Toric degenerations of weight varieties and applications

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Abstract

We show that a weight variety, which is a quotient of a flag variety by the maximal torus, admits a flat degeneration to a toric variety. In particular, we show that the moduli spaces of spatial polygons degenerate to polarized toric varieties with the moment polytopes defined by the lengths of their diagonals. We extend these results to more general Flaschka-Millson hamiltonians on the quotients of products of projective spaces. We also study boundary toric divisors and certain real loci.

1   Introduction

Let G be a complex connected semisimple Lie group and let X be a flag variety of G, parameterizing parabolic subgroups of a given type. Several authors studied flat degenerations of X to a toric variety, starting with the work of Gonciulea and Lakshmibai [9]. Caldero [3] used Kashiwara-Lusztig's canonical bases and their string parameterization in his construction, which was later extended by Alexeev-Brion [1] to the case of spherical varieties. In this paper we use their methods to construct flat degenerations of weight varieties, which are, by definition [21], quotients of the flag varieties by the action of the maximal torus. The resulting polytopes are certain "slices" of the string polytopes.

Interesting examples of weight varieties are the quotients of complex grassmannians. For example, the quotients of $\mathrm{Gr}_{\mathbb{C}}(2, n)$ can be identified [11] with the moduli spaces of spatial n-gons studied by Klyachko [20], Kapovich-Millson [16], and many other authors. There are remarkable integrable systems on these spaces, where the action variables are given by the lengths of diagonals emanating from a fixed vertex and the angle variables define the so-called bending flows, which have a transparent geometric meaning [16]. As an application, we show that there exist flat degenerations of the moduli spaces of polygons to polarized toric varieties, *toric polygon spaces*, whose moment polytopes are defined by the action variables. These action variables can be computed using the Gelfand-Tsetlin functions and the Gelfand-MacPherson correspondence [11]. Our results can also be generalized for other grassmannians and the bending flows defined by the Flaschka-Millson hamiltonians [5].

We show that there are some real cycles of codimension 2 on the moduli spaces of polygons which degenerate to the toric subvarieties corresponding to the facets of the polytopes. We also extend the work of Kamiyama-Yoshida [15] and construct topological spaces with compact torus action, which conjecturally are homeomorphic to the central fibers of the flat families.

In the last section of the paper we consider real loci of the aforementioned spaces and show that they map surjectively to the moment polytopes and compute the cardinality of the fibers of these maps.

Let $\overline{M}_{0,n}$ be the moduli space of stable n-pointed rational curves. It can be realized [17] as the Chow quotient of the Grassmannian $\mathrm{Gr}_{\mathbb{C}}(2,n)$. For an alternative construction using stable polygons, see [14]. Then $\overline{M}_{0,n}$ can be flatly degenerated to a toric moduli space whose associated fan is the common refinement of the fans of all the toric polygon spaces. This and related topics will appear in a forthcoming paper.

2   Toric degenerations of weight varieties.

In this section we use toric degenerations of flag varieties constructed by Caldero [3] to construct toric degenerations of weight varieties [21], which are defined as GIT quotients of flag varieties by the action of the maximal torus.

Let G be a connected complex semisimple group, B a Borel subgroup, U its unipotent radical, and H a Cartan subgroup such that $B = HU$. Let also $\Phi = \Phi(G,H)$ be the system of roots, $\Phi^+ = \Phi^+(B,H)$ the subset of positive roots, and $\{\alpha_1,...,\alpha_r\}$ the basis of simple roots, where r is the rank of G. Let Λ be the weight lattice of G and Λ^+ the subset of dominant weights. For $\lambda \in \Lambda^+$ we denote by $V(\lambda)$ the irreducible G-module with highest weight λ. Let $P_\lambda \supset B$ be the parabolic subgroup of G which stabilizes a highest weight vector in $V(\lambda)$. Also denote by $L_\lambda = G \times_{P_\lambda} \mathbb{C}$ the G-linearized line bundle on $X_\lambda := G/P_\lambda$ corresponding to the character λ extended to P_λ.

Let W be the Weyl group and $w_0 \in W$ the longest element of length ℓ. Choose a reduced decomposition

$$\underline{w_0} = s_{i_1} s_{i_2} \cdots s_{i_\ell}$$

into a product of simple reflections. Recall that the space $A := \mathbb{C}[G]^U$ of regular, right U-invariant functions on G, has a so-called dual canonical basis $(b_{\lambda,\phi})$, where each $b_{\lambda,\phi}$ is an eigenvector for both left and right H-action. For the right H-action it has weight λ and for the left H-action the weight is given by the formula 2.1 below. Also recall the definition of the cone $\mathcal{C}_{\underline{w_0}} \subset \Lambda_{\mathbb{R}} \times \mathbb{R}^\ell$, a rational convex polyhedral cone, given e.g. in [2]. If we look at those points in $\mathcal{C}_{\underline{w_0}}$ that have a fixed $\lambda \in \Lambda^+$, then we will obtain the *string polytope* $Q(\lambda)$, whose integral points correspond to the elements of the (dual) canonical basis (for the chosen string parameterization) $(b_{\lambda,\phi})$ for that particular $\lambda \in \Lambda^+$. For a fixed λ, we may

identify $Q(\lambda)$ with its image in $\mathbb{R}^\ell$ via the projection $\Lambda_{\mathbb{R}} \times \mathbb{R}^\ell \to \mathbb{R}^\ell$. If we denote the string parameterization by

$$b_{\lambda,\phi} \mapsto (\lambda, t_1, ..., t_\ell) \in \Lambda^+ \times \mathbb{N}^\ell,$$

then the projection

(2.1) $$\pi_\lambda : \ \mathbb{R}^\ell \to \Lambda_{\mathbb{R}}, \ \ (t_1, ..., t_\ell) \mapsto -\lambda + t_1 \alpha_{i_1} + \cdots + t_\ell \alpha_{i_\ell}$$

maps the string polytope $Q(\lambda)$ onto the convex hull of the Weyl group orbit of the dual weight $\lambda^* = -w_0 \lambda$:

$$\pi_\lambda(Q(\lambda)) = \mathrm{Conv}(W.\lambda^*) = -\mathrm{Conv}(W.\lambda) := \Delta(\lambda)$$

Caldero [3] (see also [1]) has constructed a flat deformation of the polarized flag variety (X_λ, L_λ) to a polarized toric variety $(X_{\lambda;0}, L_{\lambda;0})$ such that the corresponding moment polytope is exactly the string polytope $Q(\lambda)$. His construction is based on the key multiplicative property of the (dual) canonical basis:

$$b_{\lambda_1,\phi_1} b_{\lambda_2,\phi_2} = b_{\lambda_1+\lambda_2,\phi_1+\phi_2} + \sum_{\phi < \phi_1+\phi_2} \mathrm{coeff}.b_{\lambda_1+\lambda_2,\phi},$$

which allowed him to put a filtration on the algebra A by $H \times H$ submodules such that the associated graded ring was the algebra of the semigroup of integral points in a rational convex polyhedral cone, see [3], [1] for details.

Let $\Phi_\lambda : X_{\lambda;0} \to Q(\lambda) \subset \mathbb{R}^\ell$ be the moment map for $\mathbb{T}$-action, where $\mathbb{T}$ is the compact part of $(\mathbb{C}^*)^\ell$. Then the composition $\pi_\lambda \circ \Phi_\lambda : X_{\lambda;0} \to \Delta_\lambda \subset \Lambda_{\mathbb{R}}$ is the moment map for the torus $T \subset H$.

Recall that a *weight variety* $M_{\lambda;\mu}$ is a GIT quotient of X_λ by the action of H associated with an integral point $\mu \in \Delta(\lambda)$. More precisely, let $\mu \in \Delta(\lambda)$ be a character of H, and $L_\lambda(-\mu)$ be the linearized line bundle L_λ twisted by $-\mu$. Then $M_{\lambda;\mu} = X_\lambda^{ss}(L_\lambda(-\mu))//G$. In symplectic terms, the H- moment map for the polarized variety $(X_\lambda, L_\lambda(-\mu))$ is $\pi_\lambda \circ \Phi_\lambda - \mu$, and $M_{\lambda;\mu}$ is identified with the reduction of X_λ at the level μ. The line bundle $L_\lambda(-\mu)$ descends to the quotient $M_{\lambda;\mu}$. Let us denote the descended line bundle by $L_{\lambda;\mu}$. The following theorem shows that every Caldero toric degeneration of a flag variety descends to a toric degeneration of any of weight variety

Theorem 2.1. *Every Caldero toric degeneration of X_λ descends to a toric degeneration of $M_{\lambda;\mu}$. In particular, there exists a flat degeneration of the weight variety $M_{\lambda;\mu}$ to a projective toric variety $N_{\lambda;\mu}$ with a polarization $L_{\lambda;\mu;0}$ such that the deformation carries $L_{\lambda;\mu}^n$ to $L_{\lambda;\mu;0}^n$ for sufficiently large positive integer n. The moment polytope of the polarized toric variety $(N_{\lambda;\mu}, L_{\lambda;\mu;0})$ is $Q_{\lambda;\mu} = \pi_\lambda^{-1}(\mu) \cap Q(\lambda)$.*

Proof. By the Borel-Weil Theorem, we have the following identification of the section ring

$$R_\lambda = \bigoplus_{n=0}^{\infty} H^0(X_\lambda, L_\lambda^n) = \bigoplus_{n=0}^{\infty} V(n\lambda^*).$$

H acts on R_λ via the induced action twisted by $-\mu$. Let $R_\lambda^{H(-\mu)}$ denote the set of invariants of this twisted H-action. Using the canonical bases, we have

$$R_\lambda = \bigoplus_{n=0}^{\infty} \bigoplus_{\phi} \mathbb{C}.b_{n\lambda,\phi}.$$

Since $b_{n\lambda,\phi}$ is an eigenvector of eigenvalue $-n\lambda + \sum_k t_{i_k}\alpha_{i_k}$ with respect to the left H-action, we see that after twisting the H action on $\mathbb{C}.b_{n\lambda,\phi}$ by $-n\mu$, the eigenvalue becomes zero if $\sum_k t_{i_k}\alpha_{i_k} = n(\lambda + \mu)$. Hence,

$$R_\lambda^{H(-\mu)} = \bigoplus_{n=0}^{\infty} \bigoplus_{\sum t_{i_k}\alpha_{i_k}=n(\lambda+\mu)} \mathbb{C}.b_{\lambda,\phi}.$$

Observe that the filtration on R_λ induces a filtration on $R_\lambda^{H(-\mu)}$. Let $\mathrm{Gr}(R_\lambda^{H(-\mu)})$ be the associated graded algebra of $R_\lambda^{H(-\mu)}$. Then the weight variety $M_{\lambda;\mu} = \mathrm{Proj}(R_\lambda^H(-\mu))$ degenerates flatly to $N_{\lambda;\mu} = \mathrm{Proj}(\mathrm{Gr}R_\lambda^{H(-\mu)})$, and the degeneration carries the set of ample line bundles $L_{\lambda;\mu}^n$ to the set of ample line bundles $L_{\lambda;\mu;0}^n$ on $N_{\lambda,\mu}$ for sufficiently large n ([1]). Since in general, the moment map of a polarization L is the moment map of L^n divided by n, it follows from [18] that the moment polytope of the polarized toric variety $(N_{\lambda;\mu}, L_{\lambda;\mu;0})$ is $\pi_\lambda^{-1}(\mu) \cap Q(\lambda)$. $\quad\square$

The first part of the above theorem can also be deduced from the fact that the action of $H \times H$ on G/U extends to the total space of the Caldero degeneration and the total space of our degeneration is just a quotient of the total space of the Caldero toric degeneration of X_λ by the left action of H.

Definition 2.1. We may call the central toric fiber $N_{\lambda,\mu}$ the toric weight variety of type (λ, μ), and call the polytope $Q_{\lambda;\mu} = \pi_\lambda^{-1}(\mu) \cap Q(\lambda)$ the weight polytope of type (λ, μ). Note that the polytope $Q_{\lambda;\mu}$ is rational but need not be integral in general.

Remark 2.1. *Unlike the isomorphic type of the quotient $M_{\lambda;\mu}$ which only depends on the GIT chamber of μ, the degeneration depends on the choice of individual μ, in particular, the topological type of the central fiber $N_{\lambda;\mu}$ can change even when μ varies within its GIT chamber. See Example 3.1.*

Question 2.1. Is there a chamber structure on Δ_λ so that $N_{\lambda;\mu}$, as (unpolarized) toric variety, remain the same within a chamber?

Remark 2.2. *As noted in [1, Section 5], the Ehrhart polynomial of the polytope $Q_{\lambda;\mu}$ is given by*

$$n \mapsto \dim V(n\lambda^*)_{n\mu},$$

the multiplicity of the weight $n\mu$ for the right H-action in the irreducible G-module with highest weight $n\lambda^*$. See also discussion on Pieri's formula in [5].

3 Degeneration of polygon spaces

Let us recall the generalities about the polygon spaces. We will start with spatial polygons in Euclidean 3-space. Given an n-tuple of positive real numbers $\mathbf{r} = (r_1, ..., r_n)$ we consider the moduli space $M_{\mathbf{r}}$ of n-gons with consecutive side lengths $r_1, ..., r_n$. This space can be viewed as the symplectic quotient at the zero level of the product of two-dimensional spheres $(S^2)^n$ by the diagonal action of the group SO(3). The symplectic structure on the j-th multiple is taken to be r_j times the standard unit sphere volume form. On the algebraic geometry side, the space $M_{\mathbf{r}}$ can be realized as the GIT quotient of the n-fold product of the projective line $\mathbb{CP}^1$ by the diagonal action of the group SL(2, $\mathbb{C}$). The choice of linearization is given by r_i's (which we assume to be integers here). These spaces were studied by Klyachko [20], Kapovich-Millson [16], Hausmann-Knutson [11], and many others.

Let us now recall the Gelfand-MacPherson correspondence [8] in the form of Theorem 2.4.7 of [17]. It states that for any n-tuple of positive integers $\mathbf{r} = (r_1, ..., r_n)$, there is an isomorphism of GIT quotients

$$(\mathrm{Gr}(k, n)/\!/H_1)_{\mathcal{O}(1),\mathbf{r}} \simeq ((\mathbb{CP}^{k-1})^n /\!/ \mathrm{SL}(k, \mathbb{C}))_{\mathcal{O}(\mathbf{r}),\zeta},$$

where the line bundle $\mathcal{O}(1)$ on the grassmannian is linearized to correspond to the action of $H_1 = \{(t_1, ..., t_n) \in (\mathbb{C}^*)^n : \prod t_i = 1\}$ on $\mathbb{C}^n$ given by

$$(t_1, ..., t_n) \mapsto \mathrm{diag}(t^{\mathbf{r}}t_1, ..., t^{\mathbf{r}}t_n),$$

and $t^{\mathbf{r}} = t_1^{r_1} \cdots t_n^{r_n}$ is the character of H_1 corresponding to $\mathbf{r}$. On the other side, the line bundle $\mathcal{O}(\mathbf{r})$ on $(\mathbb{CP}^{k-1})^n$ is the tensor product of pull-backs of line bundles $\mathcal{O}(r_i)$ from the i-th multiple. This bundle has exactly one SL($k, \mathbb{C}$)-linearization, which is denoted by ζ.

Now, let us apply this correspondence to $M_{\mathbf{r}}$ using the language of hermitian matrices as was first done in [11]. Let us start with the complex vector space $\mathbb{C}^{2n}$ with coordinates $(z_1, ..., z_n, w_1, ..., w_n)$. For convenience, we arrange z_i's and w_j's in a matrix form

$$A = \begin{pmatrix} z_1 & z_2 & \cdots & z_n \\ w_1 & w_2 & \cdots & w_n \end{pmatrix}$$

If we let $P = \sum_{i=1}^n r_i/2$ stand for half the perimeter, then the moduli space $M_{\mathbf{r}}$ can be realized as the quotient of the grassmannian $\mathrm{Gr}(2, n)$ by the action of

the maximal torus. The proof goes via the so-called reduction in stages. The grassmannian $\mathrm{Gr}(2,n)$ can be realized as the symplectic quotient of the space $\mathbb{C}^{2n}$ with the standard Darboux symplectic form by the action of the group $\mathrm{U}(2)$ by left multiplication on the matrix A. The corresponding moment map takes values in 2×2 hermitian matrices and is given by

$$A\overline{A}^t = \begin{pmatrix} \sum_{i=1}^n |z_i|^2 & \sum_{i=1}^n z_i \bar{w}_i \\ \sum_{i=1}^n w_i \bar{z}_i & \sum_{i=1}^n |w_i|^2 \end{pmatrix}.$$

And we reduce at the level $\mathrm{diag}(P,P)$. Alternatively, we can reduce the space $\mathbb{C}^{2n}$ by the action of $(S^1)^n$ on the columns of the matrix A at the level $(r_1, ..., r_n)$ to get the n-fold product of two-dimensional spheres with the aforementioned symplectic structure.

Using the 2×2 hermitian matrices, we see that the moduli space of polygons can be identified with the space of solutions of matrix equation

$$A_1 + \cdots + A_n = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix},$$

where each matrix A_j has the prescribed spectrum $(0, r_j)$ and in the language of the above correspondence:

$$A_j = \begin{pmatrix} |z_j|^2 & z_j \bar{w}_j \\ \bar{z}_j w_j & |w_j|^2 \end{pmatrix}.$$

Let d_j stand for the length of the diagonal of the polygon connecting vertices 1 and j. There is a remarkable integrable system [16] on $M_\mathbf{r}$ whose action variables are these functions d_j for $3 \leq j \leq n-1$. The corresponding flows are the so-called *bending* flows and have a simple geometric description by rotating the last $n - j$ edges about the axis defined by the diagonal d_j. This integrable system was shown [11] to be the reduction of the so-called Gelfand-Tsetlin integrable system on $\mathrm{Gr}(2,n)$ [10], which we recall now briefly.

Consider the $n \times n$ hermitian matrix A^*A of rank 2. The two non-trivial eigenvalues of the $k \times k$ upper-left submatrix denote by a_j and b_j, $a_j \leq b_j$. (For completeness, we let $a_1 = 0$.) The value $b_{n-1} = P$ is predetermined by the fact that the two non-zero eigenvalues of A^*A are both equal to P. Thus we have $2(n-2)$ independent commuting hamiltonians on $\mathrm{Gr}(2,n)$, which are also called the Gelfand-Tsetlin variables and commonly written in a triangular form:

$$
\begin{array}{ccccccc}
0 & 0 & \cdots & 0 & & P & P \\
& 0 & 0 & \cdots & 0 & a_{n-1} & P \\
& & \cdots & & \cdots & & \cdots \\
& & 0 & & a_3 & & b_3 \\
& & & a_2 & & b_2 & \\
& & & & b_1 & &
\end{array}
$$

which also satisfy the interlacing inequalities.

The reduction of this system corresponds to choosing the diagonal values of the matrix A^*A to ensure the compatibility with chosen side length, which amounts to fixing the sums in rows of the Gelfand-Tsetlin diagram: $b_1 = r_1$, $a_2 + b_2 = r_1 + r_2$, ..., $a_j + b_j = \sum_{i=1}^{j} r_i$, ... , $a_{n-1} = P - r_n$. Note that the lengths of the diagonals can also easily be expressed in terms of the variables (a_i, b_i), namely:

$$d_{j+1} = b_j - a_j.$$

These variable diagonal lengths d_j's, $3 \leq j \leq n-1$, which are the action variables define a polytope $\Pi_{\mathbf{r}}$ in $\mathbb{R}^{n-3}$. This polytope is cut by $3(n-2)$ hyperplanes corresponding to the triangle inequalities, some of which are redundant, for the triangulation of the polygon by all these diagonals emanating from the first vertex. The j-th triangle has the side lengths d_j, d_{j+1}, and r_{j+1} so it contributes three inequalities:

$$d_j + d_{j+1} \leq r_{j+1}, d_j + r_{j+1} \leq d_{j+1}, d_{j+1} + r_{j+1} \leq d_j.$$

The weight polytope $\Pi_{\mathbf{r}}$ is the polytope in $\mathbb{R}^{n-3}$ defined by these triangle inequalities.

Kogan-Miller [22] and Alexeev-Brion [1] have explicitly constructed degenerations of complex flag varieties X of type A_n to polarized toric varieties which have the Gelfand-Tsetlin polytopes (denoted by $\mathrm{GT}(\lambda)$, $\lambda \in \Lambda^+$) as their moment polytopes. In the case of Alexeev-Brion degeneration, they used the simplest reduced decomposition of the longest Weyl group element in $W = S_{n+1}$

$$\underline{w}_0^{\mathrm{std}} = s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_n s_{n-1} \cdots s_1.$$

As before, we denote by $\pi_\lambda : \mathrm{GT}(\lambda) \to \Delta(\lambda) \subset \Lambda_{\mathbb{R}}$ the projection of the Gelfand-Tsetlin polytope onto the moment polytope of the flag variety X with respect to the hamiltonian $\mathrm{SU}(n)$-action and the invariant symplectic form ω on $X = G/P_\lambda$ such that $[\omega] = c_1(L_\lambda)$. From Theorem 2.1 we immediately conclude

Theorem 3.1. *There exists a flat degeneration of the moduli space of polygons $M_{\mathbf{r}}$ with integral side lengths $\mathbf{r} = (r_1, ..., r_n)$ together with a polarizing line bundle L_r on it to a polarized toric variety $N_{\mathbf{r}}$ whose moment polytope is the weight polytope $\Pi_{\mathbf{r}}$.*

The toric variety $N_{\mathbf{r}}$, even its topological type, depends on the choice of $\mathbf{r}$. In fact, the combinatorial type of the weight polytope $\Pi_{\mathbf{r}}$ may also change when $\mathbf{r}$ changes, as shown in the following examples.

Example 3.1. Consider the case of pentagons. Already here we will see how the degeneration depends on the choice of the linearized line bundle, which in terms of the polygons corresponds to choosing not only side lengths, but also the order in which they appear. Let us consider the following three quintuples of

integers: $\mathbf{r}^1 = (3,3,3,3,3)$, $\mathbf{r}^2 = (3,3,3,3,4)$, and $\mathbf{r}^3 = (3,4,3,4,3)$. For all these choices, the moduli spaces $M_{\mathbf{r}^1}$, $M_{\mathbf{r}^2}$, and $M_{\mathbf{r}^3}$ are biholomorphically isomorphic to the same space $\overline{M}_{0,5}$, the canonically compactified moduli space of projective lines marked at 5 distinct points. This surface is also isomorphic to the Del Pezzo surface of degree 5 (blow-up of $\mathbb{CP}^2$ at 4 generic points). The corresponding weight polytopes are 2-dimensional polygons. The coordinates (x, y) correspond to the lengths of the first and the second diagonals respectively.

In the case of $\mathbf{r}^1$ we have a pentagon with vertices at the points $(0,3)$, $(3,0)$, $(6,3)$, $(6,6)$, and $(3,6)$. This corresponds to a toric surface with two isolated singular points (local quotient singularities of $\mathbb{C}^2$ by the action of $\mathbb{Z}/2$).

In the case of $\mathbf{r}^2$ we have a hexagon with vertices at the points $(0,3)$, $(2,1)$, $(4,1)$, $(6,3)$, $(6,7)$, and $(4,7)$. This corresponds to a toric surface with one isolated singular point (over $(3,0)$).

In the case of $\mathbf{r}^3$ we get a heptagon with vertices at the points $(1,4)$, $(1,2)$, $(2,1)$, $(4,1)$, $(7,4)$, $(7,7)$, and $(4,7)$. The corresponding toric variety is smooth (and diffeomorphic to $\overline{M}_{0,n}$). This toric variety is a blow up of each of the previous ones at one or two singular points. Notice that we would get a different polytope with five vertices should the same side lengths be arranged as $(3,3,3,4,4)$.

Example 3.2. In the equilateral case when $n = 4, 5, 6$, or 7 and $\mathbf{r} = (1,1,...,1)$, the polytopes $\Pi_{\mathbf{r}}$ were considered in [15, Section 4] and their virtual Poincaré polynomials were computed.

Example 3.3. Now consider an example when $\mathbf{r} = (2,2,2,...,2,2n-3)$, when there is one "very long" side. In this case $\mathbf{r}$ lies in a *favorable* chamber [14, Section 3]. One can see that the polytope $\Pi_{\mathbf{r}}$ is defined by the following set of inequalities:

$$3 \leq d_3 \leq 4, \quad 5 \leq d_4 \leq d_3 + 2, \quad ..., \quad 2i - 3 \leq d_i \leq d_{i-1} + 2, \text{ for } 4 \leq i \leq n - 1.$$

This polytope is actually a simplex, and the corresponding polarized toric variety is $(\mathbb{CP}^{n-3}, \mathcal{O}(1))$. This is not surprising, because for any element $\mathbf{r}$ from the interior of a favorable chamber, the moduli space $M_{\mathbf{r}}$ is itself isomorphic to $\mathbb{CP}^{n-3}$, and thus our flat family is trivial in this case.

Example 3.4. Let us consider the case when we have three "very long" sides. More precisely, let $r_1 = r_2 = \cdots = r_{n-3} = 1$ and $r_{n-2} = r_{n-1} = r_n = n$. So the perimeter is equal to $4n - 3$ and no two long sides can ever be parallel. We will show that the moduli space $M_{\mathbf{r}}$ is isomorphic to $(\mathbb{CP}^1)^{n-3}$ in a natural way. Let us treat the moduli space $M_{\mathbf{r}}$ as the moduli space of weighted configurations of points on $\mathbb{CP}^1$. Denote these points by $z_1, ..., z_n$ respectively, where we coordinatize, as usual, by $\mathbb{C}$ the complement of a point (the "North pole" or ∞) in $\mathbb{CP}^1$. Consider the cross-ratios

$$w_i = \frac{z_{n-2} - z_n}{z_n - z_{n-1}} \cdot \frac{z_i - z_{n-1}}{z_n - z_i}, \quad \text{where} \ \ 1 \leq i \leq n - 3.$$

Since the points z_{n-2}, z_{n-1} and z_n never collide on the subset of semi-stable configurations, these cross-ratios define a global map from the moduli space $M_{\mathbf{r}}$ to $(\mathbb{CP}^1)^{n-3}$, which is an isomorphism. It is important to notice that the bending flows are **not** the standard circle actions on this space.

This example also shows that the toric variety $N_{\mathbf{r}}$ and the polytope Π_r depends on the order of side-lengths. If we had switched the first and the $(n-2)$-nd sides, so that $\mathbf{r} = (n, 1, 1, ..., 1, n, n)$, then none of the diagonals emanating from the first vertex would ever vanish, because it would not be allowed by the triangle inequalities, and the central fiber would be isomorphic to $(\mathbb{CP}^1)^{n-3}$. But if we stick with the original choice of order on the side-lengths, then the resulting toric variety will be singular, in general. This can be seen already for $n = 5$, when the resulting toric variety has an isolated quotient singularity.

In fact, since all the inequalities on the lengths of the diagonals emanating from a given vertex come from the triangles which these diagonals break the polygon into, we immediately have the following result:

Proposition 3.1. *The weight polytope $\Pi_{\mathbf{r}}$ is the compact intersection of the half spaces defined by affine hyperplanes of the following forms: $x_1 = |r_1 \pm r_2|, x_{n-3} = |r_n \pm r_{n-1}|$ or $|x_i \pm x_{i+1}| = r_{i+2}$. In particular, let $\{e_i\}$ be the coordinate basis vectors, then any facet is perpendicular to one of the following: e_1, e_{n-3}, i.e., the first or the last coordinate line; $e_i \pm e_{i+1}$, i.e., the diagonal or anti-diagonal line in the two dimensional coordinate plane spanned by $\{e_i, e_{i+1}\}$.*

In the above Proposition x_k corresponds to d_{k+2}.

Let $U_{\mathbf{r}}$ be the (non-moduli) space of polygons with side length vector $\mathbf{r}$ such that the first vertex is at the origin. Note that $M_{\mathbf{r}} = U_{\mathbf{r}}/\mathrm{SO}(3)$. We construct a new quotient space by defining a new equivalence relation on $U_{\mathbf{r}}$. For each $3 \leq i \leq n-1$, let D_i be the set of polygons with $d_i = 0$. Let D° be the set of polygons where none of the diagonals vanish. Then

$$U_{\mathbf{r}} = D^\circ \cup_I D_I,$$

where $I \subset \{3, \ldots, n-1\}$ and D_I is the set of polygons with vanishing diagonals $d_i, i \in I$ but the rest of the diagonals do not vanish. For any I, every polygon in D_I can be decomposed as polygons $P_1, \ldots, P_\ell$ mutually joining at a point, where $\ell = |I| + 1$. In this case, set

$$S_I = D_I/(\mathrm{SO}(3))^\ell$$

where $(\mathrm{SO}(3))^\ell$ acts on each factor component-wise. Let $S^\circ = D^\circ/\mathrm{SO}(3)$. We then obtain a new topological space

$$V_{\mathbf{r}} = S^\circ \cup_I S_I.$$

There is an obvious continuous collapsing map

$$f : M_{\mathbf{r}} \to V_{\mathbf{r}}.$$

There is also a continuous map

$$\Phi_{\mathbf{r}} : V_{\mathbf{r}} \to \Pi_{\mathbf{r}}.$$

The map f passes the action-angle coordinates on $M_{\mathbf{r}}$ to $V_{\mathbf{r}}$.

Conjecture. The space $V_{\mathbf{r}}$ is homeomorphic to $N_{\mathbf{r}}$.

We were able to verify this conjecture in some particular cases, however the rigorous proof evades us, because the explicit nature of the deformation family is evasive.

Remark 3.1. *Our construction of the space $V_{\mathbf{r}}$ follows Kamiyama-Yoshida [15], where a similar space, which only had a structure of a topological space stratified into a finite union of orbifolds, was constructed in the equal side length case. The bending hamiltonians descend to their space and are everywhere defined.*

3.1 Boundary toric divisors

Corresponding to each facet, there is a toric subvariety of $N_{\mathbf{r}}$ of codimension 1. Corresponding to the set of all facets perpendicular to a given vector is the union of subvarieties whose isotropy subgroup is generated by a prime vector in that direction. By the above proposition, the one dimensional isotropy subgroups are listed as below:

1. $T_1 = \{t \in (\mathbb{C}^*)^{n-3} | t_2 = \ldots = t_{n-3} = 1\}$.

2. $T_{n-3} = \{t \in (\mathbb{C}^*)^{n-3} | t_1 = \ldots = t_{n-4} = 1\}$.

3. $T_i^+ = \{t \in (\mathbb{C}^*)^{n-3} | t_i = t_{i+1}, t_k = 1 \text{ for } k \neq i, i+1\}$.

4. $T_i^- = \{t \in (\mathbb{C}^*)^{n-3} | t_{i+1} = t_i^{-1}, t_k = 1 \text{ for } k \neq i, i+1\}$. .

Corresponding to the list, we have toric divisors (in parentheses the equations of the corresponding facets of $\Pi_{\mathbf{r}}$ are given):

- (1) N_2^1 $(x_1 = r_1 + r_2)$ and N_2^2 $(x_1 = r_2 - r_1)$ or N_2^3 $(x_1 = r_1 - r_2)$

- (2) N_{n-1}^3 $(x_{n-3} = r_{n-1} + r_n)$ and N_{n-1}^1 $(x_{n-3} = r_n - r_{n-1})$ or N_{n-1}^2 $(x_{n-3} = r_{n-1} - r_n)$

- (3) N_i^1 $(x_{i-1} - x_{i-2} = r_i)$ and N_i^3 $(x_{i-2} - x_{i-1} = r_i)$

- (4) N_i^2 $(x_{i-1} + x_{i-2} = r_i)$

These are all the toric divisors, some of which are reducible. In the next section, we will show that they are the deformation images of certain subspaces of $M_{\mathbf{r}}$.

The Chow ring $A^*(M_{\mathbf{r}})$ is generated, as a ring, by degree 1 cycles corresponding to divisors Z_{ij}, $i < j$ defined by the condition that the i-th side of the polygon points in the same direction as the j-th side. We call such sides "positively parallel" (or strongly parallel), which is a stronger condition than just being parallel. This statement can be verified by considering the cycles D^S defined in [19] and which generate $A^*(\overline{M}_{0,n})$ together with the proper surjective birational morphism $\overline{M}_{0,n} \to M_{\mathbf{r}}$, and its properties [6]. It is then well-known that the map $A^*(M_{\mathbf{r}}) \to H^{2*}(M_{\mathbf{r}}, \mathbb{Z})$ is an isomorphism. There are some natural linear relations among the cycles Z_{ij}, which again can be deduced from the linear relations on the cycles D^S studied in [19] and [23]. However, these cycles are not stable under the bending flows, and will not degenerate to toric subvarieties.

Instead, let us consider subsets of the space $M_{\mathbf{r}}$ which are defined by the condition that the i-th diagonal is parallel to the $(i+1)$-st diagonal, and thus to the i-th side (we also think that a zero vector is parallel to any other vector). Let us introduce a notation for these subsets.

- denote by Y_i^1 the subset where $d_i + r_i = d_{i+1}$

- denote by Y_i^2 the subset where $d_i + d_{i+1} = r_i$

- denote by Y_i^3 the subset where $d_i - d_{i+1} = r_i$

- denote by $Y_i = Y_i^1 \cup Y_i^2 \cup Y_i^3$ their union

Here we have $2 \leq i \leq n - 1$ as we think of the first and last sides as 2-nd and n-th diagonals respectively. The number of such we get this way is bounded from above by $3n - 8$. However, depending on the choice of $\mathbf{r}$ some of these subsets might be empty.

One can easily see that Y_i^1 and Y_i^3 always have empty intersection. However, if d_i can attain zero, then Y_i^1 and Y_i^2 do intersect and when the side lengths are generic, and $3 < i < n - 1$, this intersection is homeomorphic to an S^1-bundle over an S^2-bundle over the space $M_{\mathbf{r}^{(1)}} \times M_{\mathbf{r}^{(2)}}$, where $\mathbf{r}^{(1)} = (r_1, ..., r_{i-1})$ and $\mathbf{r}^{(2)} = (r_i, ..., r_n)$.

Similarly, if d_{i+1} can attain the value of zero, then then Y_i^2 and Y_i^3 do intersect and when side lengths are generic, and $2 < i < n - 2$, this intersection is homeomorphic to an S^1-bundle over an S^2-bundle over the space $M_{\mathbf{r}^{(1)}} \times M_{\mathbf{r}^{(2)}}$, where $\mathbf{r}^{(1)} = (r_1, ..., r_i)$ and $\mathbf{r}^{(2)} = (r_{i+1}, ..., r_n)$.

Note that when $r_1 = r_2$ then the subsets Y_2^2 and Y_2^3 coincide and are equivalently defined by $d_3 = 0$. The same goes for the case $r_{n-1} = r_n$, subsets Y_{n-1}^1 and Y_{n-1}^2 coincide and are defined by $d_{n-1} = 0$. Unless stated otherwise, we would like to exclude these from consideration in what follows.

We denote by $M_{\mathbf{r}}^0$ the "toric" open submanifold of $M_{\mathbf{r}}$, where none of the diagonals vanishes. We say that a subset $Q \subset M_{\mathbf{r}}$ *degenerates to* $Q_0 \subset N_{\mathbf{r}}$ if

$Q \cap M_{\mathbf{r}}^0$ (resp Q_0) is preserved by the bending flows (resp. compact torus action), and they have the same image in $\Pi_{\mathbf{r}}$.

Lemma 3.1. *1. If $M_{\mathbf{r}}$ is smooth, then $Y_i \cap M_{\mathbf{r}}^0$, for $2 \leq i \leq n-1$, is a smooth closed submanifold of $M_{\mathbf{r}}^0$ of real codimension 2.*

2. Each Y_i degenerates to the union of the toric divisors N_i^1, N_i^2, and N_i^3 of $N_{\mathbf{r}}$. All toric divisors of $N_{\mathbf{r}}$ arise this way.

Proof. (1) The intersections

$$Y_i \cap M_{\mathbf{r}}^0$$

are clearly non-empty. Each connected components of these intersections is also a connected component of the fixed point set of some one-dimensional torus (see the remark after Proposition 3.1), so it is smooth.

(2) is clear. $\square$

Remark 3.2. *The intersection $N_i^1 \cap N_i^2$ is a toric subvariety which is isomorphic to the product of the toric degenerations of polygon spaces with side lengths $(r_1, ..., r_{i-1})$ and $(r_i, ..., r_n)$ respectively. Analogously, the intersection N_i^2 and N_i^3, if non-empty, is a toric subvariety which is isomorphic to the product of the toric degenerations of polygon spaces with side lengths $(r_1, ..., r_i)$ and $(r_{i+1}, ..., r_n)$ respectively. It is clear that these divisors $\{N_i^j\}$ for $j = 1, 2,$ and 3 and $2 \leq i \leq n-1$ generate $A^*(N_{\mathbf{r}})$. Although the additive structure on homology of $M_{\mathbf{r}}$ can readily be understood if we work with divisors $\{Z_{ij}\}$, there are too many generators and linear relations between these cycles. Using the real cycles represented by connected components of the Y_i's, the homology of $M_{\mathbf{r}}$ can be understood with more ease, as we can see explicitly the toric subvarieties that these cycles degenerate to in $N_{\mathbf{r}}$.*

Question 3.1. Do the cycles of the (connected components of) subspaces Y_i for $2 \leq i \leq n-1$, generate $H_*(M_{\mathbf{r}}, \mathbb{Z})$?

In some examples that we were able to compute, this indeed was the case. However, in general due to the fact that some of the Y_i's degenerate to the sum of two or three irreducible toric divisors, the answer to this question might not be entirely obvious.

As follows from [1, Proposition 2.2(ii)], our family is isomorphic to the trivial family over $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. In the future we hope to establish an explicit trivialization, so the degeneration of the aforementioned cycles will be transparent.

4 Degeneration of Flaschka-Millson integrable systems

In [5] the authors considered moduli spaces of weighted configurations on $\mathbb{CP}^m$. Given an n-tuple of positive integers $\mathbf{r} = (r_1, ..., r_n)$ their moduli space is, again, identified with the GIT quotient of $(\mathbb{CP}^m)^n$ by the diagonal action of $\mathrm{SL}(m+1, \mathbb{C})$, with the linearizing line bundle $\mathcal{O}(\mathbf{r})$ is given by the tensor product of pull-backs of the bundles $\mathcal{O}(r_i)$ from the i-th factor. It has a unique $\mathrm{SL}(m+1, \mathbb{C})$ linearization. In order for this moduli space to be nonempty, we need to impose, in addition to the obvious condition $n > m+1$, the so-called strong triangle inequalities, one for each $1 \leq i \leq n$:

$$r_i \leq P, \quad \text{where} \quad P := \frac{1}{m+1} \sum_{j=1}^{n} r_j.$$

Note that when $m = 2$, these are just the usual triangle inequalities. On the symplectic side, this moduli space is identified with the moduli space of solution to the matrix equation:

$$A_1 + \cdots + A_n = P.\mathrm{Id},$$

where each A_i is a rank one $(m+1) \times (m+1)$ hermitian symmetric matrix, with spectrum $\{r_i, 0, 0, ..., 0\}$, and we mod out by the diagonal action of the group $\mathrm{U}(m+1)$. Then the Flaschka-Millson bending hamiltonians, by definition, are the non-trivial eigenvalues of the partial sums

$$\sum_{i=1}^{k} A_i.$$

Using the Gelfand-MacPherson correspondence [17], the above moduli space $M_{\mathbf{r}}$ can be identified with a GIT quotient of the grassmannian $\mathrm{Gr}(m+1, n)$ with respect to the torus action. The choice of the linearized line bundle is analogous to the case $m = 1$.

These hamiltonians also appear if we take the Gelfand-Tsetlin integrable system on the grassmannian and consider its reduction. More precisely, consider the Gelfand-Tsetlin triangle for this case. The top line consists of $(n - m - 1)$ zeroes and $(m + 1)$ P's. (For the purposes of drawing the diagram we assumed that

$m < n/2$, but we don't hold this assumption in general):

$$
\begin{array}{ccccccccccc}
0 & 0 & \cdots & & 0 & & 0 & P & & P & \cdots & P & & P \\
 & 0 & 0 & \cdots & 0 & & & 0 & a_1 & & P & & P & \cdots & P & & P \\
 & & 0 & & \cdots & & 0 & & a_2 & & b_2 & & & P & & \cdots & & P \\
 & & & \cdots & & & & \cdots & & & & & & & \cdots \\
 & & & & \cdots & & 0 & & a_{n-2m} & & \cdots & & p_{n-2m} & & P \\
 & & & & & \cdots & & & & \cdots & & & & \cdots \\
 & & & & & & 0 & & a_{m+1} & & & & \cdots & & q_{m+1} \\
 & & & & & & & \cdots & & & & \cdots & & \cdots \\
 & & & & & & & & o_3 & & p_3 & & q_3 \\
 & & & & & & & & & p_2 & & q_2 \\
 & & & & & & & & & & q_1
\end{array}
$$

There are a total of $(m + 1) \times (n - m - 1)$ indeterminants a_1, ..., q_{m+1} in the diagram. The reduction at the level $\mathbf{r} = (r_1, ..., r_n)$ means that there are the following condition imposed, analogous to Section 3, where we considered the case $m = 1$:

$$q_1 = r_1, \quad p_2 + q_2 = r_1 + r_2, \quad ..., \quad a_{m+1} + \cdots + q_{m+1} = r_1 + \cdots + r_{m+1}, \quad ... \text{ etc.}$$

meaning that the sum of all elements in row k from the bottom equals $\sum_{i=1}^{k} r_i$. The Flaschka-Millson hamiltonians in this presentation are the differences of adjacent elements in each row:

$$b_2 - a_2, \quad b_3 - a_3, \quad c_3 - b_3, \quad , ..., \quad q_2 - p_2.$$

There are precisely

$$(m + 1)(n - m - 1) - (n - 1) = mn - 2m - m^2$$

of them, which is exactly the complex dimension of the moduli space. The polygon defined by the natural inequalities on these hamiltonians is, as before, denoted by $\Pi_{\mathbf{r}}$.

Proceeding in a completely analogous way to Section 3, we can see that there exists a flat degeneration of the moduli space $M_{\mathbf{r}}$ to a polarized toric variety with the moment polytope $\Pi_{\mathbf{r}}$.

Remark 4.1. *If we denote by $P - \mathbf{r}$ the n-tuple of weights $(P - r_1, P - r_2, ..., P - r_n)$, then Howard-Millson [13] showed that the GIT quotients*

$$((\mathbb{CP}^m)^n // \mathrm{SL}(m + 1, \mathbb{C}))_{\mathcal{O}(\mathbf{r})} \quad \text{and} \quad ((\mathbb{CP}^{n-m-2})^n // \mathrm{SL}(n - m - 1, \mathbb{C}))_{\mathcal{O}(P - \mathbf{r})}$$

are isomorphic. Applying the Gelfand-MacPherson correspondence, this duality translates to the isomorphism of GIT quotients

$$(\mathrm{Gr}(m + 1, n) // H_1)_{\mathcal{O}(1), \mathbf{r}} \quad \text{and} \quad (\mathrm{Gr}(n - m - 1, n) // H_1)_{\mathcal{O}(1), P - \mathbf{r}} \quad ,$$

*which they established by using the complex Hodge * operator. This duality clearly induces an isomorphism between polytopes $\Pi_{\mathbf{r}}$ and $\Pi_{P - \mathbf{r}}$ and the toric varieties $N_{\mathbf{r}}$ and $N_{P - \mathbf{r}}$.*

5 Real loci

First, let us have a brief discussion on real loci in general weight varieties. Let τ be an involution of G (group automorphism of order 2, and we treat G as a real Lie group with the Iwasawa decomposition $G = KAU$) satisfying the following conditions:

- τ commutes with the Cartan involution θ

- τ maps H to H

- τ is anti-holomorphic

- if G^τ, K^τ, and A^τ stand for the subgroups of G, K, and A respectively fixed by τ and U^τ is the subset of elements in U fixed by τ, then $G^\tau = K^\tau A^\tau U^\tau$. (Note that we do not require that τ maps U to U.)

In fact, given a conjugacy class of real forms of G, one can always find an involution τ as above so this conjugacy class is represented by G^τ. Alternatively, given a Satake diagram Σ [12], we can always find an involution τ such that Σ corresponds to the real form G^τ.

Now let us take a subset of simple roots J such that it contains all the black simple roots in the Satake diagram Σ and perhaps some white roots. If there are white roots connected by an arrow, then we either include both of them in J or neither. Let P_J be the corresponding parabolic subgroup containing B (the minimal among all parabolics containing B and all the root vectors $E_{-\alpha}$, $\alpha \in J$). The following is straightforward:

Lemma 5.1. *The parabolic subgroup P_J is τ-stable.*

This immediately implies that there is an induced involution, also denoted by τ, on the flag manifold $X = G/P$, and its fixed point set is the real flag manifold $X^\tau \simeq G^\tau/P^\tau$. In fact there exists an integral symplectic structure [7] ω (corresponding to a polarizing line bundle L_λ) such that τ is anti-symplectic with respect to ω and X^τ is a lagrangian submanifold in X.

In what follows we denote by $T \subset H$ the connected component of the maximal torus containing the identity element. The involution τ maps T to T and let us denote by T^+ the connected subgroup of T on which τ acts identically and by T^- - where it acts by inverting all elements, so that $T = T^+ \times T^-$. Let us also denote by Q a (not necessarily connected) subgroup of T consisting of *all* elements on which τ acts by $\tau(t) = t^{-1}$. If the dimension of T^+ is denoted by a, then

$$Q \simeq (\mathbb{Z}/2)^a \times T^-,$$

and has 2^a connected components. Let us also denote by T^τ the subgroup of T fixed by τ. If the dimension of T^- is b, then

$$T^\tau \simeq T^+ \times (\mathbb{Z}/2)^b.$$

Note that the involution τ on X satisfies the following property with respect to K-action:

$$\tau(k.x) = \tau(k)\tau(x).$$

We can choose the moment map Φ for the T-action in such a way that $\Phi(\tau(x)) = -\tau(\Phi(x))$, where in the left hand side the involution on $\mathfrak{t}^*$ is the one induced from τ on T, and also denoted by τ. According to a theorem of O'Shea-Sjamaar [24] which generalizes Duistermaat's [4], the image of X^τ under the moment map Φ for the T-action is the same as the intersection of X with the annihilator $(\mathfrak{t}^-)^\perp \subset \mathfrak{t}^*$ of $\mathfrak{t}^-$. Let us choose a regular value μ of Φ such that $\mu \in (\mathfrak{t}^-)^\perp$. Then the fixed point set of τ in $\Phi^{-1}(\mu)$, which is the same as $\Phi^{-1}(\mu) \cap X^\tau$ is non-empty and T^τ-stable. If we reduce at μ, and denote the induced involution on $X//T$ also by τ, then the natural map

$$\psi : \ X^\tau//T^\tau := (\Phi^{-1}(\mu) \cap X^\tau)/T^\tau \to (X//T)^\tau$$

has the following properties. The map ψ is surjective onto a (finite number of) connected component of $(X//T)^\tau$. The map ψ is a finite map, and is injective if T acts freely on $\Phi^{-1}(\mu)$. Replacing the involution τ acting on X by $s\tau$, where $s \in Q$ we can get all other connected components of $(X//T)^\tau$ be in the image of maps analogous to ψ. The induced involution on $X//T$ will still be the same, yet if s acts non-trivially on X and belongs to a different connected component of Q than the identity, then the T^τ-orbits on $X^{s\tau}$ are actually disjoint from those on X^τ and if $X^{s\tau}$ is non-empty, then $X^{s\tau} \cap \Phi^{-1}(\mu)/T^\tau$ will map onto different connected components of $(X//T)^\tau$. To get all the connected components of $(X//T)^\tau$, it is enough to choose one s from each connected component of Q, see details in [7].

When the Satake diagram Σ does not contain any black roots (when G^τ is so-called quasi-split), then B itself is τ-stable and the action of τ descends to the full flag manifold G/B. And when G^τ is actually a split real form (no arrows in Σ), then each $P \supset B$ is τ-stable and, moreover, $T = Q = T^-$ and T^τ is finite and isomorphic to $(\mathbb{Z}/2)^r$, where $r = \mathrm{rank}(G)$.

We will postpone a more detailed treatment of real loci in general weight varieties and their degenerations to a future paper, and now will concentrate on the case when $G = \mathrm{SL}(n, \mathbb{C})$, τ is the standard complex conjugation and X is a complex grassmannian. First, let us consider the case of spatial polygons. Then the real quotient $X^\tau//T^\tau$ is the moduli space of planar polygons (more precisely, its quotient by the mirror reflections), which we denote by $M_{\mathbf{r}}^{(2)}$. If we, in addition, fix admissible non-zero values of diagonals, then a polygon can have only finitely many shapes. Their number generically equals 2^{n-3}, because we can do a 180° "bending" about each of the diagonals. Thus we have shown that the map $M_{\mathbf{r}}^{(2)} \to \Pi_{\mathbf{r}}$ is surjective, and generically 2^{n-3}-to-one. However, this map is not globally finite, and only becomes such at the special fiber $N_{\mathbf{r}}$. It is easy to see that the involution τ on general fiber $M_{\mathbf{r}}$ extends to a complex conjugate involution on the special fiber, where its fixed point set is a lagrangian locus

$N_{\mathbf{r}}^{(2)}$ which maps surjectively and finitely onto $\Pi_{\mathbf{r}}$. This statement has a natural generalization to the moduli spaces $M_{\mathbf{r}} = (\mathbb{CP}^m)^n // \mathrm{SL}(m+1, \mathbb{C})$ considered in [5]:

Proposition 5.1. *Let* $\mathbf{r} = (r_1, ..., r_n)$ *be an admissible n-tuple of positive numbers. Let τ be a complex conjugate involution, with fixed point set $\mathrm{SL}(n, \mathbb{R})$ in $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{Gr}_{\mathbb{R}}(m+1, n)$ in $\mathrm{Gr}_{\mathbb{C}}(m+1, n)$. Then τ extends to the special fiber $N_{\mathbf{r}}$ of the flat family and the fixed point set $N_{\mathbf{r}}^{\tau}$ maps finitely and surjectively under the moment map to the polytope $\Pi_{\mathbf{r}}$, and the generic fiber has cardinality $2^{mn-2m-m^2}$.*

Acknowledgments

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