$ such that $(E, [\![\cdot,\cdot]\!], \rho)$ is a Courant algebroid, where $E = L \oplus L^*$, $\rho = \rho_L + \rho_{L^*}$ and $[\![\cdot,\cdot]\!]$ is given by (3.3). If $(L^*, [\cdot,\cdot]_{L^*}, \rho_{L^*})$ is a Lie algebroid, then we call the pair (L, L^*) a **Lie bialgebroid** [12, 16].

Example 3.4 In the case of $E = TM \oplus T^*M$ with bracket $[\![\cdot,\cdot]\!]_M$ as in Example 3.2, both TM and T^*M are Dirac subbundles of E, so they form a Lie bialgebroid. (For the "twisted" Courant algebroids of [20], only T^*M is integrable, so (T^*M, TM) is a Lie quasi-bialgebroid [19]).

Let us consider an element $\Lambda \in \Gamma(\wedge^2 L^*)$ and the associated bundle map $\Lambda^\sharp : L \to L^*$. Let $L_\Lambda \subset L \oplus L^* = E$ be given by the graph of $\Lambda^\sharp$.

Proposition 3.5 *L_Λ is a Dirac structure if and only if Λ satisfies*

$$d_L\Lambda + \frac{1}{2}[\Lambda,\Lambda]_{L^*} = \Lambda^\sharp(\Phi). \tag{3.5}$$

Proposition 3.5 can be proven along the same lines of [12, Thm. 6.1], which is the particular case where $\Phi = 0$; see also [19].

4 Quasi-Poisson actions as Dirac structures

In this section we consider the Courant algebroid given by the direct sum of the Courant algebroids in Examples 3.1 and 3.2,

$$E := (\mathfrak{g} \oplus \mathfrak{g}^*) \oplus (TM \oplus T^*M), \tag{4.1}$$

with bracket

$$[\![(a_1, b_1), (a_2, b_2)]\!] := [(u_1, \mu_1), (u_2, \mu_2)]_{\mathfrak{d}} + [\![(X_1, \alpha_1), (X_2, \alpha_2)]\!]_M, \tag{4.2}$$

where $a_i = (u_i, \mu_i) \in \mathfrak{g} \oplus \mathfrak{g}^*$, $b_i = (X_i, \alpha_i) \in \Gamma(TM \oplus T^*M)$, $i = 1, 2$ (we regard a_i as constant sections and the bracket is extended to arbitrary sections in $C^\infty(M, \mathfrak{g} \oplus \mathfrak{g}^*)$ by the Leibniz rule), and anchor

$$\rho : E \to TM, \tag{4.3}$$

given by the natural projection of E onto TM. Note that $E = L \oplus L^*$, where $L = \mathfrak{g} \oplus T^*M$ is a Dirac structure and $L^* = \mathfrak{g}^* \oplus TM$ is an isotropic complement.

We now show that quasi-Poisson spaces can be naturally identified with certain Dirac structures in E. Suppose that $(\mathfrak{g}, F, \chi)$ is a Lie quasi-bialgebra, $\pi \in \mathfrak{X}^2(M)$ is a bivector field on M and $\rho_M : \mathfrak{g} \to \mathfrak{X}^1(M)$ is a linear map. It follows from the natural identification

$$\Gamma((\wedge^2\mathfrak{g}^*) \oplus (\mathfrak{g}^* \otimes TM) \oplus (\wedge^2 TM)) \xrightarrow{\sim} \Gamma(\wedge^2(\mathfrak{g}^* \oplus TM)) = \Gamma(\wedge^2 L^*) \tag{4.4}$$

that the pair (ρ_M, π) defines an element $\Lambda \in \Gamma(\wedge^2 L^*)$. As before, let $\Lambda^\sharp : L \to L^*$ be the associated bundle map.

We have the following quasi-Poisson counterpart of Prop. 1.1:

Theorem 4.1 *The following are equivalent:*

1. $L_\Lambda = \mathrm{graph}(\Lambda^\sharp)$ *is a Dirac structure in* E*;*
2. $\rho_M : \mathfrak{g} \to \mathfrak{X}^1(M)$ *defines a quasi-Poisson action on* (M, π)*;*
3. $(\mathfrak{g} \oplus T^*M, r, [\cdot,\cdot]_A)$ *is a Lie algebroid (with* r *defined by* (2.6) *and* $[\cdot,\cdot]_A$ *defined by* (2.7),(2.8) *and* (2.9)*).*

Proof: By Proposition 3.5, condition 1. is equivalent to the Maurer-Cartan type equation (3.5). In order to explicitly identify its terms, let us write $\rho_M = \sum_{i,j} e^i \otimes \rho_{ij}\partial x_j$, where e^i is a basis for $\mathfrak{g}^*$, and $\pi = \sum_{k,m} \pi_{km}\partial x_k \wedge \partial x_m$. The corresponding element $\Lambda \in \Gamma(\wedge^2(\mathfrak{g}^* \oplus TM))$ is

$$\Lambda = \sum_{i,j} (e^i, 0) \wedge \rho_{ij}(0, \partial x_j) + \sum_{k,m} \pi_{km}(0, \partial x_k) \wedge (0, \partial x_m). \tag{4.5}$$

Writing the Courant bracket (4.4) in the standard form (3.3), one sees that $\Phi = \chi$ (regarded as an element in $\Gamma(\wedge^3 L)$), and one checks that $\Lambda^\sharp : \Gamma(\mathfrak{g} \oplus T^*M) \to \Gamma(\mathfrak{g}^* \oplus TM)$ is given by

$$\Lambda^\sharp(v,\alpha) = (-\rho_M^*(\alpha), \rho_M(v) + \pi^\sharp(\alpha)), \quad v \in \mathfrak{g}, \alpha \in \Omega^1(M). \tag{4.6}$$

It follows that the right-hand side of (3.5) becomes

$$\Lambda^\sharp(\Phi) = \rho_M(\chi). \tag{4.7}$$

In order to identify the term $d_L\Lambda$, note that $d_L = \partial_{\mathfrak{g}}$, the Chevalley-Eilenberg operator of $\mathfrak{g}$ (since the differential on $\mathfrak{X}^1(M)$ is zero). It is then simple to check that $d_L\Lambda \in \wedge^2\mathfrak{g}^* \otimes \mathfrak{X}^1(M)$ is defined by

$$d_L\Lambda(u,v) = -\rho_M([u,v]_{\mathfrak{g}}), \quad \text{for } u, v \in \mathfrak{g}. \tag{4.8}$$

The remaining term in (3.5) is

$$\frac{1}{2}[\Lambda,\Lambda]_{L^*} \in (\mathfrak{g}^* \otimes \mathfrak{X}^2(M)) \oplus (\wedge^2\mathfrak{g}^* \otimes \mathfrak{X}^1(M)) \oplus (\mathfrak{X}^3(M)). \tag{4.9}$$

The bracket $[\cdot,\cdot]_{L^*}$ on $\Gamma(\mathfrak{g}^* \oplus TM)$ is $F^* + [\cdot,\cdot]$, where $[\cdot,\cdot]$ is the Lie bracket of vector fields; using (4.5) and the graded Leibniz identity for $[\cdot,\cdot]_{L^*}$, we obtain the following results: the component of (4.9) in $\mathfrak{g}^* \otimes \mathfrak{X}^2(M)$ is given by

$$v \mapsto \mathcal{L}_{\rho_M(v)}\pi + \rho_M(F(v)), \quad v \in \mathfrak{g}; \tag{4.10}$$

the component of (4.9) in $\wedge^2\mathfrak{g}^* \otimes \mathfrak{X}^1(M)$ is

$$(u,v) \mapsto [\rho_M(u), \rho_M(v)], \quad u, v \in \mathfrak{g}, \tag{4.11}$$

and the component in $\mathfrak{X}^3(M)$ is $\frac{1}{2}[\pi,\pi]$. Separating the terms by degrees, we find that

$$d_L\Lambda + \frac{1}{2}[\Lambda,\Lambda]_{L^*} = \rho_M(\chi)$$

is equivalent to the three equations:

$$\rho_M([u,v]_{\mathfrak{g}}) = [\rho_M(u), \rho_M(v)], \quad \frac{1}{2}[\pi,\pi] = \rho_M(\chi) \ \text{ and } \ \mathcal{L}_{\rho_M(v)}\pi = -\rho_M(F(v)),\ u, v \in \mathfrak{g}.$$

Hence conditions 1. and 2. are equivalent.

In order to show that 1. and 3. are equivalent, we observe that L_Λ is a Dirac structure if and only if $(L_\Lambda, \rho|_{L_\Lambda}, [\![\cdot,\cdot]\!]|_{L_\Lambda})$ is a Lie algebroid. So it suffices to prove that r and $[\cdot,\cdot]_A$ agree with $\rho|_{L_\Lambda}$ and $[\![\cdot,\cdot]\!]|_{\Gamma(L_\Lambda)}$ under the identification

$$L = \mathfrak{g} \oplus T^*M \xrightarrow{\sim} L_\Lambda, \quad (v,\alpha) \mapsto ((v,\alpha), (-\rho_M^*(\alpha), \rho_M(v) + \pi^\sharp(\alpha)))$$

(analogous to (1.3)). For the anchor map, we have

$$\rho((v,\alpha),(-\rho_M^*(\alpha),\rho_M(v)+\pi^\sharp(\alpha))) = \rho_M(v)+\pi^\sharp(\alpha) = r(v,\alpha).$$

For the bracket of elements of type $(u,0)$, $(v,0)$, we have

$$[\![((u,0),(0,\rho_M(u))),((v,0),(0,\rho_M(v)))]\!] = (([u,v]_{\mathfrak{g}},0),(0,[\rho_M(u),\rho_M(v)])),$$

hence the projection to $\Gamma(L)=\Gamma(\mathfrak{g}\oplus T^*M)$ is just $[u,v]_{\mathfrak{g}}$. For elements $(u,0)$ and $(0,\alpha)$, we get

$$\begin{aligned}[\![((u,0),(0,\rho_M(u))),((0,\alpha),(-\rho_M^*(\alpha),\pi^\sharp(\alpha)))]\!] &= [(u,0),(0,-\rho_M^*(\alpha))]_{\mathfrak{d}} \\ &\quad + [\![(\rho_M(u),0),(\pi^\sharp(\alpha),\alpha)]\!]_M,\end{aligned}$$

which equals $((\mathrm{ad}^*_{\rho_M^*(\alpha)}u, -\mathrm{ad}^*_u\rho_M^*(\alpha)),([\rho_M(u),\pi^\sharp(\alpha)],\mathcal{L}_{\rho_M(u)}\alpha))$; its projection to $\Gamma(L)$ is

$$(\mathrm{ad}^*_{\rho_M^*(\alpha)}u,\mathcal{L}_{\rho_M(u)}\alpha) = (-i_{\rho_M^*(\alpha)}F(u),\mathcal{L}_{\rho_M(u)}\alpha).$$

Finally, for elements $(0,\alpha)$, $(0,\beta)$, we similarly find that the projection of

$$[\![((0,\alpha),(-\rho_M^*(\alpha),\pi^\sharp(\alpha))),((0,\beta),(-\rho_M^*(\beta),\pi^\sharp(\beta)))]\!]$$

on $\Gamma(L)$ is $(i_{\rho^*(\alpha\wedge\beta)}\chi,[\alpha,\beta]_\pi)$. □

For a Lie quasi-bialgebra $(\mathfrak{g},F,\chi)$, the extreme cases of $F=0$ or $\chi=0$ are of interest:

Example 4.2 Let $\mathfrak{g}$ be a quadratic Lie algebra, and consider the Lie quasi-bialgebra structure for which $F=0$ and $\chi\in\wedge^3\mathfrak{g}$ is the Cartan trivector [1, Ex. 2.1.5]; in this case, the Lie algebroids of Thm. 4.1 coincide with the ones defined in [5] for quasi-Poisson $\mathfrak{g}$-manifolds.

Example 4.3 A Lie quasi-bialgebra for which $\chi=0$ is a Lie bialgebra; in this case the Lie algebroids of Thm. 4.1 are the same as the ones studied by Lu [13] in the context of Poisson actions.

5 Final remarks

We conclude the paper with some remarks and questions:

First of all, the equivalence of conditions 1. and 2. in Thm. 4.1 leads to a "gauge-invariant" definition of quasi-Poisson structure on a manifold M associated with a Manin *pair* $(\mathfrak{g},\mathfrak{d})$ [1, 11], rather than a quasi-triple: this is a Dirac structure in the Courant algebroid $E=\mathfrak{d}\oplus(TM\oplus T^*M)$ which intersect TM trivially and whose intersection with $\mathfrak{d}\oplus TM$ projects to $\mathfrak{g}$ under the natural map $E\to\mathfrak{d}$. For any choice of isotropic complement of $\mathfrak{g}$, $\mathfrak{h}\subset\mathfrak{d}$, this recovers the usual notion of quasi-Poisson structure on M associated with the Lie quasi-bialgebra defined by the quasi-triple $(\mathfrak{g}, \mathfrak{h}, \mathfrak{d})$.

Second, the identification of quasi-Poisson structures with Dirac structures in the Courant algebroid (4.1) indicates some other generalizations: since quasi-Poisson structures correspond to special elements in $\Gamma(\wedge^2 L^*)$ (those whose first component vanish under (4.4)), it could be interesting to understand what kind of structures correspond to more general elements; In another direction, the construction of the Lie algebroids of quasi-Poisson spaces can be extended to manifolds carrying quasi-Poisson actions of Lie quasi-bialgebr*oids*.

Third, as mentioned in Example 4.3, when $\chi = 0$ we are in the situation of a Poisson action of a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ on a Poisson manifold M; in this case, the Lie algebroid of Thm. 4.1 is obtained by a generalized semi-direct product involving the Lie algebroids $\mathfrak{g} \ltimes M$ and T^*M, as well as algebroid actions of each one on the other [13]. This is an example of a *matched pair of Lie algebroids*, in the sense of [17]. If $\chi \neq 0$, then T^*M fails to be a Lie algebroid in a way "controlled" by the action of $\mathfrak{g} \ltimes M$ on it in such a way that, by Thm. 4.1, $\mathfrak{g} \oplus T^*M$ still acquires a Lie algebroid structure. This suggests a corresponding notion of "quasi" matched pair.

Another remark, yet to be explored, is that the Lie algebroid $A = \mathfrak{g} \oplus T^*M$ associated with a quasi-Poisson action is naturally part of a Lie quasi-bialgebroid: the dual $\mathfrak{g}^* \oplus TM$ is equipped with the bracket $\mathrm{pr}_{L^*}([\![\cdot,\cdot]\!]|_{\Gamma(L^*)})$ and anchor $\rho|_{L^*}$ inherited from (4.1). This observation is immediate from the geometric construction in Thm. 4.1, though it is not evident from the algebraic approach of [5]. In particular, when $\chi = 0$, (A, A^*) is a Lie bialgebroid.

Finally, there are interesting global versions of these structures. As we just observed, the Lie algebroid A of a quasi-Poisson structure fits into a Lie quasi-bialgebroid, so its global counterpart is a *quasi-Poisson groupoid*. This shows how to associate quasi-Poisson groupoids to quasi-Poisson spaces and fits well with the theory of [10]. In particular, when $\chi = 0$, the Lie groupoid integrating A is a Poisson groupoid [16]. This Poisson groupoid is built out of the Poisson-Lie group of $(\mathfrak{g}, \mathfrak{g}^*)$ and the symplectic groupoid of T^*M, as well as actions of each one on the other; it is an example of a *matched pair of Lie groupoids* [14]. This indicates a general construction of (quasi)Poisson groupoids as (quasi)matched pairs. It would be interesting to find the precise relationship between these "doubles" and the ones e.g. in [15].

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