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Frame-independent mechanics: geometry on affine bundles [1]

by

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Abstract

Main ideas of the differential geometry on affine bundles are presented. Affine counterparts of Lie algebroid and Poisson structures are introduced and discussed. The developed concepts are applied in a frame-independent formulation of the time-dependent and the Newtonian mechanics.

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1 Introduction

Frame-independent formulation of the Newtonian analytical mechanics can be achieved, like in the case of a relativistic charged particle, by increasing the dimension of the configuration space (cf. the Kaluza theory). There is an alternative approach, based on ideas of Tulczyjew, in which the four-dimensional space-time is used as the configuration space. The phase space is no longer a cotangent bundle, but an affine bundle, modeled on the cotangent bundle. Also time-dependent mechanics requires affine objects. Here Lagrangian is a function on a space of first-jets of motions, interpreted as an affine subbundle of the tangent bundle to the space-time. The phase bundle is a vector bundle, however the Hamiltonian is not a function, but a section of a bundle over the phase manifold. The structure on the phase bundle, which makes possible the Hamiltonian formulation of the dynamics, is no longer a standard Poisson structure, but its affine counterpart.

It is known that Lagrangian formulation of the dynamics of an autonomous system is based on the Lie algebroid structure of the tangent bundle. The associated linear Poisson structure on the dual (cotangent) bundle provides a framework

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for the Hamiltonian formulation of the dynamics. In this expository note we show how this correspondence looks like in the affine setting (sections 3, 4). First, we introduce the notion of a special affine bundle and its special affine dual (section 2). In section 3 we present basic constructions of the geometry of affine values, i.e. the geometry based on sections of certain affine bundle, instead of the algebra of smooth functions. In section 4 we establish relation between Lie affgebroids (affine version of Lie algebroids) and aff-Jacobi brackets on the dual bundle. These results are applied in a frame-independent formulation of the time-dependent and the Newtonian mechanics (section 5).

2 The affine geometry

2.1 Affine spaces and affine bundles

We start with a short review of basic notions in affine geometry. We recall some definitions and set the notation. An *affine space* is a triple (A, V, α), where A is a set, V is a vector space over a field $\mathbb{K}$ and α is a mapping $\alpha\colon A \times A \to V$ such that

- $\alpha(a_3, a_2) + \alpha(a_2, a_1) + \alpha(a_1, a_3) = 0$;
- the mapping $\alpha(\cdot, a)\colon A \to V$ is bijective for each $a \in A$.

We shall also write simply A to denote the affine space (A, V, α) and $\mathsf{V}(A)$ to denote its *model vector space* V. The *dimension of an affine space* A we will call the dimension of its model vector space $\mathsf{V}(A)$. A mapping φ from the affine space A_1 to the affine space A_2 is *affine* if there exists a linear map $\varphi_{\mathsf{V}} : \mathsf{V}(A_1) \to \mathsf{V}(A_2)$ such that, for every $a \in A_1$ and $u \in \mathsf{V}(A_1)$,

$$\varphi(a + u) = \varphi(a) + \varphi_{\mathsf{V}}(u).$$

The mapping φ_{V} is called the *linear part* of φ. We will need also multi-affine, especially bi-affine, maps. Let A, A_1, A_2 be affine spaces. A map

$$\Phi : A_1 \times A_2 \longrightarrow A$$

is called *bi-affine* if it is affine with respect to every argument separately. One can also define linear parts of a bi-affine map. By Φ^1_{V} we will denote the linear-affine map

$$\Phi^1_{\mathsf{V}} : \mathsf{V}(A_1) \times A_2 \longrightarrow \mathsf{V}(A),$$

where the linear part is taken with respect to the first argument. Similarly, for the second argument we have the affine-linear map

$$\Phi^2_{\mathsf{V}} : A_1 \times \mathsf{V}(A_2) \longrightarrow \mathsf{V}(A)$$

and finally the *bilinear part of* Φ:

$$\Phi_{\mathsf{V}} : \mathsf{V}(A_1) \times \mathsf{V}(A_2) \longrightarrow \mathsf{V}(A).$$

For an affine space A we define its *vector dual* $A^\dagger$ as a set of all affine functions $\phi : A \to \mathbb{K}$. The dimension of $A^\dagger$ is greater by 1 than the dimension of A. By 1_A we will denote the element of $A^\dagger$ being the constant function on A equal to 1. Observe that A is naturally embedded in the space $\widehat{A} = (A^\dagger)^*$. The affine space A can be identified with the one-codimensional affine subspace of $\widehat{A}$ of those linear functions on $A^\dagger$ that evaluated on 1_A give 1. Similarly, $\mathsf{V}(A)$ can be identified with the vector subspace of $\widehat{A}$ of those elements that evaluated on 1_A give 0. The above observation justifies the name *vector hull of* A for $\widehat{A}$.

In the following we will widely use *affine bundles* which are smooth, locally trivial bundles of affine spaces. The notation will be, in principle, the same for affine spaces and affine bundles. For instance, $\mathsf{V}(A)$ denotes the vector bundle which is the model for an affine bundle $\zeta : A \to M$ over a base manifold M. By Sec we denote the spaces of sections, e.g. $\mathsf{Sec}(A)$ (or sometimes $\mathsf{Sec}(\zeta)$) is the affine space of sections of the affine bundle $\zeta : A \to M$. The difference of two sections a, a' of the bundle A is a section of $\mathsf{V}(A)$. Equivalently, we can add sections of $V(A)$ to sections of A:

$$\mathsf{Sec}(A) \times \mathsf{Sec}(\mathsf{V}(A)) \ni (a, v) \mapsto a + v \in \mathsf{Sec}(A).$$

Every $a \in \mathsf{Sec}(A)$ induces a 'linearization'

$$I_a : A \to \mathsf{V}(A), \quad a'_p \mapsto a'_p - a(p).$$

The *bundle of affine morphisms* $\mathsf{Aff}_M(A_1; A_2)$ from A_1 to A_2 is a bundle which fibers are spaces of affine maps from fibers of A_1 to fibers of A_2 over the same point in M. The space of sections of the bundle we will denote by $\mathsf{Aff}(A_1; A_2)$. It consists of actual morphisms of affine bundles. $\mathsf{Aff}_M(A_1; A_2)$ is an affine bundle modelled on $\mathsf{Aff}_M(A_1; \mathsf{V}(A_2))$ and $\mathsf{Aff}(A_1; A_2)$ is an affine space modelled on $\mathsf{Aff}(A_1; \mathsf{V}(A_2))$. Like for affine maps, we can define a *linear part of an affine morphism*: If $\varphi \in \mathsf{Aff}(A_1; A_2)$ then $\varphi_{\mathsf{V}} \in \mathsf{Hom}(\mathsf{V}(A_1); \mathsf{V}(A_2))$ and for any $a \in A_1$ and $u \in V(A_1)$, both over the same point in M, we have

$$\varphi_{\mathsf{v}}(u) = \varphi(a + u) - \varphi(a).$$

For an affine bundle A we define also its *vector dual* $A^\dagger = \mathsf{Aff}_M(A; M \times \mathbb{R})$ and its *vector hull*, i.e. the vector bundle $\widehat{A} = (A^\dagger)^*$.

2.2 Special affine bundles

A vector space with a distinguished non-zero vector will be called *a special vector space*. A canonical example of a special vector space is $(\mathbb{R}, 1)$. Another example of a special vector space is $\mathbf{A}^\dagger = (A^\dagger, 1_A)$. A *special affine* space is an affine space modelled on a special vector space. Similarly, a vector bundle $\mathbf{V}$ with a distinguished non-vanishing section v will be called *a special vector bundle.* An affine bundle modelled on a special vector bundle will be called *a special affine bundle.* If $\mathbf{I}$ denotes the special vector space $(\mathbb{R}, 1)$ understood also as a special affine space, then $M \times \mathbf{I}$ is a special affine bundle over M. With some abuse of notation this bundle will be also denoted by $\mathbf{I}$. If $\mathbf{A}_i$, $i = 1, 2$, denote special affine bundles modelled on special vector bundles $\mathbf{V}_i$ with distinguished sections v_i then an affine bundle morphism $\varphi : \mathbf{A}_1 \to \mathbf{A}_2$ is a *morphism of special affine bundles* if

$$\varphi_{\mathsf{v}}(v_1) = v_2,$$

i.e. its linear part is a *morphism of special vector bundles.* The set of morphisms of special affine bundles $\mathbf{A}_i$ will be denoted by $\mathbf{Aff}(\mathbf{A}_1, \mathbf{A}_2)$. Similarly, bi-affine morphisms are called *special bi-affine* if they are special affine with respect to every argument separately.

In the category of special affine bundles there is a canonical notion of duality. The *special affine dual* $\mathbf{A}^\#$ of a special vector bundle $\mathbf{A} = (A, v_A)$ is an affine subbundle in $A^\dagger$ that consists of those affine functions on fibers of A the linear part of which maps v_A to 1, i.e. those that are special affine morphisms between $\mathbf{A}$ and $\mathbf{I}$:

$$\mathbf{A}^\# = \mathbf{Aff}(\mathbf{A}, \mathbf{I}).$$

The model vector bundle for the dual $\mathbf{A}^\#$ is the vector subbundle of $\mathbf{A}^\dagger$ of those functions which linear part vanishes on v_A. We see therefore that 1_A is a section of the model vector bundle since its linear part is identically 0. Therefore we can consider $\mathbf{A}^\#$ as a special affine bundle. The special affine duality is a true duality: in finite dimensions we have a canonical identification

$$(\mathbf{A}^\#)^\# \simeq \mathbf{A}.$$

Given a special affine bundle $\mathbf{A} = (A, v_A)$, the distinguished section v_A gives rise to an $\mathbb{R}$ action on A, $a_p \mapsto a_p + tv_A(p)$, whose fundamental vector field will be denoted by $\chi_{\mathbf{A}}$.

3 The geometry of affine values

3.1 The affine phase bundle

In the standard differential geometry many constructions are based on the algebra $C^\infty(M)$ of smooth functions on the manifold M. In the AV-geometry we replace $C^\infty(M)$ by the space of sections of certain affine bundle over M. Let $\zeta : \mathbf{Z} \to M$ be a one-dimensional affine bundle over the manifold M modelled on the trivial bundle $M \times \mathbb{R}$. Such a bundle will be called an *AV-bundle.* Since $\mathbf{Z}$ is modelled on the trivial bundle $M \times \mathbb{R}$, there is an action of $\mathbb{R}$ in every fiber of $\mathbf{Z}$. Therefore $\mathbf{Z}$ is also a principal $\mathbb{R}$-bundle. The fundamental vector field induced by the action will be denoted by $\chi_{\mathbf{Z}}$. The affine space $\mathsf{Sec}(\mathbf{Z})$ is modelled on $C^\infty(M)$. Any section σ of $\mathbf{Z}$ induces a trivialization

$$I_\sigma : \mathbf{Z} \ni z \longmapsto (\zeta(z), z - \sigma(\zeta(z))) \in M \times \mathbb{R}.$$

The above trivialization induces the identification between $\mathsf{Sec}(\mathbf{Z})$ and $C^\infty(M)$. We can go surprisingly far in many constructions replacing the ring of smooth functions with the affine space of sections of an AV-bundle. It is possible because many objects of standard differential geometry (like the cotangent bundle with its canonical symplectic form) have many properties that are conserved by certain affine transformations. To build an AV-analog of the cotangent bundle T^*M, let us define an equivalence relation in the set of all pairs (m, σ), where m is a point in M and σ is a section of ζ. Two pairs (m, σ) and (m', σ') are equivalent if $m' = m$ and $\mathrm{d}(\sigma' - \sigma)(m) = 0$. We have identified the section $\sigma' - \sigma$ of $pr_M : M \times \mathbb{R} \to M$ with a function on M for the purpose of evaluating the differential $\mathrm{d}(\sigma' - \sigma)(m)$. We denote by $\mathsf{P}\mathbf{Z}$ the set of equivalence classes. The class of (m, σ) will be denoted by $\mathbf{d}\sigma(m)$ and will be called the *differential* of σ at m. We will write $\mathbf{d}$ for the affine exterior differential to distinguish it from the standard d. We define a mapping $\mathsf{P}\zeta \colon \mathsf{P}\mathbf{Z} \to M$ by $\mathsf{P}\zeta(\mathbf{d}\sigma(m)) = m$. The bundle $\mathsf{P}\zeta$ is canonically an affine bundle modelled on the cotangent bundle $\pi_M \colon \mathsf{T}^*M \to M$ with the affine structure

$$\mathbf{d}\sigma_2(m) - \mathbf{d}\sigma_1(m) = \mathrm{d}(\sigma_2 - \sigma_1)(m).$$

This affine bundle is called the *phase bundle* of ζ. A section of $\mathsf{P}\zeta$ will be called an *affine 1-form*. Let $\alpha \colon M \to \mathsf{P}\mathbf{Z}$ be an affine 1-form and let σ be a section of ζ. The differential $\mathrm{d}(\alpha - \mathbf{d}\sigma)(m)$ does not depend on the choice of σ and will be called the *differential of α at m*. We will denote it by $\mathbf{d}\alpha(m)$. The differential of an affine 1-form $\alpha \in \mathsf{Sec}(\mathsf{P}\mathbf{Z})$ is an ordinary 2-form $\mathbf{d}\alpha \in \Omega^2(M)$. The corresponding *affine de Rham complex* looks now like

$$\mathsf{Sec}(\mathbf{Z}) \xrightarrow{\mathbf{d}} \mathsf{Sec}(\mathsf{P}\mathbf{Z}) \xrightarrow{\mathbf{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \Omega^3(M) \xrightarrow{\mathrm{d}} \dots$$

and consists of affine maps. This is an *affine complex* in the sense that its linear part is a complex of linear maps.

Like the cotangent bundle T^*M itself, its AV-analog $\mathsf{P}\mathbf{Z}$ is equipped with a canonical symplectic structure. For a chosen section σ of ζ we have the isomorphisms

$$\begin{aligned} I_\sigma &\colon \mathbf{Z} \to M \times \mathbb{R}, \\ I_{\mathbf{d}\sigma} &\colon \mathsf{P}\mathbf{Z} \to \mathsf{T}^*M, \end{aligned}$$

and for two sections σ, σ' the mappings $I_{\mathbf{d}\sigma}$ and $I_{\mathbf{d}\sigma'}$ differ by the translation by $\mathrm{d}(\sigma - \sigma')$, i.e.

$$I_{\mathbf{d}\sigma'} \circ I_{\mathbf{d}\sigma}^{-1} \colon \mathsf{T}^*M \to \mathsf{T}^*M$$

$$: \alpha_m \mapsto \alpha_m + \mathrm{d}(\sigma - \sigma')(m).$$

Now we use an affine property of the canonical symplectic form ω_M on the cotangent bundle: **translations in T^*M by a closed 1-form are symplectomorphisms**, to conclude that the two-form $I^*_{\mathbf{d}\sigma}\omega_M$, where ω_M is the canonical symplectic form on T^*M, does not depend on the choice of σ and therefore it is a canonical symplectic form on $\mathsf{P}\mathbf{Z}$. We will denote this form by $\omega_{\mathbf{Z}}$.

There is no canonical Liouville 1-form on $\mathsf{P}\mathbf{Z}$ (in the standard sense) which is the potential of the canonical symplectic form $\omega_{\mathbf{Z}}$ but there is such a form in the affine sense. The details can be found in [GGU2].

3.2 The canonical identification

Every special affine bundle $\mathbf{A} = (A, v_A)$ gives rise to a certain bundle of affine values $\mathsf{AV}(\mathbf{A})$. The total space of the bundle $\mathsf{AV}(\mathbf{A})$ is A and the base is $A/\langle v_A\rangle$ with the canonical projection from the space to the quotient. The meaning of the quotient is obvious: the class $[a_p]$ of $a_p \in A_p$ is the orbit $\{a_p + tv_A(p)\}$ of $\chi_{\mathbf{A}}$. For the reason of a further application we choose the distinguished section $v_{\mathsf{AV}(\mathbf{A})}$ of the model vector bundle characterized by $\chi_{\mathsf{AV}(\mathbf{A})} = -\chi_{\mathbf{A}}$. In the space of sections of $\mathsf{AV}(\mathbf{A})$ one can distinguish *affine sections*, i.e. such sections

$$\sigma : \mathbf{A}/\langle v_A\rangle \to \mathbf{A}$$

which are affine morphisms. The space of affine sections will be denoted by $\mathsf{AffSec}(\mathsf{AV}(\mathbf{A}))$.

In the theory of vector bundles there is an obvious identification between sections of the bundle E^* and functions on E which are linear along fibres. If φ is a section of E^*, then the corresponding function ι_φ is defined by the canonical pairing

$$\iota_\varphi(X) = \langle \varphi, X\rangle.$$

In the theory of special affine bundles we have an analog of the above identification:

$$\mathsf{Sec}(\mathbf{A}^{\#}) \simeq \mathsf{AffSec}(\mathsf{AV}(\mathbf{A})), \quad \mathbf{F}_\sigma \leftrightarrow \sigma,$$

where $\mathbf{F}_\sigma$ is the unique (affine) function on $\mathbf{A}$ such that $\chi_{\mathbf{A}}(\mathbf{F}_\sigma) = -1$, $\mathbf{F}_\sigma \circ \sigma = 0$. In local coordinates (x^i, s) on A, adapted to the structure of the bundle $\mathsf{AV}(\mathbf{A})$, i.e. such that (x^i) are the coordinates on $A/\langle v_A\rangle$ and $\partial_s = -\chi_{\mathbf{A}}$ (remember that the fundamental vector field of Y is the generator of $\exp(-tY)$), we have

$$\mathbf{F}_\sigma(x, s) = s - \sigma(x).$$

4 Brackets on affine bundles

A *Lie affgebra* is an affine space A with a bi-affine operation

$$[\cdot,\cdot] : A \times A \to \mathsf{V}(A)$$

satisfying the following conditions:

- skew-symmetry: $[a_1, a_2] = -[a_2, a_1]$
- Jacobi identity: $[a_1, [a_2, a_3]]^2_{\mathsf{V}} + [a_2, [a_3, a_1]]^2_{\mathsf{V}} + [a_3, [a_1, a_2]]^2_{\mathsf{V}} = 0.$

In the above formula $[\cdot,\cdot]^2_{\mathsf{V}}$ denotes the affine-linear part of the bracket $[\cdot,\cdot]$, i.e. we take a linear part with respect to the second argument. One can show that any skew-symmetric bi-affine bracket as above is completely determined by its affine-linear part. The detailed description of the concept of the Lie affgebra can be found in [GGU1].

A *Lie affgebroid* is an affine bundle with a Lie affgebra bracket on the space of sections

$$[\cdot,\cdot] : \mathsf{Sec}(A) \times \mathsf{Sec}(A) \to \mathsf{Sec}(\mathsf{V}(A))$$

and a morphism of affine bundles

$$\rho : A \to \mathsf{T}M$$

inducing a map from $\mathsf{Sec}(A)$ into vector fields on M such that

$$[a, fv]^2_{\mathsf{V}} = f[a, v]^2_{\mathsf{V}} + \rho(a)(f)v$$

for any smooth function f on M. Note that the same concept has been introduced by E. Martínez, T. Mestdag and W. Sarlet under the name of *affine Lie algebroid* [MMS].

Example 4.1. Given a fibration $\xi \colon M \to \mathbb{R}$, take the affine subbundle $A \subset \mathsf{T}M$ characterized by $\xi_*(X) = \partial_t$ for $X \in \mathsf{Sec}(A)$. Then the standard bracket of vector fields in A and $\rho \colon A \to \mathsf{T}M$, being just the inclusion, define on A a structure of a Lie affgebroid. This is the basic example of the concept of *affine Lie algebroid* developed in [SMM, MMS].

Let us remind that any affine bundle A is canonically embedded in the vector bundle $\widehat{A}$ being its vector hull. It turns out that there is a one-to-one correspondence between Lie affgebroid structures on A and Lie algebroid structures on the vector hull. We have therefore the following theorem (for the proof see [GGU1]):

Theorem 4.1. *For an affine bundle A the following are equivalent:*

- $[\cdot,\cdot] : \mathsf{Sec}(A) \times \mathsf{Sec}(A) \to \mathsf{Sec}(\mathsf{V}(A))$ *is a Lie affgebroid bracket on A;*

- $[\cdot,\cdot]$ *is the restriction of a Lie algebroid bracket* $[\cdot,\cdot]^\wedge\colon \mathsf{Sec}(\widehat{A}) \times \mathsf{Sec}(\widehat{A}) \to \mathsf{Sec}(\widehat{A})$.

Note that $1_A \in \mathsf{Sec}(A^\dagger)$ is a closed 'one-form' for $\widehat{A}$ which is therefore a particular *Jacobi algebroid* as defined in [GM] (or *generalized Lie algebroid* in the terminology of D. Iglesias and J. C. Marrero, see [IM]).

Definition 4.1. Given an AV-bundle $\mathbf{Z}$ over M, an *aff-Poisson (resp. aff-Jacobi) bracket* on $\mathbf{Z}$ is a Lie affgebra bracket

$$\{\cdot,\cdot\}\colon \mathsf{Sec}(\mathbf{Z}) \times \mathsf{Sec}(\mathbf{Z}) \to C^\infty(M)$$

such that

$$X_\sigma = \{\sigma,\cdot\}^2_{\mathsf{v}}\colon C^\infty(M) \to C^\infty(M)$$

is a derivation (resp., a first-order differential operator) for every $\sigma \in \mathsf{Sec}(\mathbf{Z})$. In the Poisson case, X_σ is a vector field on M called *the Hamiltonian vector field of* σ.

In the linear case there is a correspondence between Lie algebroid brackets and linear Poisson structures on the dual bundle. In the affine setting we have an analog of this correspondence. Let X be a section of $\mathsf{V}(\mathbf{A})$. Since $\mathsf{V}(\mathbf{A})$ is a vector subbundle in $\widehat{\mathbf{A}}$, the section X corresponds to a linear function $\iota^\dagger_X$ on $\mathbf{A}^\dagger$. The function $\iota^\dagger_X$ is invariant with respect to the vertical lift of the distinguished section 1_A of $\mathbf{A}^\dagger$. We see that the function $\iota^\dagger_X$ restricted to $\mathbf{A}^\#$ is constant on fibres of the projection $\mathbf{A}^\# \to \mathbf{A}^\#/\langle 1_A\rangle$ and defines an affine function $\iota^\#_X$ on the base of $\mathsf{AV}(\mathbf{A}^\#)$. Hence we have a canonical identification between

- sections X of $\mathsf{V}(\mathbf{A})$,
- linear functions $\iota^\dagger_X$ on $\mathbf{A}^\dagger$ which are invariant with respect to the vertical lift of 1_A,
- affine functions $\iota^\#_X$ on $\mathbf{A}^\#/\langle 1_A\rangle$.

Using the above identification we can formulate the following theorem (for the proof see [GGU2]):

Theorem 4.2. *There is a canonical one-to-one correspondence between Lie affgebroid brackets* $[\cdot,\cdot]_{\mathbf{A}}$ *on* $\mathbf{A}$ *and affine aff-Jacobi brackets* $\{\cdot,\cdot\}_{\mathbf{A}^\#}$ *on* $\mathsf{AV}(\mathbf{A}^\#)$*, uniquely defined by:*

$$\{\sigma,\sigma'\}_{\mathbf{A}^\#} = \iota^\#_{[\mathbf{F}_\sigma,\mathbf{F}_{\sigma'}]_{\mathbf{A}}}.$$

Moreover, $\mathbf{A}$ *is a special Lie affgebroid, i.e.* v *is a central section for the Lie algebroid bracket of the vector hull* $\widehat{\mathbf{A}}$*, exactly when the bracket* $\{\cdot,\cdot\}_{\mathbf{A}^\#}$ *is aff-Poisson.*

Example 4.2. Let $\mathbf{Z} = (Z, v)$ be an AV-bundle. The tangent bundle $\mathsf{T}\mathbf{Z}$ is equipped with the tangent $\mathbb{R}$-action. Dividing $\mathsf{T}\mathbf{Z}$ by the action we obtain the Atiyah algebroid of the principal $\mathbb{R}$-bundle $\mathbf{Z}$ which we denote by $\widetilde{\mathsf{T}}\mathbf{Z}$. It is a special Lie algebroid. The distinguished section of $\widetilde{\mathsf{T}}\mathbf{Z}$ is represented by the fundamental vector field $\chi_{\mathbf{Z}}$. The AV-bundle $\mathsf{AV}(\widetilde{\mathsf{T}}\mathbf{Z})$ is a bundle over $\mathsf{T}M$. The special affine dual for $\widetilde{\mathsf{T}}\mathbf{Z}$ is $\mathsf{P}\mathbf{Z} \times \mathbf{I}$. The AV-bundle $\mathsf{AV}((\widetilde{\mathsf{T}}\mathbf{Z})^{\#})$ is the trivial bundle over the affine phase bundle $\mathsf{P}\mathbf{Z}$ and the correspoding aff-Poisson bracket is the standard Poisson bracket on $\mathsf{P}\mathbf{Z}$ associated with the canonical symplectic form $\omega_{\mathbf{Z}}$ on $\mathsf{P}\mathbf{Z}$.

5 Examples

In the following we would like to present two examples of application of the AV-differential geometry to classical mechanics. In both examples the Lagrangian mechanics is associated with a certain special Lie affgebroid $\mathbf{A}$ and lagrangians are sections of $\mathsf{AV}(\mathbf{A})$. The Hamiltonian mechanics in turn is associated with the dual bundle $\mathbf{A}^{\#}$ and hamiltonians are sections of $\mathsf{AV}(\mathbf{A}^{\#})$.

5.1 Time-dependent mechanics

The space of events for the inhomogeneous formulation of the time-dependent mechanics is the space-time M fibrated over the time $\mathbb{T}$ being the affine $\mathbb{R}$. The fibration will be denoted by ξ. First-jets of this fibration form the infinitesimal (dynamical) configuration space. Since there is the distinguished vector field ∂_t on $\mathbb{T}$, the first-jets of the fibration over the time can be identified with those vectors tangent to M which project on ∂_t. Such vectors form an affine subbundle A of the tangent bundle $\mathsf{T}M$, modelled on the bundle $\mathsf{V}M$ of vertical vectors. In the standard formulation the bundle V^*M, dual to the bundle of vectors which are vertical with respect to the fibration over time, is the phase space for the problem. The phase space carries a canonical Poisson structure, but Hamiltonian fields for this structure are vertical with respect to the projection on time, so they cannot describe the dynamics. To solve this problem one needs the distinguished vector field ∂_t to be added to the Hamiltonian vector field to obtain the dynamics. This can be done correctly when the fibration over the time is trivial, i.e. when $M = Q \times \mathbb{T}$. Then we have the phase space $\mathsf{V}^*M = \mathsf{T}^*Q \times \mathbb{T}$ and hamiltonian

$$H : \mathsf{T}^*Q \times \mathbb{R} \ni (\alpha, t) \longmapsto H(\alpha, t) \in \mathbb{R}.$$

The phase dynamics is given by the Hamiltonian vector field X_{H_t} of $H_t = H(\cdot, t)$ calculated separately for every t and corrected by ∂_t:

$$X_{H_t} + \partial_t.$$

When the fibration is not trivial one has to choose a reference vector field that projects onto ∂_t. Changing the reference vector field means changing the hamiltonian.

The Lagrange formalism in the affine formulation originates on the trivial AV-bundle $\mathbf{A} = A_0 \times \mathbb{R}$ which is a subbundle in $\mathsf{T}M \times \mathbb{R}$. The subbundle A_0 is an affine subbundle in $\mathsf{T}M$ of those vectors that project onto ∂_t. Since $\mathsf{T}M \times \mathbb{R}$ is canonically a Lie algebroid with distinguished 1-cocycle dt, the bundle $\mathbf{A}$ is a special Lie affgebroid. Lagrangians are sections of $\mathsf{AV}(A)$ and, since the latter bundle is trivial, they are ordinary functions on A_0.

The Hamilton formalism now takes place not on the dual vector bundle V^*M of $\mathsf{V}M$, like in the classical approach, but on the dual AV-bundle $\mathsf{AV}(\mathbf{A}^\#) = \mathsf{AV}(\mathbf{A}_0^\dagger)$ which can be recognized as

$$\zeta : \mathsf{T}^*M \to \mathsf{T}^*M/\langle \xi^*(dt)\rangle$$

and which carries a canonical aff-Poisson structure induced from the canonical symplectic Poisson bracket on T^*M:

$$\{\sigma, \sigma'\} \circ \zeta = \{\mathbf{F}_\sigma, \mathbf{F}_{\sigma'}\}_M, \tag{5.1}$$

where $\{\cdot,\cdot\}_M$ is the canonical Poisson bracket on T^*M. The hamiltonians are sections of the bundle ζ.

To compare this with the standard approach, let us assume that we have a trivialization $M = Q \times \mathbb{T}$ of the space-time into a product of the space and the time. This induces the decomposition

$$\mathsf{T}^*M = \mathsf{T}^*Q \times \mathsf{T}^*\mathbb{T}$$

and the bundle ζ becomes

$$\zeta : \mathsf{T}^*M = \mathsf{T}^*Q \times \mathsf{T}^*\mathbb{T} \to \mathsf{T}^*Q \times \mathbb{T} = \mathsf{T}^*M/\langle \xi^*(dt)\rangle.$$

A time-dependent hamiltonian, which is a section of ζ, has the form:

$$\widehat{H}(\alpha, t) = (\alpha, t, -H(\alpha, t))$$

and the function on T^*M, that corresponds to it, reads

$$F_{\widehat{H}}(\alpha, t, s) = s + H(\alpha, t).$$

Finally, we obtain the dynamics

$$\Gamma = X_{\widehat{H}} = \zeta_*(X_{F_{\widehat{H}}}) = \zeta_*(X_{H_t} + \partial_t - \frac{\partial H}{\partial t}\partial_s) = X_{H_t} + \partial_t.$$

We see that we have recovered the correct dynamics. However, in our picture, the term ∂_t is not added 'by hand' but it is generated from $\widehat{H}$ by means of the aff-Poisson structure. Of course, if we have no decomposition into space and time, there is no canonical ∂_t on M and nothing canonical can be added by hand in the standard approach. This problem disappears in the aff-Poisson formulation. In this example, hamiltonians are sections of an AV-bundle and lagrangians are ordinary functions, however not on a vector but on an affine bundle.

Remark 5.1. The formula (5.1) gives an example of an affine Poisson reduction. The standard Poisson reduction deals with a fibration $\rho\colon M \to N$ and Poisson brackets $\{\cdot,\cdot\}_M$, $\{\cdot,\cdot\}_N$ on M and N, respectively. We say that $\{\cdot,\cdot\}_N$ is a reduction of $\{\cdot,\cdot\}_M$ if

$$\{\rho^* f, \rho^* g\}_M = \rho^*\{f, g\}_N. \tag{5.2}$$

In the AV geometry we have two AV-bundles $\mathbf{Z}, \mathbf{Y}$ with aff-Poisson brackets $\{\cdot,\cdot\}_{\mathbf{Z}}$, $\{\cdot,\cdot\}_{\mathbf{Y}}$ on sections of Z, Y, respectively. A fibration $\rho\colon Z \to Y$ is assumed to be an AV-morphism. The pull-back $\rho^*\sigma$ of a section is well defined and the condition (5.2) can be replaced by

$$\{\rho^*\sigma, \rho^*\sigma'\}_{\mathbf{Z}} = \rho^*\{\sigma, \sigma'\}_{\mathbf{Y}}.$$

We say that $\{\cdot,\cdot\}_{\mathbf{Y}}$ is an affine reduction (with respect to ρ) of the aff-Poisson structure $\{\cdot,\cdot\}_{\mathbf{Z}}$. In our example 5.1 we have $Z = \mathsf{T}^*M \times \mathbb{R}$, $Y = \mathsf{T}^*M$, and $\rho(a, r) = a - r dt$.

5.2 The inhomogeneous formulation of the Newtonian mechanics

The Newtonian space-time is a four-dimensional affine space N with the model vector space $\mathsf{V}(N)$. There is a distinguished non-zero covector $\tau \in \mathsf{V}(N)^*$ that measures time intervals between events:

$$\Delta t(x_1, x_2) = \langle \tau, x_2 - x_1 \rangle.$$

There is also an Euclidean metric g defined on the subspace $E_0 = \ker\tau$. It serves for measuring the spatial distance between symultaneous events. In the standard approach the dynamics of a massive particle is described in an inertial frame. An inertial frame is a class of inertial observers that move in the space-time with the same constant velocity. A level-1 set of τ represents velocities of inertial observers and velocities of particles.

When we fix an inerial frame u, the model vector space $\mathsf{V}(N)$ splits into $E_0 \times \mathbb{R}$:

$$\mathsf{V}(N) \ni v \longmapsto (v - \langle \tau, v\rangle u, \ \ \langle \tau, v\rangle)$$

and the dual $\mathsf{V}(N)^*$ splits into $E_0^* \times \mathbb{R}$:

$$\mathsf{V}(N)^* \ni \alpha \longmapsto (\iota^*(\alpha), \ \ \langle \alpha, u\rangle) \in E_0^* \times \mathbb{R},$$

where ι^* is the canonical projection which is dual to the embedding $\iota : E_0 \hookrightarrow \mathsf{V}(N)$. In the standard approach the phase space of a massive particle is $N \times E_0^*$, that is interpreted as the bundle which is dual to the bundle $N \times E_0$ of vectors tangent to N and vertical with respect to the projection on the time given by τ. The phase of a particle with a mass m moving with a relative velocity $v \in E_0$ in the field of forces with a potential $\varphi(x)$ is $(x,p) = (x, mg(v)) \in N \times E_0^*$ and the Hamilton function is

$$H_u = H_u(x,p) = \frac{1}{2m}p^2 + \varphi(x),$$

with $p^2 = \langle p, g^{-1}(p)\rangle$. The dynamics is generated from the hamiltonian H_u by means of the canonical Poisson structure of $N \times E_0^*$ obtained by reduction from $\mathsf{T}^*N \simeq N \times \mathsf{V}(N)^*$. We meet here the same problem as in the previous example: the Hamiltonian vector field generated from H_u is vertical with respect to the projection on the time. Here we have to add the constant vector field equal to u at every point to obtain the correct phase equations. Using the AV-geometry we can get rid of the inertial observer and find the correct aff-Poisson structure that generates the correct phase equations.

Analyzing transformation rules for the energy and the momentum while changing the observer, we obtain the gauge-independent (observer-independent) phase AV-bundle

$$\widehat{\mathsf{P}} \longrightarrow \mathsf{P}$$

where

$$\widehat{\mathsf{P}} = (N \times E_0^* \times \mathbb{R} \times E_1)/E_0 \quad \text{and} \quad \mathsf{P} = (N \times E_0^* \times E_1)/E_0$$

with respect to the gauge equivalence

$$\begin{aligned}(x,p,s,u'+v) \;\sim\; & (x',p',s,u') \Leftrightarrow x = x', \\ & p = p' + mg(v), \\ & s = s' + \langle p, v\rangle + \frac{1}{2}m\langle g(v), v\rangle.\end{aligned}$$

The phase AV-bundle comes together with a canonical aff-Poisson structure. This structure is inherited from the AV bundle $\mathsf{T}^*N \to \mathsf{T}^*N/\langle\tau\rangle \simeq N \times E_0^*$. We have seen in the previous example that such a bundle has a canonical aff-Poisson structure coming from the symplectic structure of the cotangent bundle. What we claim here is that the aff-Poisson structure survives the procedure of collecting phase bundles for all inertial frames and then dividing by transformation rules.

The potential $\varphi(x)$ gives rise to a well-defined Hamiltonian section $\widehat{H}$ of $\mathsf{AV}(\widehat{\mathsf{P}})$ which for the trivialization given by the inertial frame u

$$I_u\colon\ \widehat{\mathsf{P}} \longrightarrow N \times E_0^* \times \mathbb{R},$$

reads

$$H(x,p) = \frac{1}{2m}p^2 + \varphi(x).$$

The observed dynamics on the phase space $\mathsf{P} \simeq N \times E_0$ is

$$\dot{x} \;=\; \left(g^{-1}\left(\frac{p}{m}\right)\right) + u,$$

$$\dot{p} \;=\; -\frac{\partial \varphi}{\partial q}(q,t).$$

If we fix additionally an inertial observer from the class characterized by u, specifying a point x_0 in N, then N splits into $E_0 \times \mathbb{R}$ and the dynamics on $E_0 \times \mathbb{R} \times E_0^*$, with elements denoted by (q,t,p), reads

$$\begin{aligned}
\dot{q} &= \left(g^{-1}\left(\frac{p}{m}\right)\right), \\
\dot{t} &= 1, \\
\dot{p} &= -\frac{\partial \varphi}{\partial q}(q,t),
\end{aligned}$$

that agrees with the Newtonian picture.

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