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Nonlinearizability of certain Poisson structures near a symplectic leaf

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Abstract

We give an intrinsic proof that Vorobjev's first approximation of a Poisson manifold near a symplectic leaf is a Poisson manifold. We also show that Conn's linearization results cannot be extended in Vorobjev's setting.

1 Introduction

A Poisson structure on a smooth n-dimensional manifold M is a bivector field $\pi \in \Gamma(\Lambda^2 TM)$ such that the Schouten bracket $[\pi, \pi] = 0$. It is known that every Poisson structure π on M gives rise to a foliation by symplectic leaves. The study of the local structure of a Poisson manifold near a zero-dimensional leaf (i.e., a point m for which $\pi_m = 0$) leads to the linearization problem, which we now describe. Recall that the *linear approximation* of a Poisson structure π at a zero-dimensional leaf m is determined by a Lie algebra $\mathfrak{g}$, called the *transverse Lie algebra or isotropy Lie algebra* at m. More precisely, the linear approximation coincides with the canonical linear Poisson structure $\pi^{(1)}$ defined on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. In [20], Weinstein showed that, if π is a Poisson structure on M which vanishes at a point m and whose transverse Lie algebra at m is semi-simple, then π is formally isomorphic to its linear approximation $\pi^{(1)}$. The analytic and the smooth versions of this result were proved by Conn in [4] and [5]. Other partial results on the linearization of Poisson structures can be found in [16] and [7]. A survey of the literature on linearization of Poisson brackets may be found in [10].

Less studied is the local structure of a Poisson manifold near a symplectic leaf with a nonzero dimension. Results on the Poisson topology of neighborhoods of symplectic leaves have been obtained by Ginzburg and Golubev [11], Crainic [6], and Fernandes [8], [9]. The study of linearization near symplectic leaves of nonzero dimension was initiated by Vorobjev (see [19]), who defined the analogue of the linear approximation $(\mathfrak{g}^*, \pi^{(1)})$ for a symplectic leaf of dimension $d > 0$. His goal was to generalize Conn's local linearization result to nonzero dimensional symplectic leaves.

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In the present work, we give complete proofs of some results (see Theorems 2.1 and 3.2 below) which were stated in [19] without proofs. We also give examples of linearizable and nonlinearizable Poisson structures near a symplectic leaf having a nonzero dimension. In particular, we show that Conn's linearization result cannot be extended in a straightforward way to the semi-local context. To our knowledge, Example 3 (see below) is the first counter-example to the semi-local linearization question.

2 Integrable geometric data

Let S be an embedded submanifold of a smooth n-dimensional manifold P. Fix a tubular neighborhood $\mathcal{N}$ of S. This corresponds to a vector bundle $p : \mathcal{N} \to S$. We identify S with the zero section of $\mathcal{N}$. Moreover, we denote $Vert = \ker p_*$. An *Ehresmann connection* on $\mathcal{N}$ is a projection map $\Gamma : T\mathcal{N} \to Vert$. Equivalently, we have a smooth vector subbundle $Hor \subset T\mathcal{N}$ such that

$$T_x\mathcal{N} = Hor_x \oplus Vert_x \quad \forall\, x \in \mathcal{N}.$$

For any vector field $X \in \chi(S)$, there is a unique horizontal vector field $\overline{X} \in \chi(Hor)$ which is called the *horizontal lift* of X, and satisfies

$$p_*(\overline{X}) = X.$$

Define the curvature of Γ by the formula

$$Curv_\Gamma(X,Y) = [\overline{X}, \overline{Y}] - \overline{[X,Y]}, \qquad \text{for any} \;\; X, Y \in \chi(S).$$

Suppose that S is equipped with a symplectic form $\omega \in \Omega^2(S)$. Now, we consider a triple (Γ, ν, φ) called *geometric data* and formed by

- an Ehresmann connection Γ,
- a vertical bivector field $\nu \in \Gamma(\Lambda^2 Vert)$,
- and a nondegenerate 2-form $\varphi \in \Omega^2(S) \otimes \mathcal{C}^\infty(\mathcal{N})$ given by

$$\varphi = \omega \otimes 1 + R,$$

where ω is the symplectic form on the zero section S, and R vanishes on S, i.e.

$$\varphi_x(X,Y) = \omega_x(X,Y), \quad \forall\, x \in S, \quad \forall X, Y \in \chi(S).$$

The projection $p : \mathcal{N} \to S$ induces a map $p^* : \Omega^2(S) \otimes \mathcal{C}^\infty(\mathcal{N}) \to \Omega^2(\mathcal{N})$ by pulling back the first factor of the tensor product. Define the *coupling 2-form* $\widehat{\omega} \in \Omega^2(\mathcal{N})$ by $\widehat{\omega} = p^*\varphi$. In a sufficiently small tubular neighborhood we have that, for any $x \in \mathcal{N}$, the linear map $\widehat{\omega}_{|Hor}(x) : Hor_x \to \mathrm{ann}(Vert_x)$ is an isomorphism, where $\mathrm{ann}(Vert_x)$ is the annihilator of $Vert_x$.. If necessary, we may work with a smaller tubular neighborhood. Define the *horizontal coupling bivector* by

$$\mu = \widehat{\omega}^{-1}_{|Hor}. \tag{2.1}$$

This gives the identity

$$\widehat{\omega}(\mu\alpha, \mu\beta) = -\mu(\alpha, \beta).$$

With respect to the above notations we have the following theorem:

Theorem 2.1. *The bivector field $\pi = \mu + \nu$ is a Poisson bivector field if and only if the following* integrability conditions *are satisfied*

1. *ν is a Poisson bivector field (i.e. $[\nu, \nu] = 0$);*
2. *$[\overline{X},\ \nu] = 0$, for all $X \in \chi(S)$;*
3. *$Curv_\Gamma(X, Y) = \nu(d\widehat{\omega}(\overline{X}, \overline{Y}))$;*
4. *$d\widehat{\omega}(\overline{X}_1, \overline{X}_2, \overline{X}_3) = 0$, for all $X_1, X_2, X_3 \in \chi(S)$.*

The proof relies on the following three lemmas.

Lemma 2.1. *Let M be a smooth manifold and let Λ be a bivector field on M. Then,*

$$-\frac{1}{2}[\Lambda, \Lambda](\alpha, \beta, \gamma) = \Big(\Lambda(d\Lambda(\alpha, \beta), \gamma) + \langle \alpha, [\Lambda\beta, \Lambda\gamma]\rangle\Big) + c.p.,$$

for any $\alpha, \beta, \gamma \in \Omega^1(M)$).

Proof. We have the following formula (see [13]):

$$\Lambda(\mathcal{L}_{\Lambda\alpha}\beta - \mathcal{L}_{\Lambda\beta}\alpha - d(\Lambda(\alpha, \beta))) = [\Lambda\alpha,\ \Lambda\beta] + \frac{1}{2}[\Lambda, \Lambda](\alpha, \beta, \cdot).$$

There follows

$$\frac{1}{2}[\Lambda, \Lambda](\alpha, \beta, \gamma) = -\mathcal{L}_{\Lambda\alpha}\langle\beta,\ \Lambda\gamma\rangle - \langle\alpha,\ [\Lambda\beta,\ \Lambda\gamma]\rangle + \text{c.p.}$$

□

Lemma 2.2. *For any horizontal 1-forms α, β, and γ, we have*

$$-\frac{1}{2}[\mu, \mu](\alpha, \beta, \gamma) = d\widehat{\omega}(\mu\alpha, \mu\beta, \mu\gamma).$$

Proof. Indeed,

$$\begin{aligned} d\widehat{\omega}(\mu\alpha, \mu\beta, \mu\gamma) &= \mathcal{L}_{\mu\alpha}\Big(\widehat{\omega}(\mu\beta, \mu\gamma)\Big) + \widehat{\omega}(\mu\alpha, [\mu\beta, \mu\gamma]) + \text{c.p.} \\ &= \mu(d\mu(\beta, \gamma), \alpha) + \langle\alpha, [\mu\beta, \mu\gamma]\rangle + \text{c.p.} \\ &= -\frac{1}{2}[\mu, \mu](\alpha, \beta, \gamma) \text{ by Lemma 2.1.} \end{aligned}$$

There follows the lemma. □

Lemma 2.3. *For any horizontal 1-form α and for any vertical 1-forms β, γ, we have*

$$-\frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma) = [\overline{X},\nu](\alpha,\beta),$$

where $\overline{X} = \mu\alpha$.

Proof. By Lemma 2.1, we have that

$$\begin{aligned} -\frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma) &= \pi\Big(d(\pi(\alpha,\beta)),\gamma\Big) + \pi\Big(d(\pi(\beta,\gamma)),\alpha\Big) + \pi\Big(d(\pi(\gamma,\alpha)),\beta\Big) \\ &\quad +\langle\alpha,[\pi\beta,\pi\gamma]\rangle + \langle\beta,[\pi\gamma,\pi\alpha]\rangle + \langle\gamma,[\pi\alpha,\pi\beta]\rangle. \\ &= \pi(d\pi(\beta,\gamma),\alpha) + \langle\beta,[\pi\gamma,\pi\alpha]\rangle + \langle\gamma,[\pi\alpha,\pi\beta]\rangle \\ &= -\mathcal{L}_{\overline{X}}(\pi(\beta,\gamma)) + \langle\beta,[\pi\gamma,\pi\alpha]\rangle + \langle\gamma,[\pi\alpha,\pi\beta]\rangle \ (\star) \end{aligned}$$

since $\pi(\alpha,\beta) = 0$ and $\pi(\gamma,\alpha) = 0$, and the Lie bracket of two vertical vector fields is again vertical implying that $\langle\alpha,[\pi\beta,\pi\gamma]\rangle = 0$. On the other hand, by the product rule,

$$\mathcal{L}_{\overline{X}}(\pi(\beta,\gamma)) = (\mathcal{L}_{\overline{X}}\pi)(\beta,\gamma) + \pi(\mathcal{L}_{\overline{X}}\beta,\gamma) + \pi(\beta,\mathcal{L}_{\overline{X}}\gamma)$$

and so

$$(\mathcal{L}_{\overline{X}}\pi)(\beta,\gamma) = \mathcal{L}_{\overline{X}}(\pi(\beta,\gamma)) - \pi(\mathcal{L}_{\overline{X}}\beta,\gamma) - \pi(\beta,\mathcal{L}_{\overline{X}}\gamma). \, (\star\star)$$

Concentrating now on the second term of $(\star\star)$, we set $Z = \pi\gamma$ and compute using the product rule that

$$\begin{aligned} -\pi(\mathcal{L}_{\overline{X}}\beta,\gamma) &= (\mathcal{L}_{\overline{X}}\beta)(Z) \\ &= \overline{X}\cdot\pi(\gamma,\beta) - \beta([\pi\alpha,\pi\gamma]) \\ &= -\mathcal{L}_{\overline{X}}(\pi(\beta,\gamma)) + \beta([\pi\gamma,\pi\alpha]). \end{aligned}$$

Performing a similar computation on the third term of $(\star\star)$, we find that

$$(\mathcal{L}_{\overline{X}}\pi)(\beta,\gamma) = -\mathcal{L}_{\overline{X}}(\pi(\beta,\gamma)) + \langle\beta,[\pi\gamma,\pi\alpha]\rangle + \langle\gamma,[\pi\alpha,\pi\beta]\rangle. \, (\star\star\star)$$

Combining $(\star)$ and $(\star\star\star)$, we see that

$$\begin{aligned} -\tfrac{1}{2}[\pi,\pi](\alpha,\beta,\gamma) &= (\mathcal{L}_{\overline{X}}\pi)(\beta,\gamma), \\ &= (\mathcal{L}_{\overline{X}}(\mu+\nu))(\beta,\gamma,) \\ &= (\mathcal{L}_{\overline{X}}\mu)(\beta,\gamma), \\ &= 0, \end{aligned}$$

since β and γ are vertical. □

Proof of Theorem 2.1: The splitting $T\mathcal{N} = Hor \oplus Vert$ induces the splittings

$$\bigwedge^k T\mathcal{N} = \bigoplus_{i+j=k} Hor^i \wedge Vert^j,$$

where $Hor^i \wedge Vert^j$ is the wedge product of i copies of the vector bundle Hor with j copies of the vector bundle $Vert$. A section of $Hor^i \wedge Vert^j$ is said to be a *multivector field of degree* (i, j).

Thus, the trivector field $[\pi, \pi]$ is a sum of component trivector fields of degrees (3,0), (2,1), (1,2), and (0,3). We will show that conditions 1-4 given in Theorem 2.1 are equivalent to the vanishing of $[\pi, \pi]$ in degree $(n, 3-n)$, where $n = 0, .., 3$. We have

$$[\pi, \pi] = [\mu + \nu, \mu + \nu] = [\mu, \mu] + 2[\mu, \nu] + [\nu, \nu].$$

In general, the horizontal distribution is not necessarily integrable, but $Vert$ is always integrable.

Degree (0,3): Thus, the degree (0,3) component of $[\pi, \pi]$ is exactly $[\nu, \nu]$.

Degree (1,2): Let α be a horizontal 1-form, and let β, γ be two vertical 1-forms. By lemma 2.3, we get

$$[\pi, \pi](\alpha, \beta, \gamma) = -2[\overline{X}, \nu](\alpha, \beta), \tag{2.2}$$

where $\overline{X} = \mu\alpha$.

Degree (2,1): Suppose α, β are horizontal 1-forms and γ is a vertical 1-form. By Lemma 2.1, we have that

$$\begin{aligned} -\frac{1}{2}[\pi, \pi](\alpha, \beta, \gamma) &= \pi\Big(d\pi(\alpha, \beta), \gamma\Big) + \pi\Big(d\pi(\beta, \gamma), \alpha\Big) + \pi\Big(d\pi(\gamma, \alpha), \beta\Big) \\ &\quad + \langle \alpha, [\pi\beta, \pi\gamma] \rangle + \langle \beta, [\pi\gamma, \pi\alpha] \rangle + \langle \gamma, [\pi\alpha, \pi\beta] \rangle. \end{aligned}$$

Several terms vanish automatically. In particular $\pi(\beta, \gamma) = 0$ and $\pi(\gamma, \alpha) = 0$ since π vanishes in degree (1,1). Without loss of generality, we can suppose that $\overline{X} = \mu\alpha$, $\overline{Y} = \mu\beta$ are horizontal lifts of vector fields $X, Y \in \chi(S)$. Setting $Z := \pi\gamma$, we get

$$\langle \alpha, [\pi\beta, \pi\gamma] \rangle = \langle \alpha, \mathcal{L}_{\overline{Y}} Z \rangle = 0$$

and

$$\langle \beta, [\pi\gamma, \pi\alpha] \rangle = \langle \beta, -\mathcal{L}_{\overline{X}} Z \rangle = 0$$

since the Lie derivative of a vertical vector field by the horizontal lift of a vector field on S is again vertical. Thus,

$$\begin{aligned} -\frac{1}{2}[\pi, \pi](\alpha, \beta, \gamma) &= \pi(d(\pi(\alpha, \beta)), \gamma) + \langle \gamma, [\pi\alpha, \pi\beta] \rangle \\ &= \nu(d(\mu(\alpha, \beta)), \gamma)) + \langle \gamma, [\overline{X}, \overline{Y}] \rangle \\ &= -\nu(d(\widehat{\omega}(\overline{X}, \overline{Y})), \gamma) + \langle \gamma, [\overline{X}, \overline{Y}] \rangle \end{aligned}$$

But

$$[\overline{X}, \overline{Y}] = \overline{[X, Y]} + Curv_\Gamma(X, Y) \quad \text{and} \quad \langle \gamma, \overline{[X, Y]} \rangle = 0.$$

It follows

$$-\frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma) = -\nu(d(\widehat{\omega}(\overline{X},\overline{Y})),\gamma) + \langle\gamma, Curv_\Gamma(X,Y)\rangle. \tag{2.3}$$

Degree (3,0): We have seen earlier that the degree (3,0) components of $[\pi,\pi]$ and $[\mu,\mu]$ are equal, thus

$$[\pi,\pi](\alpha,\beta,\gamma) = [\mu,\mu](\alpha,\beta,\gamma).$$

By Lemma 2.2,

$$d\widehat{\omega}(\mu\alpha,\mu\beta,\mu\gamma) = -\frac{1}{2}[\mu,\mu](\alpha,\beta,\gamma). \tag{2.4}$$

This completes the proof of Theorem 2.1.

□

Definition. A geometric data (Γ,ν,φ) satisfying conditions 1-4 of Theorem 2.1 is said to be *integrable*. In this case, the corresponding bivector field $\pi=\mu+\nu$ is called the *coupling Poisson bivector field associated with* (Γ,ν,φ).

Remark 1. Let X_1, X_2, X_3 be vector fields on the base manifold S, and let $\overline{X}_1,\overline{X}_2,\overline{X}_3$ be their horizontal lifts respectively. Using the above notations, we have

$$d\widehat{\omega}(\overline{X}_1,\overline{X}_2,\overline{X}_3) = \mathcal{L}_{\overline{X}_1}\Big(\widehat{\omega}(\overline{X}_2,\overline{X}_3)\Big) - \widehat{\omega}([\overline{X}_1,\overline{X}_2],\overline{X}_3) + c.p.$$

The curvature $Curv_\Gamma$ takes values in the vertical bundle $Vert$. It follows that

$$\widehat{\omega}\Big([\overline{X}_1,\overline{X}_2],\overline{X}_3\Big) = \widehat{\omega}\Big(\overline{[X_1,X_2]},\overline{X}_3\Big).$$

Consequently,

$$\begin{aligned} d\widehat{\omega}(\overline{X}_1,\overline{X}_2,\overline{X}_3) &= \mathcal{L}_{\overline{X}_1}\Big(\widehat{\omega}(\overline{X}_2,\overline{X}_3)\Big) - \widehat{\omega}\Big(\overline{[X_1,X_2]},\overline{X}_3\Big) + c.p. \\ &= \mathcal{L}_{\overline{X}_1}\Big(\varphi(X_2,X_3)\Big) - \varphi\Big([X_1,X_2],X_3\Big) + c.p. \\ &= (\partial_\Gamma\varphi)(X_1,X_2,X_3), \end{aligned}$$

where

$$\partial_\Gamma : \Omega^k(S)\otimes\mathcal{C}^\infty(\mathcal{N}) \to \Omega^{k+1}(S)\otimes\mathcal{C}^\infty(\mathcal{N})$$

is the operator defined by the formula

$$\begin{aligned} \partial_\Gamma\mathbb{F}(X_0,\ldots,X_k) &= \sum_{i=0}^{n}(-1)^i\mathcal{L}_{\overline{X}_i}\Big(\mathbb{F}(X_0,\ldots,\hat{X}_i,\ldots,X_k)\Big) \\ &\quad + \sum_{i<j}(-1)^{i+j}\mathbb{F}([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k). \end{aligned}$$

Hence integrability condition 4 in Theorem 2.1 can be replaced by $\partial_\Gamma \varphi = 0$.

Remark 2. We have shown that, given a manifold P, any integrable geometric data (Γ, ν, φ) with respect to a tubular neighborhood $\mathcal{N}$ of a symplectic submanifold (S, ω) of P induces a Poisson bivector field π on $\mathcal{N}$, which admits S as a symplectic leaf. The converse is also true. Indeed, suppose π is a Poisson structure on the total space $\mathcal{N}$ of a vector bundle over a symplectic manifold (S, ω), where S is identified with the zero section of $\mathcal{N}$, and S is a symplectic leaf of $\mathcal{N}$. Let $Vert = \ker p_*$, where p is the natural projection of $\mathcal{N}$ onto S. In addition, we assume that π is *horizontally nondegenerate*, that is $\pi_{|\mathrm{ann}(Vert)}$ is nondegenerate. Then, there is a natural Ehresmann connection associated with π which is determined by the horizontal subbundle

$$Hor = \pi(\mathrm{ann}(Vert)).$$

The bivector field π can be decomposed into $\pi = \mu + \nu$, where μ and ν are horizontal and vertical, respectively. Furthermore, we can define

$$\varphi(X, Y) = \pi^{-1}_{|Hor}(\overline{X}, \overline{Y}),$$

where $\overline{X}, \overline{Y}$ are the lifts of the vector fields $X, Y \in \chi(S)$. Using Equations (1)–(4), we conclude that integrability conditions 1–4 are satisfied.

Remark 3. Brahic [1] and Vaisman [18] have each given independent proofs of Theorem 2.1. The proof given here is intrinsic, and has the advantage of clearly relating Vorobjev's linearization formula to geometric identities for transitive Lie algebroids (see below).

3 Transitive Lie algebroids

A *Lie algebroid* over a manifold S is a real vector bundle $p : E \to S$ together with a bundle map $\rho : E \to TS$ and a real Lie algebra structure $[\cdot, \cdot]_E$ on $\Gamma(E)$ such that the following *Leibniz rule* holds

$$[v, fw]_E = f[v, w]_E + (\rho(v) \cdot f)w,$$

for any $f \in \mathcal{C}^\infty(S)$ and $v, w \in \Gamma(E)$. The map ρ is called the *anchor* of the Lie algebroid. Using the Jacobi identity for $[\cdot, \cdot]_E$ and the Leibniz identity, one can show that the induced map $\rho : \Gamma(E) \to \chi(S)$ is a Lie algebra homomorphism, i.e.

$$\rho[u, v]_E = [\rho(u), \rho(v)].$$

Given any Lie algebroid, the distribution spanned by the image of the anchor is integrable, and the leaves of the resulting foliation are called the *orbits* of the Lie algebroid. When the anchor is surjective, we call the Lie algebroid *transitive*.

If $E \xrightarrow{\rho} S$ is a transitive Lie algebroid with anchor map ρ, then we obtain a short exact sequence of vector bundles

$$0 \to I \xrightarrow{v} E \xrightarrow{\rho} TS \to 0,$$

where $I = \ker \rho$ is called the *isotropy bundle*.

Recall that a *connection* for a transitive Lie algebroid $E \xrightarrow{\rho} S$ is a splitting $\sigma : TS \to E$ of the sequence above. Its *curvature* $R_\sigma \in \Omega^2(S) \otimes \Gamma(I)$ is given by

$$R_\sigma(X,Y) = [\sigma X, \sigma Y] - \sigma[X,Y], \quad \text{for all } X, Y \in \chi(S).$$

Any connection σ for a transitive Lie algebroid induces a covariant derivative for the isotropy bundle I defined by

$$\nabla^\sigma_X s = [\sigma X, s], \quad \text{for all } X \in \chi(S) \text{ and } s \in \Gamma(I).$$

By a *covariant derivative* for a vector bundle $\mathcal{N} \to S$, we mean a linear map $\nabla : \Gamma(\mathcal{N}) \to \Omega^1(S) \otimes \Gamma(\mathcal{N})$ satisfying the following properties:

$$\nabla_X fs = X(f) \cdot s + f\nabla_X s \quad \text{and} \quad \nabla_{fX} s = f\nabla_X s$$

for all $X \in \chi(S)$, $f \in \mathcal{C}^\infty(S)$, and $s \in \Gamma(\mathcal{N})$. Let σ be a connection for a transitive Lie algebroid E. For simplicity, the corresponding covariant derivative will be denoted by ∇ instead of ∇^σ when there is no ambiguity. Define the *curvature of* ∇ by

$$R_\nabla(X_1, X_2) = [\nabla_{X_1},\ \nabla_{X_2}] - \nabla_{[X_1,X_2]},$$

for all $X_1, X_2 \in \chi(S)$. The Jacobi identity yields geometric identities when evaluated on isotropic and coisotropic sections.

Theorem 3.1. *Let $\sigma : TS \to E$ be a connection for a transitive Lie algebroid and let ∇ be the induced covariant derivative for the isotropy bundle I. Then, for all $X, X_1, X_2, X_3 \in \chi(S)$ and for all $s, s_1, s_2 \in \Gamma(I)$, we have*

- **(i)** *The isotropy bundle I is a Lie algebroid.*
- **(ii)** $\nabla_X[s_1, s_2] = [\nabla_X s_1, s_2] + [s_1, \nabla_X s_2]$,
- **(iii)** $[R_\sigma(X_1, X_2), s] - R_\nabla(X_1, X_2)(s) = 0$,
- **(iv)** $\nabla_{X_1} R_\sigma(X_2, X_3) + R_\sigma(X_1, [X_2, X_3]) + c.p. = 0$.

Proof. Simply use the definitions of the connection, the covariant derivative and their respective curvatures, and the fact that

$$\mathcal{J}(\sigma X, s_1, s_2) = 0, \quad \mathcal{J}(\sigma X_1, \sigma X_2, s) = 0, \quad \textit{and} \quad \mathcal{J}(\sigma X_1, \sigma X_2, \sigma X_3) = 0,$$

where $\mathcal{J}$ is the jacobiator of the bracket on sections of E. $\square$

Versions of this theorem have been noted by several authors (see [14],[8],[19]).

Any connection σ for a transitive Lie algebroid $E \to S$ gives rise to a *homogeneous* Ehresmann connection Γ on the dual I^* of the isotropy bundle, i.e. the horizontal lift of every vector field $X \in \chi(S)$ preserves the space $\mathcal{C}^\infty_{\mathrm{lin}}(I^*)$ of fiberwise linear functions on I^*. To define the horizontal lift of X, we introduce the natural isomorphism

$$\ell : \Gamma(I) \to \mathcal{C}^\infty_{\mathrm{lin}}(I^*) \tag{3.1}$$

given by the natural pairing. The horizontal lift $\overline{X}$ of a vector $X \in \chi(S)$ satisfies

$$\mathcal{L}_{\overline{X}}(\ell(s)) = \ell(\nabla_X s).$$

Let $Hor \subset TI^*$ be the corresponding horizontal subbundle. Then

$$TI^* = Hor \oplus Vert,$$

where $Vert = \ker p_*$ with $p_* : TI^* \to TS$. Define the *Lie-Poisson bivector* ν by the following formula:

$$\nu(du_1, du_2) = \ell([s_1, s_2]), \quad \text{where } \ell(s_i) = u_i. \tag{3.2}$$

The Lie-Poisson bivector defines a Poisson structure on TI^* (see Section 16.5 of [3]). Now, we suppose that S is equipped with a symplectic form ω. Let

$$\varphi = \omega \otimes 1 + \ell \circ R_\sigma. \tag{3.3}$$

Theorem 3.2. *Given a transitive Lie algebroid E over a symplectic manifold S together with a connection $\sigma : TS \to E$, the geometric data (Γ, ν, φ) defined as above is integrable.*

Proof. We will show that geometric identities **(i)**–**(iv)** in Theorem 3.1 are equivalent to integrability conditions 1–4 in Theorem 2.1.
1. Immediate by Equation 3.2.
2. By Theorem 3.1**(ii)**, we have

$$\nabla_X[s_1, s_2] - [\nabla_X s_1, s_2] - [s_1, \nabla_X s_2] = 0.$$

But

$$\ell(\nabla_X[s_1, s_2]) = \langle \overline{X}, d\ell([s_1, s_2])\rangle = \mathcal{L}_{\overline{X}}(\nu(du_1, du_2)),$$

where $\ell(s_i) = u_i$. Similarly

$$\ell([\nabla_X s_1, s_2] + [s_1, \nabla_X s_2]) = \nu(d(\mathcal{L}_{\overline{X}} u_1), du_2) + \nu(du_1, d(\mathcal{L}_{\overline{X}} u_2)).$$

It follows that

$$\ell(\nabla_X[s_1,s_2]-[\nabla_X s_1,s_2]-[s_1,\nabla_X s_2])=[\overline{X},\nu](du_1,du_2).$$

Hence,

$$\nabla_X[s_1,s_2]-[\nabla_X s_1,s_2]+[s_1,\nabla_X s_2]=0 \iff [\overline{X},\nu]=0.$$

3. By Theorem 3.1**(iii)**, we have

$$\begin{array}{rcl} 0 &=& \ell\Big([R_\sigma(X_1,X_2),s]-R_\nabla(X_1,X_2)(s)\Big)\\ &=& \nu\Big(d(\ell(R_\sigma(X_1,X_2)),\ d(\ell(s))\Big)-\langle[\overline{X}_1,\overline{X}_2]-\overline{[X_1,X_2]},d\ell(s)\rangle. \end{array}$$

Since $\varphi(X_1,X_2)-\ell(R_\sigma(X_1,X_2))$ is constant on each fiber, we have

$$\nu\Big(d(\ell(R_\sigma(X_1,X_2)),\ d(\ell(s))\Big)=\nu\Big(d(\varphi(X_1,X_2)),\ d(\ell(s))\Big).$$

Therefore,

$$\begin{array}{rcl} 0 &=& \ell\Big([R_\sigma(X_1,X_2),s]-R_\nabla(X_1,X_2)(s)\Big)\\ &=& \nu\Big(d(\varphi(X_1,X_2)),\ d(\ell(s))\Big)-\langle Curv_\Gamma(X_1,X_2),\ d\ell(s)\rangle\\ &=& \nu\Big(d(\widehat{\omega}(\overline{X}_1,\overline{X}_2)),\ d(\ell(s))\Big)-\langle Curv_\Gamma(X_1,X_2),\ d\ell(s)\rangle. \end{array}$$

We obtain that

$$ad_{R_\sigma(X_1,X_2)}-R_\nabla(X_1,X_2)=0 \iff Curv_\Gamma(X_1,X_2)=\nu^\sharp d(\widehat{\omega}(\overline{X}_1,\overline{X}_2)).$$

4. One has

$$\begin{array}{rcl} \ell\Big(\nabla_{X_1}R_\sigma(X_2,X_3)+R_\sigma(X_1,[X_2,X_3])\Big) &=& \mathcal{L}_{\overline{X_1}}\Big(\ell\circ R_\sigma(X_2,X_3)\Big)\\ && +\ell\circ R_\sigma(X_1,[X_2,X_3]). \end{array}$$

Therefore,

$$\begin{array}{rcl} \ell\Big(\nabla_{X_1}R_\sigma(X_2,X_3)+R_\sigma(X_1,[X_2,X_3])\Big)+c.p. &=& \mathcal{L}_{\overline{X_1}}\Big(\varphi(X_2,X_3)\Big)\\ && +\varphi(X_1,[X_2,X_3])+c.p\\ &=& \partial_\Gamma\varphi(X_1,X_2,X_3). \end{array}$$

Applying Theorem 3.1**(iv)**, one gets $\partial_\Gamma\varphi=0$. This completes the proof. $\square$

The resulting coupling Poisson structure on I^* depends on the choice of the connection, but is unique up to isomorphism by the following proposition of Vorobjev proved in [19].

Proposition 3.1. *Let E_1 and E_2 be two isomorphic transitive Lie algebroids over the same symplectic base manifold (S, ω), and let $\sigma_1 : TS \to E_1$, $\sigma_1 : TS \to E_2$ be two connections. There exists a diffeomorphism ψ from a neighborhood $\mathcal{V}_1$ of the zero section $S \subset I_1^*$ onto a neighborhood $\mathcal{V}_2$ of the zero section $S \subset I_2^*$ such that $\psi_{|S} = id$ and $\psi_* \pi_1 = \pi_2$, where I_i is the isotropy bundle of E_i and π_i is the coupling Poisson bivector field associated with σ_i, for $i = 1, 2$.*

4 Vorobjev (non)linearizability

In this section, we are interested in the particular case where the isotropy bundle is the conormal bundle N^*S to a symplectic leaf S of a Poisson manifold. Given a Poisson manifold (P, Λ), the Poisson tensor induces a natural map from the cotangent bundle to the tangent bundle by the formula

$$\begin{array}{rcl} \Lambda^\sharp : T^*P & \to & TP \\ \alpha & \mapsto & \Lambda(\alpha, \cdot). \end{array}$$

It is known that $T^*P \to P$ is a Lie algebroid, called the *Poisson algebroid* (see [3],[17]) whose anchor map is $\Lambda^\sharp$ and whose Lie bracket is given by

$$[\alpha, \beta]_{T^*P} = \mathcal{L}_{\Lambda^\sharp \alpha} \beta - \mathcal{L}_{\Lambda^\sharp \beta} \alpha - d\Lambda(\alpha, \beta).$$

The restriction to a symplectic leaf (S, ω) give a transitive Lie algebroid $T^*P|_S \to S$. Furthermore, the kernel of the anchor map ρ of this transitive Lie algebroid coincides with the conormal bundle N^*S of S. Let E be a tubular neighborhood of S, and let $p : E \to S$ be the corresponding vector bundle. The derivative of p gives a connection for the transitive Lie algebroid $T^*P|_S \to S$, namely, $\sigma : TS \to T^*P|_S$ defined by

$$\sigma(X) = p^* \omega(X),$$

where $\omega \in \Omega^2(S)$ is symplectic form of S. This connection is called the *pull-back connection* induced by the tubular neighborhood. We have the splitting

$$TP|_S = TS \oplus NS.$$

Moreover, we know that the normal bundle NS is endowed with a canonical Poisson structure ν. Theorem 3.2 says that, up to a shrinking of the tubular neighborhood E of S, there is a coupling Poisson bivector field π on E having S as a symplectic leaf.

Definition.

• The Vorobjev-Poisson structure $\pi = \mu + \nu$ is called the *first approximation* of Λ at the symplectic leaf (S, ω) with respect to the neighborhood E. This naturally extends the classical linear approximation of a Poisson structure at a zero-dimensional leaf $S = \{s_0\}$ to higher dimensions.

• We say that Λ is *Vorobjev linearizable at the symplectic leaf* S if there is a tubular neighborhood $E \subset P$ of S with fibers tangent to the normal bundle NS and a diffeomorphism $\psi : E \to U \subset P$ such $\psi|_S = id$ and $\psi_* \pi = \Lambda$.

We now introduce a family of Poisson manifolds called Casimir-weighted products and use them to give some concrete examples of Vorobjev (non)linearizability. Let (P_1, π_1) and (P_2, π_2) be Poisson manifolds with Casimir functions f_1 and f_2, respectively. The *Casimir-weighted product* is the Poisson manifold $(P_1 \times P_2, f_2\pi_1 + f_1\pi_2)$. That $f_2\pi_1 + f_1\pi_2$ is Poisson is an easy computation using Schouten brackets. The next proposition describes the symplectic leaves of a Casimir-weighted product.

Proposition 4.1. *Suppose that (P_1, π_1) and (P_2, π_2) are Poisson manifolds with smooth nowhere vanishing Casimir functions f_1 and f_2, respectively. Then every symplectic leaf of the Casimir-weighted product is of the form $(S_1 \times S_2, \frac{1}{f_2}\omega_1 + \frac{1}{f_1}\omega_2)$, where (S_1, ω_1) and (S_2, ω_2) are symplectic leaves of (P_1, π_1) and (P_2, π_2), respectively.*

We now compute Vorobjev linearizations of Casimir-weighted products of symplectic Poisson manifolds by Lie-Poisson manifolds.

Theorem 4.1. *Let (S, π_S) be a symplectic Poisson manifold, and let f be a Casimir for a Lie-Poisson manifold $(\mathfrak{g}^*, \pi_{\mathfrak{g}^*})$ such that $f(0) = 1$. Then the Vorobjev linearization of the Casimir-weighted product at the leaf $S \times \{0\}$ is*

$$\left(S \times \mathfrak{g}^*, \frac{1}{J_0^1 f} \pi_S + \pi_{\mathfrak{g}^*} \right),$$

where $J_0^1 f = 1 + d_0 f \in \mathcal{C}^\infty(S \times \mathfrak{g}^*)$ denotes the first jet of f at 0.

Proof. Projection to the first factor makes the Casimir-weighted product into a vector bundle $pr_1 : S \times \mathfrak{g}^* \to S$ which is canonically isomorphic to the normal bundle of the zero-section, i.e. $S \times \mathfrak{g}^* \tilde{\to} NS$ and so we have a canonical tubular neighborhood of the symplectic leaf $S \times \{0\}$.

We now find the coupling Poisson bivector $\pi = \mu + \nu$ induced by this tubular neighborhood. By Equation (3.2), the Lie-Poisson bivector is the Poisson bivector of the direct product of $(S, 0)$ and the Lie-Poisson manifold $(\mathfrak{g}^*, \pi_{\mathfrak{g}^*})$. Essentially this means that $\nu = \pi_{\mathfrak{g}^*}$.

By Equation (2.1), the computation of the horizontal coupling form μ is a two step processes: First we must compute the horizontal distribution Hor of the connection, and then we must compute the coupling 2-form $\widehat{\omega}$.

We now show that the covariant derivative ∇ on the normal bundle NS is flat and that the horizontal distribution Hor has leaves $S \times \{\xi\}$ for each $\xi \in \mathfrak{g}^*$.

Let $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ be a Darboux chart on S. Let $y_1, \ldots, y_k$ for $\mathfrak{g}$ be any basis. Each y_j is a linear function on $\mathfrak{g}^*$ representing a section dy_j of N^*S by pulling-back by the composition of maps $S \times \mathfrak{g}^* \to \mathfrak{g}^* \tilde{\to} T_0\mathfrak{g}^*$. Then

$$(pr_1^* q_1, \ldots, pr_1^* q_n, pr_1^* p_1, \ldots, pr_1^* p_n, y_1, \ldots, y_m)$$

are coordinates on $S \times \mathfrak{g}^*$. We see that each dy_j is a ∇-parallel section by the computation

$$\begin{aligned} \nabla_{X_{q_i}} dy_j &= [\sigma(X_{q_i}), dy_j] \\ &= [pr_1^* dq_i, dy_j] \\ &= d\{pr_1^* q_i, y_j\} \\ &= 0, \end{aligned}$$

since q_i and y_j are coordinates on the first and second factor of $S \times \mathfrak{g}^*$, respectively, and the bracket $\{\, , \}$ is computed using the bivector $f\pi_S + \pi_{\mathfrak{g}^*}$. Similarly, $\nabla_{X_{p_i}} dy_j = 0$. Thus, $\{dy_1, \ldots, dy_k\}$ is a ∇-parallel frame field for N^*S. The dual frame $\{\partial y_1, \ldots, \partial y_k\}$ is necessarily parallel for the dual covariant derivative on the dual bundle NS. Consequently, the horizontal vector bundle Hor is spanned by $\{\partial q_1, \ldots, \partial q_n, \partial p_1, \ldots, \partial p_n\}$.

Equation (3.3) for the horizontal coupling form is $\varphi = \omega \otimes 1 + \ell \circ R_\sigma$. Thus, we must compute the curvature of the pull-back connection, R_σ. In particular,

$$\begin{aligned} \ell \circ R_\sigma(X_{q_i}, X_{p_j}) &= \ell\Big([\sigma(X_{q_i}), \sigma(X_{p_j})] - \sigma([X_{q_i}, X_{p_j}])\Big) \\ &= \ell([pr_1^* dq_i,\ pr_1^* dp_j]) \\ &= \ell(d\{pr_1^* q_i, pr_1^* p_j\}) \\ &= \delta_{ij}\ell(df) \\ &= \delta_{ij}\Big(\tfrac{\partial f}{\partial y_k}(0)\ell(dy_k)\Big), \end{aligned}$$

showing that $\ell \circ R_\sigma = \omega \otimes (J_0^1 f - 1)$, where $J_0^1 f$ is the 1-jet of f at 0. Consequently, the coupling 2-form is $\varphi = \omega \otimes 1 + \ell \circ R_\sigma = \omega \otimes J_0^1 f$, and so the coupling bivector is $\mu = \frac{1}{J_0^1 f}\pi_S$. □

Example 4.1. Let $(T, \pi_T = \partial_u \wedge \partial_v)$ be the *unit symplectic torus* obtained by identifying opposite sides of the unit square. Let $(\mathfrak{g}^*, 0) = (\mathbb{R}, 0)$ be the 1-dimensional Lie-Poisson manifold, and let $f(z) = e^z$ be a Casimir. The Casimir-weighted product

$$(T \times \mathbb{R}, e^z \partial_u \wedge \partial_v)\,,$$

has Vorbjev linearization given by

$$\left(T \times \mathbb{R}, \left(\frac{1}{1+z}\right)\partial_u \wedge \partial_v\right),$$

A linearizing isomorphism is given by $\psi(x, y, z) = (x, y, -1 + e^{-z})$.

Example 4.2. Let $(T, \pi_T = \partial_u \wedge \partial_v)$ be the unit symplectic torus. Let $(\mathfrak{g}^*, \pi_{\mathfrak{g}}) = (\mathbb{R}, 0)$ be the 1-dimensional Lie-Poisson manifold, and let $f(z) = 1 + z^2$ be a Casimir. The Casimir-weighted product

$$\left(T \times \mathbb{R}, (1+z^2)\partial_u \wedge \partial_v\right),$$

has Vorobjev linearization given by

$$(T \times \mathbb{R}, \partial_u \wedge \partial_v).$$

There is no Poisson isomorphism from the Casimir-weighted product to the Vorobjev linearization. To see this, note that any such isomorphism must induce isomorphisms of symplectic leaves. Recall that an invariant of a compact 2n-dimensional symplectic manifold is the *symplectic volume*,

$$\mathrm{Vol}(S, \omega) = \int_S \omega^n.$$

The Casimir-weighted product has leaves of non-constant symplectic volume $1+z^2$, but the Vorobjev linearization has leaves of constant symplectic volume 1.

The following example shows that a Poisson manifold is not necessarily Vorobjev linearizable at a symplectic leaf possessing a semisimple transverse Lie algebra of compact type. In other words, Conn's theorem [5] cannot be extended in this context.

Example 4.3. Let $(T, \pi_T = \partial_u \wedge \partial_v)$ be the unit symplectic torus. Consider the Lie-Poisson manifold $(\mathfrak{g}^*, \pi_{\mathfrak{g}^*}) = (\mathfrak{so}(3)^*, x\partial_y \wedge \partial_z + y\partial_z \wedge \partial_x + z\partial_x \wedge \partial_y)$. Recall that the symplectic leaves of $\mathfrak{so}(3)^*$ are the origin together with the spheres centered at the origin, and so $f = 1 + x^2 + y^2 + z^2$ is a Casimir. Moreover, the sphere at radius r has symplectic form $\omega = \frac{1}{r} dA$, where $dA = dx \wedge dy + dy \wedge dz + dz \wedge dx$ is the standard area form on $\mathbb{R}^3$ (see[15], pp.457-458). By Proposition 4.1, $S = T \times \{0\}$ is a symplectic leaf of the Casimir-weighted product

$$(P, \pi) = (T \times \mathfrak{so}(3)^*, f\pi_T + \pi_{\mathfrak{g}^*}).$$

We will show that (P, π) is not Vorobjev linearizable at S. By Theorem 4.1, the Vorobjev linearization of (P, π) at S is the direct product

$$(T \times \mathfrak{so}(3)^*, \pi_T + \pi_{\mathfrak{g}^*}).$$

Suppose for a contradiction that

$$\psi : (T \times \mathfrak{so}(3)^*, f\pi_T + \pi_{\mathfrak{g}}) \tilde{\rightarrow} (T \times \mathfrak{so}(3)^*, \pi_T + \pi_{\mathfrak{g}^*})$$

is an isomorphism of Poisson manifolds. The map ψ induces isomorphisms of symplectic leaves. By Proposition 4.1,

$$(S_1, \omega_1) = (T \times S^2_{r_1}, \frac{1}{1+r_1^2}\omega_T + \frac{1}{r_1} dA)$$

is a 4-dimensional symplectic leaf of (P, π) when $r_1 > 0$. Again by Proposition 4.1, the only 4-dimensional leafs of the direct product are of the form

$$(S_2, \omega_2) = (T \times S^2_{r_2}, \omega_T + \frac{1}{r_2} dA)$$

when $r_2 > 0$. Since ψ restricts to a symplectomorphism $(S_1, \omega_1) \tilde{\to} (S_2, \omega_2)$, there must be leaves of equal symplectic volume. The symplectic volume of a product is the product of the symplectic volumes, thus,

$$\mathrm{Vol}(S_1, \omega_1) = \mathrm{Vol}\left(T, \frac{\omega_T}{1 + r_1^2}\right) \mathrm{Vol}\left(S^2_{r_1}, \frac{dA}{r_1}\right) = \left(\frac{1}{1 + r_1^2}\right)\left(\frac{4\pi r_1^2}{r_1}\right) = \frac{4\pi r_1}{1 + r_1^2},$$

$$\mathrm{Vol}(S_2, \omega_2) = \mathrm{Vol}\left(T, \omega_T\right) \mathrm{Vol}\left(S^2_{r_2}, \frac{dA}{r_2}\right) = (1)\left(\frac{4\pi r_2^2}{r_2}\right) = 4\pi r_2,$$

and so

$$r_2 = \frac{r_1}{1 + r_1^2}. \tag{4.1}$$

On the other hand, the second homotopy group $\pi_2(T \times S^2) = \mathbb{Z}$ with generator $\phi : S^2 \to \{t\} \times S^2$, where $t \in T$ is any point. The diffeomorphism ψ of $T \times S^2$ induces an isomorphism ψ_* of homotopy groups, and so $\psi_* \phi = \pm\phi$. By assumption, ψ is a symplectomorphism $\psi^* \omega_2 = \omega_1$, thus

$$4\pi r_1 = \int_\phi \omega_1 = \int_\phi \psi^* \omega_2 = \int_{\psi_* \phi} \omega_2 = \int_{\pm\phi} \omega_2 = \pm 4\pi r_2.$$

But this contradicts Equation (4.1).

Remark 4. Let Λ be a Poisson structure horizontally nondegenerate defined on the total space E of a vector bundle $p : E \to S$ over a compact base manifold S. We identify S with the zero section and suppose that S is a symplectic leaf of (E, Λ). We denote by (Γ, ν, φ) the geometric data associated with the first approximation π of Λ. In [2], the author shows that the germ of Λ along S is isomorphic to a Poisson bivector field Λ' admitting (Γ, ν, φ') as associated geometric data. The above example shows that $\varphi' = \varphi$ cannot occur in that case.

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