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Poisson structures on foliated manifolds

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Abstract

We present a survey of results on relationships between a Poisson structure and a foliation. Namely, we discuss transversally-Poisson structures, leaf-wise Poisson structures and coupling Poisson structures. The latter extend Sternberg's symplectic form, which describes coupling between a particle and a field. The survey contains no new results, and is based on [4, 8, 9, 10].

1 Introduction

We present a survey of results on relationships between a Poisson structure and a foliation. This survey contains no new results, and is based on the following papers: [4, 8, 9, 10].

We suggest that it is interesting to study Poisson and related structures on foliated manifolds since these may be relevant to the study of physical systems depending on gauge parameters, which are the coordinates along the leaves of a foliation. In what follows we are interested only in mathematical aspects.

2 Calculus on foliated manifolds

In this preliminary section we prepare the necessary computational formulas. All the objects that we consider are differentiable of class C^∞.

Let us recall that a p-dimensional foliation $\mathcal{F}$ on a n-dimensional manifold M consists of the partition of M into maximal integral submanifolds (*leaves*) of an integrable, p-dimensional subbundle $F = T\mathcal{F}$, of the tangent bundle TM. By the classical Frobenius theorem, M is covered by cubical, coordinate domains with *adapted coordinates* (x^a, y^u) $(a = 1, ..., q = n - p;\ u = p + 1, ..., p + q)$, where the *slices* of $\mathcal{F}$ have the local equations $x^a = const.$, and y^u are coordinates along the leaves.

In foliation theory it is important to study the geometric objects that depend on the leaves i.e., locally, depend on the coordinates (x^a). They are called *basic*, *projectable* or *foliated* since they may be seen as either lifts from or projections on the space of leaves $M/\mathcal{F}$. Since the latter may be topologically complicated, its differential geometry may be defined as the study of projectable objects on the *foliated manifold* $(M, \mathcal{F})$. For instance, one may speak of projectable (foliated, basic) functions, differential forms, vector and tensor fields, vector and principal bundles, etc.

On a foliated manifold, we can do differential calculus in a way that takes into account the foliation. For this purpose, we choose a *normal bundle* $H = H\mathcal{F}$ of the foliation, such that

$$TM = H \oplus F. \tag{2.1}$$

Obviously H is isomorphic with the *transversal bundle* TM/F, and it is a foliated vector bundle. With adapted coordinates we also associate *adapted bases* of F, H:

$$F = span\left\{\frac{\partial}{\partial y^u}\right\}, \quad H = span\left\{X_a = \frac{\partial}{\partial x^a} - t_a^u \frac{\partial}{\partial y^u}\right\}, \tag{2.2}$$

where t_a^u are some local coefficients and the Einstein summation convention is used. By duality, we get

$$T^*M = H^* \oplus F^*, \tag{2.3}$$

and *adapted cobases*

$$H^* = ann\, F = span\{dx^a\}, \quad F^* = ann\, H = span\{\theta^u = dy^u + t_a^u dx^a\} \tag{2.4}$$

(by *ann* we denote the annihilator of a vector bundle).

Now, let $\Omega(M), \mathcal{V}(M)$ be the exterior algebras of differential forms and multivector fields on M, respectively. Then, decompositions (2.1), (2.3) yield bigradings

$$\Omega(M) = \sum_{k=1}^{n} \sum_{s+t=k} \Omega^{st}(M), \; \mathcal{V}(M) = \sum_{k=1}^{n} \sum_{s+t=k} \mathcal{V}^{st}(M), \tag{2.5}$$

where s is the H-degree and t is the F-degree.

The calculus operations become $\mathcal{F}$-related if we use bigradings (2.5). The operations we will need are the exterior differential and the Schouten-Nijenhuis bracket. If we look at a form $\omega \in \Omega^{st}$ and evaluate $d\omega$ on arguments $X_1, ..., X_\alpha \in \mathcal{V}^{10}$, $Y_1, ..., Y_\beta \in \mathcal{V}^{01}$, we see that non zero results may be obtained only for $(\alpha = s+1, \beta = t)$, $(\alpha = s, \beta = t+1,\ \alpha = s+2, \beta = t-1)$. Therefore (e.g., [6]),

$$d = d'_{10} + d''_{01} + \partial_{2,-1}, \tag{2.6}$$

where the lower indices are the degrees of the corresponding component operators. (It is essential to use the fact that the Lie bracket of tangent to the leaves vector fields also is tangent to the leaves.) Furthermore, the coboundary condition $d^2 = 0$ is equivalent to

$$\begin{gathered} d''^2 = 0,\ \partial^2 = 0,\ d'^2 + d''\partial + \partial d'' = 0, \\ d'd'' + d''d' = 0,\ \partial d' + d'\partial = 0. \end{gathered} \tag{2.7}$$

The operator d'' is the coboundary of the *leaf-wise de Rham cohomology*, which computes the cohomology of M with coefficients in the sheaf of germs of foliated functions [6]. On the other hand, one has the *basic de Rham cohomology*, which is the cohomology of the complex of basic forms with coboundary d.

Now, let us refer to the Schouten-Nijenhuis bracket. We recall the following general formula of Lichnerowicz:

$$i([P,Q])\varphi = (-1)^t(s+1)i(P)d(i(Q)\varphi) \\ +(-1)^s i(Q)d(i(P)\varphi) - i(Q)(i(P)d\varphi), \tag{2.8}$$

where $P \in \mathcal{V}^s(M), Q \in \mathcal{V}^t(M)$, and $\varphi \in \Omega^{s+t-1}(M)$. The operator i is the *interior product*.

Now, if P is of bidegree (a,b) $(a+b=s)$ and Q is of bidegree (h,k) $(h+k=t)$, the component $[P,Q]^{uv}$ $(u+v=s+t-1)$ is provided by formula (2.8) where $\varphi \in \Omega^{uv}(M)$. With (2.6), we see that the only possibilities to get non-zero components correspond to the replacement of d by d', d'', ∂ in (2.8), which leads to the cases

$$u=a+h-1, v=b+k;\ u=a+h, v=b+k-1; \\ u=a+h-2,\ v=b+k+1. \tag{2.9}$$

All the other components vanish because of degree incompatibility.

We will need more precise formulas for a Schouten-Nijenhuis bracket of two bivector fields.

Lemma 2.1. *For any bivector field $P \in \mathcal{V}^2(M)$ one has*

$$[P,P](\alpha,\beta,\gamma) = 2[d\gamma(\sharp_P\alpha, \sharp_P\beta) - (L_{\sharp_P\gamma}P)(\alpha,\beta)], \quad \alpha,\beta,\gamma \in \Omega^1(M), \tag{2.10}$$

where L denotes the Lie derivative and $\sharp_P : T^*M \to TM$ is defined by $\beta(\sharp_P\alpha) = P(\alpha,\beta)$.

Proof. It is easy to see that the two sides of (2.10) are tensor fields. Hence, it is enough to check the formula for α,β,γ equal to differentials of local coordinates on M, which is simple. □

Corollary 2.1. *For any two bivector fields $P_1, P_2 \in \mathcal{V}^2(M)$ one has*

$$[P_1,P_2](\alpha,\beta,\gamma) = d\gamma(\sharp_{P_1}\alpha, \sharp_{P_2}\beta) + d\gamma(\sharp_{P_2}\alpha, \sharp_{P_1}\beta) \\ -(L_{\sharp_{P_1}\gamma}P_2)(\alpha,\beta) - (L_{\sharp_{P_2}\gamma}P_1)(\alpha,\beta). \tag{2.11}$$

Proof. Polarize formula (2.10). □

If the bivector field $P \in \mathcal{V}^2(M)$ is *regular*, i.e., $s = rank\, P = dim\, D = const.$, $D = im\, \sharp_P$, we may compute $[P, P]$ as follows. Choose a decomposition

$$TM = E \oplus D,\ T^*M = E^* \oplus D^* \quad (E^* = ann\, D,\ D^* = ann\, E). \tag{2.12}$$

Then, we have an isomorphism $\sharp_P : ann\, E \to D$, with an inverse $-\flat_P : D \to ann\, E$, and there exists a well defined differential 2-form $\theta \in \Gamma \wedge^2 D^*$ of rank s (Γ denotes the space of cross sections of a vector bundle) defined by

$$\theta(X, Y) = P[\flat_P(p_D X), \flat_P(p_D Y)] \tag{2.13}$$

(p denotes projections). Conversely, (2.13) allows us to reconstruct P from θ, such that $ker\, \sharp_P = ann\, D$. We will say that θ is *equivalent to P modulo E*.

Lemma 2.2. *If P is a regular bivector field and θ is equivalent to P modulo E, then*

$$[P, P](\alpha, \beta, \gamma) = \begin{cases} 0 & \text{if } \beta, \gamma \in ann\, D, \\ 2\gamma([\sharp_P \alpha, \sharp_P \beta]) & \text{if } \alpha, \beta \in ann\, E,\ \gamma \in ann\, D, \\ 2d\theta(\sharp_P \alpha, \sharp_P \beta, \sharp_P \gamma) & \text{if } \alpha, \beta, \gamma \in ann\, E. \end{cases} \tag{2.14}$$

Proof. We use the decomposition (2.12). If not all the arguments are in $ann\, E$ the result immediately follows from either (2.10) or the following Gelfand-Dorfman expression of the Schouten-Nijenhuis bracket of two bivector fields [1]

$$[P, P](\alpha, \beta, \gamma) = 2 \sum_{Cycl(\alpha,\beta,\gamma)} < \gamma, \sharp_P(L_{\sharp_P \alpha} \beta) > . \tag{2.15}$$

For arguments in $ann\, E$, we have $\alpha = \flat_P X, \beta = \flat_P Y, \gamma = \flat_P Z$, with $X, Y, Z \in \Gamma D$, and (2.15) yields

$$[P, P](\alpha, \beta, \gamma) = -2 \sum_{Cycl(\alpha,\beta,\gamma)} < L_X \beta, Z >$$

$$= 2 \sum_{Cycl(X,Y,Z)} \{X(\theta(Y, Z)) - \theta([X, Y], Z)\}.$$

□

In the case of a foliated manifold $(M, \mathcal{F})$ with a normal bundle H, if $P \in \mathcal{V}^2(M)$ is an arbitrary bivector field, it has a decomposition

$$P = P'_{20} + \bar{P}_{11} + P''_{02}, \tag{2.16}$$

where the indices denote the bidegree of the components, and we can compute the corresponding decomposition of the Schouten-Nijenhuis bracket $[P, P]$ by applying formula (2.10) to 1-forms $\alpha, \beta, \gamma \in \Omega^{10}(M)$ and $\lambda, \mu, \nu \in \Omega^{01}(M)$. The results are contained in the following formulas

$$\begin{aligned}
&[P', P']_{30}(\alpha, \beta, \gamma) = 2[d'\gamma(\sharp_{P'}\alpha, \sharp_{P'}\beta) - (L_{\sharp_{P'}\gamma}P')(\alpha, \beta)], \\
&[P', P']_{21}(\alpha, \beta, \lambda) = 2\partial\lambda(\sharp_{P'}\alpha, \sharp_{P'}\beta) = -2\lambda([\sharp_{P'}\alpha, \sharp_{P'}\beta]), \\
&[P', P']_{12}(\alpha, \lambda, \mu) = 0, \quad [P', P']_{03}(\lambda, \mu, \nu) = 0,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
&[\bar{P}, \bar{P}]_{30}(\alpha, \beta, \gamma) = 0, \\
&[\bar{P}, \bar{P}]_{21}(\alpha, \beta, \lambda) = 2[d''\lambda(\sharp_{\bar{P}}\alpha, \sharp_{\bar{P}}\beta) - (L_{\sharp_{\bar{P}}\lambda}\bar{P})(\alpha, \beta)], \\
&[\bar{P}, \bar{P}]_{12}(\alpha, \lambda, \mu) = -2[d'\mu(\sharp_{\bar{P}}\lambda, \sharp_{\bar{P}}\alpha) + (L_{\sharp_{\bar{P}}\mu}\bar{P})(\alpha, \lambda)], \\
&[\bar{P}, \bar{P}]_{03}(\lambda, \mu, \nu) = 2[\partial\nu(\sharp_{\bar{P}}\lambda, \sharp_{\bar{P}}\mu) - (L_{\sharp_{\bar{P}}\nu}\bar{P})(\lambda, \mu)],
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
&[P'', P'']_{30}(\alpha, \beta, \gamma) = 0, \quad [P'', P'']_{21}(\alpha, \beta, \lambda) = 0, \\
&[P'', P'']_{12}(\alpha, \lambda, \mu) = 0, \\
&[P'', P'']_{03}(\lambda, \mu, \nu) = 2[d''\nu(\sharp_{P''}\lambda, \sharp_{P''}\mu) - (L_{\sharp_{P''}\nu}P'')(\lambda, \mu)],
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
[P', \bar{P}]_{30}(\alpha, \beta, \gamma) &= d''\gamma(\sharp_{P'}\alpha, \sharp_{\bar{P}}\beta) - d''\gamma(\sharp_{P'}\beta, \sharp_{\bar{P}}\alpha) \\
&\quad -(L_{\sharp_{P'}\gamma}\bar{P})(\alpha, \beta) - (L_{\sharp_{\bar{P}}\gamma}P')(\alpha, \beta), \\
[P', \bar{P}]_{21}(\alpha, \beta, \lambda) &= d'\lambda(\sharp_{P'}\alpha, \sharp_{\bar{P}}\beta) - d'\lambda(\sharp_{P'}\beta, \sharp_{\bar{P}}\alpha) \\
&\quad -(L_{\sharp_{\bar{P}}\lambda}P')(\alpha, \beta), \\
[P', \bar{P}]_{12}(\alpha, \lambda, \mu) &= \partial\mu(\sharp_{P'}\alpha, \sharp_{\bar{P}}\lambda) - (L_{\sharp_{\bar{P}}\mu}P')(\alpha, \lambda) \\
&= -(L_{\sharp_{P'}\alpha}\bar{P})(\lambda, \mu), \quad [P', \bar{P}]_{03}(\lambda, \mu, \nu) = 0,
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
&[P', P'']_{30}(\alpha, \beta, \gamma) = 0, \quad [P', P'']_{03}(\lambda, \mu, \nu) = 0, \\
&[P', P'']_{21}(\alpha, \beta, \lambda) = -(L_{\sharp_{P''}\lambda}P')(\alpha, \beta) \\
&[P', P'']_{12}(\alpha, \lambda, \mu) = d'\mu(\sharp_{P'}\alpha, \sharp_{P''}\lambda) - (L_{\sharp_{P''}\mu}P')(\alpha, \lambda) \\
&\qquad = (L_{\sharp_{P'}\alpha}P'')(\lambda, \mu),
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
&[\bar{P}, P'']_{30}(\alpha, \beta, \gamma) = 0, \quad [\bar{P}, P'']_{21}(\alpha, \beta, \lambda) = 0, \\
&[\bar{P}, P'']_{12}(\alpha, \lambda, \mu) = d''\mu(\sharp_{\bar{P}}\alpha, \sharp_{P''}\lambda) - (L_{\sharp_{\bar{P}}\mu}P'')(\alpha, \lambda) \\
&\qquad -(L_{\sharp_{P''}\mu}\bar{P})(\alpha, \lambda), \\
&[\bar{P}, P'']_{03}(\lambda, \mu, \nu) = d'\nu(\sharp_{\bar{P}}\lambda, \sharp_{P''}\mu) - d'\nu(\sharp_{\bar{P}}\mu, \sharp_{P''}\lambda) \\
&\qquad -(L_{\sharp_{\bar{P}}\nu}P'')(\lambda, \mu) - (L_{\sharp_{P''}\nu}\bar{P})(\lambda, \mu).
\end{aligned} \tag{2.22}$$

3 Transversally-Poisson structures

In discussing Poisson geometry on a foliated manifold $(M, \mathcal{F})$ it is natural to look at:

a) structures that have the Poisson property in the transversal geometry of $\mathcal{F}$;

b) leaf-wise Poisson structures;

c) Poisson structures on M with special relationships to the foliation.

In the present section we consider transversally-Poisson structures for a given foliation $\mathcal{F}$. We described such structures in [4] and [8].

Definition 3.1. A *transversally-Poisson structure* of the foliation $\mathcal{F}$ on the manifold M is a bivector field $P \in \mathcal{V}^2(M)$ such that

$$\{f, g\} = P(df, dg) \qquad f, g \in C^\infty(M) \tag{3.1}$$

restricts to a Lie algebra bracket on $C^\infty_{pr}(M)$, where the index *pr* denotes projectability.

Proposition 3.1. *The bivector field $P \in \mathcal{V}^2(M)$ defines a transversally-Poisson structure of the foliation $\mathcal{F}$ iff*

$$(L_Y P)|_{ann\, F} = 0 \;\; [P, P]|_{ann\, F} = 0 \tag{3.2}$$

for all $Y \in \Gamma F$.

Proof. If $f, g \in C^\infty_{pr}(M)$, $\forall\, Y \in \Gamma F$, $Yf = 0$, and one has

$$(L_Y P)(df, dg) = Y(P(df, dg)) = Y\{f, g\}\ ,$$

hence, the first condition (3.2) is equivalent with $\{f, g\} \in C^\infty_{pr}(M)$.

The second condition (3.2) is a direct consequence of the formula

$$[P, P](df, dg, dh) = 2 \sum_{Cycl(f,g,h)} \{\{f, g\}, h\}, \tag{3.3}$$

which is a consequence of Lemma 2.1. □

Remark 3.1. If we choose a normal bundle of $\mathcal{F}$ and write P under the form (2.16), the first condition (3.2) is equivalent with $(L_Y P')|_{ann\, F} = 0$. Furthermore, formulas (2.17)-(2.22) show that the second condition (3.2) is equivalent with $[P', P']|_{ann\, F} = 0$, modulo the first condition (3.2). Thus, transversally-Poisson is a property of the transversal component P' of the bivector field P.

We may define the Hamiltonian vector field of a foliated function f with respect to a transversally-Poisson structure P by $X_f = i(df)P$, and, because of (3.2), it will be a projectable vector field.

Furthermore, we give the following definition.

Definition 3.2. The generalized distribution $\mathcal{D}$ defined by

$$\mathcal{D}_x = span\{\ Y(x), X_f(x)\ /\ Y \in \Gamma F,\ f \in C^\infty_{pr}(M)\} \quad (x \in M)$$

is called the *characteristic distribution* of the transversally-Poisson structure P.

The following proposition shows the geometry behind a transversally-Poisson structure.

Proposition 3.2. *The characteristic distribution $\mathcal{D}$ of a transversally-Poisson structure of a foliation is completely integrable, and each leaf Σ of $\mathcal{D}$ is a presymplectic manifold, with a presymplectic $2-$form of kernel $F|_\Sigma$.*

Proof. Brackets of the form $[Y_1, Y_2]$, $[Y, X_f]$, $Y_1, Y_2, Y \in \Gamma F$, $f \in C^\infty_{pr}(M)$ belong to $\mathcal{D}$ because $\mathcal{F}$ is a foliation, and because X_f is projectable. Then, the Jacobi identity for the Poisson structure on $C^\infty_{pr}(M)$ yields $([X_f, X_g] - X_{\{f,g\}})(h) = 0$, $\forall f, g, h \in C^\infty_{pr}(M)$, whence

$$[X_f, X_g] = X_{\{f,g\}} + Y, \quad Y \in \Gamma F\ . \tag{3.4}$$

Hence, the distribution $\mathcal{D}$ is involutive.

Furthermore, let U be an $\mathcal{F}-$adapted coordinate neighborhood, and $p : U \longrightarrow V$, $V = U/U \cap \mathcal{F}$ the submersion onto the corresponding space of slices. The distribution $\mathcal{D}|_U$ projects onto the symplectic distribution $p_*(\mathcal{D})$ of the Poisson structure induced by the first term of (2.16) on V. It follows that $p_*(\mathcal{D})$ has a constant dimension along the integral paths of the vector fields $p_* X_f$ $(f \in C^\infty_{pr}(M))$. Hence $\mathcal{D} = p_*^{-1}(p_*(\mathcal{D}))$ has a constant dimension along the integral paths of the vector fields X_f. $\mathcal{D}$ also has a constant dimension along the integral paths of vector fields $Y \in \Gamma F$ because $p_*(\mathcal{D})$ does not change along such paths.

Accordingly, the complete integrability of $\mathcal{D}$ follows from a version of the Frobenius-Sussmann-Stefan theorem (see Theorem 2.9$''$ of [7]).

The leaves Σ of the characteristic distribution $\mathcal{D}$ are immersed submanifolds of M,foliated by the restriction of $\mathcal{F}$, and are sent by the submersion p above to open symplectic submanifolds of the symplectic leaves σ of the projection of the first term of (2.16). The symplectic form of σ lift to a global, closed $2-$form λ of Σ with the kernel $F|_\Sigma$. $\square$

Now, let us notice that the Hamiltonian vector fields of foliated functions do not depend on the leaf-wise component P'' of the decomposition (2.16), and the same holds for the characteristic distribution $\mathcal{D}$.

The idea of associating a Hamiltonian vector field to a foliated function is natural, if we intend to see the leaf-wise coordinates as gauge parameters of a dynamical system, where motion affects the gauge parameters but is not affected by the latter. (For instance, the temperature of a body moving with high friction may be seen as a gauge parameter of this kind.) Indeed, then, the Hamiltonian function of the system will have to be a projectable function (independent of the gauge parameters) but, the dynamical vector field has to be a usual vector field on phase space in order to also define the evolution of the gauge parameters.

The previous considerations lead to [8]

Definition 3.3. On a foliated manifold $(M, \mathcal{F})$, two transversally-Poisson structures P_1, P_2 are *transversally equivalent* if $\sharp_{P_1}|_{ann\,F} = \sharp_{P_2}|_{ann\,F}$. A family of transversally equivalent transversally-Poisson structures is a *Hamiltonian structure* of the foliation $\mathcal{F}$.

It is easy to understand that a Hamiltonian structure of $\mathcal{F}$ is equivalent with a vector bundle morphism $\chi : ann\,F \to TM$ such that, if we define the Hamiltonian vector fields of foliated functions by $X_f = \chi(df)$, the formula

$$\{f, g\} = X_f g \tag{3.5}$$

defines a Poisson algebra structure on $C^\infty_{pr}(M)$. (In particular, χ is skew symmetric.) Namely, $\chi = \sharp_P|_{ann\,F}$ for any structure P of the equivalence class that defines the Hamiltonian structure.

A Hamiltonian structure has a characteristic distribution $\mathcal{D}$, which is the common characteristic distribution of all the corresponding equivalent transversally-Poisson structures P, and $\mathcal{D} = F + im\,\chi$. It is easy to see that $F \cap im\,\chi = \chi(ann\,\mathcal{D})$. (Since $F \subseteq \mathcal{D}$, $ann\,\mathcal{D} \subseteq ann\,F$.) Indeed, $\alpha \in ann\,\mathcal{D}$ iff $\alpha = p^*(\lambda)$ for some $\lambda \in ker\,\sharp_{p_*(P')}$, where P' is defined by (2.16), and then

$$p_*(\chi(\alpha)) = p_*(i(\alpha)P') = i(\lambda)(p_*(P')) = 0.$$

This implies $\chi(ann\,\mathcal{D}) \subseteq (im\,\chi) \cap F$. On the other hand, if $\chi(\alpha) \in F$, we must have $\alpha = p^*(\lambda)$ where $\lambda \in ker\,\sharp_{P'}$, and this justifies the converse inclusion.

Example 3.1. Let $\mathcal{D}$ be a coisotropic foliation of dimension $n + k$ $(k \leq n)$ of a symplectic manifold M of dimension $2n$, with the symplectic form ω. It is well known that the ω-orthogonal distribution of $\mathcal{D}$ is tangent to a foliation $\mathcal{F}$, and that, $\forall x \in M$, there exist local coordinates (x^a, x^u, y^i) around x such that $a = 1, \ldots, p := n - k$, $u = p + 1, \ldots, n$, $i = 1, \ldots, n$, $x^a = const.$ are the local equations of $\mathcal{D}$, and the symplectic form has the canonical expression

$$\omega = \sum_{a=1}^{p} dx^a \wedge dy^a + \sum_{u=p+1}^{n} dx^u \wedge dy^u. \tag{3.6}$$

(This result is a theorem of Lie [2].) The local equations of the foliation $\mathcal{F}$ are $x^a = const.$, $x^u = const.$, $y^u = const,$, and the computation of the hamiltonian vector field X^ω_f of an $\mathcal{F}$-foliated function (via (3.6)) shows that X^ω_f is an $\mathcal{F}$-foliated vector field tangent to the leaves of $\mathcal{D}$. Therefore, $\chi = -\flat_\omega^{-1}|_{ann\,F}$ is a hamiltonian structure of the foliation $\mathcal{F}$ with the presymplectic foliation $\mathcal{D}$. Moreover, in this case we have $F \subseteq im\,\chi$. The Poisson structure defined by the original symplectic structure ω is one of the $\mathcal{F}$-transversally-Poisson structures that defines χ.

The following definition introduces the case where the evolution of the gauge parameters is subject to linear constraints.

Definition 3.4. A Hamiltonian structure χ of a foliation $\mathcal{F}$ is *transversal* (to $\mathcal{F}$) if there exists a normal bundle H of $\mathcal{F}$ such that $im\,\chi \subseteq H$. The distribution H will be called an *image extension* of χ. (It is possible to have more than one image extension.) A transversal Hamiltonian structure of $\mathcal{F}$ is a *tame* structure if all the brackets of differentiable vector fields that belong to $im\,\chi$ are contained in an image extension H.(In the tame case, only such image extensions will be used.)

Proposition 3.3. *Let χ be a transversal Hamiltonian structure of the foliation $\mathcal{F}$ with image extension H. Then χ is tame with image extension H iff the Nijenhuis tensor N_H of the projection $p_H : TM \to TM$ of $TM = H \oplus F$ onto H satisfies the condition*

$$N_H(\chi\alpha, \chi\beta) = 0, \qquad \forall \alpha, \beta \in \Omega^{10}(M). \tag{3.7}$$

Proof. Since $p_H^2 = p_H$, the required Nijenhuis tensor is

$$N_H(X, Y) = [p_H X, p_H Y] - p_H[p_H X, Y] - p_H[X, p_H Y] + p_H[X, Y], \tag{3.8}$$

where $X, Y \in \mathcal{V}(M)$. Generally, χ has local equations

$$\chi(dx^a) = h^{ab} X_b + k^{au} \frac{\partial}{\partial y^u}, \tag{3.9}$$

and, if H is an image extension, $k^{au} = 0$. In view of (3.4) and of Definition 3.4, χ is tame iff

$$\chi(dh^{ab}) = [\chi(dx^a), \chi(dx^b)],$$

which is equivalent to

$$h^{ac} h^{be} \tau^u_{ce} = 0, \quad \tau^u_{ce} = \frac{\partial t^u_c}{\partial x^e} - \frac{\partial t^u_e}{\partial x^c} + t^v_c \frac{\partial t^u_e}{\partial y^v} - t^v_e \frac{\partial t^u_c}{\partial y^v}. \tag{3.10}$$

Condition (3.7) is the invariant form of (3.10). □

Transversal Hamiltonian structures χ admit an extended Hamiltonian formalism.

Fix an image extension H of χ, and use the decomposition (2.6) of the exterior differential. χ is defined for any differential form $\alpha \in \Omega^{10}(M)$ and $\chi(\alpha) \in \Gamma H$. Accordingly, $\forall f \in C^\infty(M)$, we get a Hamiltonian vector field

$$X'_f = \chi(d'f), \tag{3.11}$$

and $\forall f, g \in C^\infty(M)$ we get an *extended Poisson bracket*

$$\{f, g\}' := X'_f g = < \chi(d'f), dg > = < \chi(d'f), d'g > \tag{3.12}$$

$$= - < d'f, \chi(d'g) >= -\{g, f\}'.$$

On the other hand, for $X \in \Gamma H$ and $\alpha \in \Omega^{10}(M)$, we may decompose the Lie derivative as

$$L_X\alpha = L'_X\alpha + L''_X\alpha, \tag{3.13}$$

where

$$L'_X = i(X)d' + d'i(X), \; L''_X = i(X)d'' + d''i(X), \tag{3.14}$$

and we can extend the Gelfand-Dorfman-Schouten-Nijenhuis bracket [1] to arbitrary forms $\alpha, \beta, \gamma \in \Omega^{10}(M)$ by putting

$$[h, k]'(\alpha, \beta, \gamma) := \sum_{Cycl(\alpha,\beta,\gamma)} \{< kL'_{h(\alpha)}\beta, \gamma > + < hL'_{k(\alpha)}\beta, \gamma >\}, \tag{3.15}$$

where $\chi, \kappa : ann\, F \to H$ are skew symmetric morphisms. The extended bracket is trilinear over $C^\infty(M)$. Hence, for a Hamiltonian structure χ we have

$$[\chi, \chi]'(\alpha, \beta, \gamma) = 0 \quad \forall \alpha, \beta, \gamma \in \Omega^{10}(M),$$

since this is true for basic forms. Using (3.12) and (3.14), we see that the previous property implies that, $\forall f, g, l \in C^\infty(M)$, one has

$$\sum_{Cycl(f,g,l)} [\{\{f, g\}', l\}' + d'^2 f(X'_g, X'_l)] = \frac{1}{2}[\chi, \chi]'(d'f, d'g, d'l) = 0. \tag{3.16}$$

Proposition 3.4. *If χ is a tame Hamiltonian structure on $(M, \mathcal{F})$ the Poisson bracket $\{\ ,\ \}'$ is a Poisson bracket on M.*

Proof. For any normal bundle H of the foliation $\mathcal{F}$ one gets

$$d'^2 f(X, Y) =< d''f, N_H(X, Y) >, \quad \forall f \in C^\infty(M), \forall X, Y \in \Gamma E, \tag{3.17}$$

where N_H is the Nijenhuis tensor (3.8). Indeed, if $X, Y \in \Gamma H$, (3.8) yields

$$N_H(X, Y) = p_F[X, Y], \tag{3.18}$$

where p_F is the projection onto the second term of the decomposition $TM = H \oplus F$. On the other hand,

$$d'^2 f(X, Y) = d(d'f)(X, Y) = XYf - YXf - < d'f, [X, Y] >$$

$$= [X, Y]f - (p_H[X, Y])f =< df, p_F[X, Y] >=< d''f, p_F[X, Y] > .$$

Thus, (3.17) is justified, and the conclusion follows from the characterization (3.7) of the tame hamiltonian structures and formula (3.16). □

Now, back to a general, transversal, Hamiltonian structure χ of the foliation $\mathcal{F}$, with the image extension H, we can use the component L'_X ($X \in \mathcal{V}^{10}(M)$) in order to define a bracket of 1-forms $\alpha, \beta \in \mathcal{V}^{10}$ similar to that encountered on a Poisson manifold. Namely,

$$\{\alpha,\beta\}' = L'_{\chi\alpha}\beta - L'_{\chi\beta}\alpha - d' < \chi\alpha, \beta > . \tag{3.19}$$

Formula (3.19) implies

$$\{f\alpha, g\beta\}' = fg\{\alpha,\beta\}' + f(h(\alpha)g)\beta - g(h(\beta)f)\alpha \quad (f,g \in C^\infty(M)), \tag{3.20}$$

whence we see that the bracket (3.19) is skew symmetric because it is such for foliated 1-forms, where it reduces to the Poisson bracket of 1-forms on a local transversal submanifold of $\mathcal{F}$.

If we take $\alpha = d'f, \beta = d'g$ in (3.19), and evaluate on an argument $X \in \mathcal{V}^{10}(M)$, after some technical calculations, we get

$$\{d'f, d'g\}' = d'\{f,g\}' + L_{X'_g} d''f - L_{X'_f} d''g. \tag{3.21}$$

Definition 3.5. Let χ be a transversal Hamiltonian structure of the foliation $\mathcal{F}$, and H an image extension of χ. The function $f \in C^\infty(M)$ will be called a *distinguished function* for χ if i) $d'f$ is a foliated 1-form, ii) $im\,\chi \subseteq ker\,d'^2 f$.

We denote by $\Omega^0_d(M)$ the space of distinguished functions. Any foliated function is distinguished, $\forall f,g \in \Omega^0_d(M)$, $\{f,g\}' \in C^\infty_{pr}(M)$, and, in view of (3.16), the extended Poisson bracket of distinguished functions satisfies the Jacobi identity. Therefore, $\Omega^0_d(M)$ is a Poisson algebra and $C^\infty_{pr}(M)$ is an ideal of the former. Furthermore, if $f,g \in C^\infty_{pr}(M)$, one gets $L_{X'_g} d''f = 0$, and (3.21) implies

$$\{d'f, d'g\}' = d'\{f,g\}', \quad \forall f,g \in \Omega^0_d. \tag{3.22}$$

Then, if we take $f,g \in \Omega^0_d(M)$, $l \in C^\infty(M)$ in (3.16) and use (3.18), we get

$$X'_{\{f,g\}'} = p_H[X'_f, X'_g] \quad f,g \in \Omega^0_d(M). \tag{3.23}$$

We may say that the transversal Hamiltonian structure χ defines a Poisson structure on the *non-holonomic submanifold* (i.e., a non-completely integrable distribution) H of M.

Proposition 3.5. *Let χ be a tame hamiltonian structure of the foliation $\mathcal{F}$, H an image extension of χ, and P' the Poisson structure defined by the brackets $\{\ ,\ \}'$. Then, the triple $(ann\,F, \{\ ,\ \}', \chi)$, with the bracket* (3.19)*, is a Lie subalgebroid of the cotangent Lie algebroid $(T^*M, \{\ ,\ \}_{P'}, \sharp_{P'})$.*

Proof. The bracket $\{\ ,\ \}_{P'}$ is given by (3.19), without accents and with χ replaced by $\sharp_{P'}$. Since $\sharp_{P'}|_{ann\,F} = \chi$,

$$\{\alpha,\beta\}' = \{\alpha,\beta\}_{P'}, \qquad \forall \alpha,\beta \in \Omega^1_{pr}(M).$$

Then, (3.20) implies

$$\{f\alpha, g\beta\}' = \{f\alpha, g\beta\}_{P'}, \quad \forall f,g \in C^\infty(M), \forall \alpha,\beta \in \Omega^1_{pr}(M).$$

□

4 Leaf-tangent Poisson structures

In this section we discuss Poisson structures along the leaves of a foliation [9].

Definition 4.1. The Poisson structure defined by the bivector field $P \in \mathcal{V}^2(M)$ is *leaf-tangent* to $\mathcal{F}$ if its symplectic leaves are submanifolds of the leaves of $\mathcal{F}$, equivalently, if the leaves of $\mathcal{F}$ are Poisson submanifolds of (M, P).

Obviously, P is $\mathcal{F}$-leaf-tangent iff $P \in \Gamma \wedge^2 F$ ($F = T\mathcal{F}$), equivalently, for any normal bundle H, in (2.16), one has $P'_{1,1} = 0$, $\bar{P}_{1,2} = 0$.

Accordingly, formulas (2.17)-(2.22) show that $P \in \Gamma \wedge^2 F$ is a Poisson bivector field on M iff

$$d''\nu(\sharp_P\lambda, \sharp_P\mu) - (L_{\sharp_P\nu}P)(\lambda, \mu) = 0, \quad \forall \lambda, \mu, \nu \in \Omega^{01}(M), \tag{4.1}$$

i.e., iff the restrictions of P to the leaves are Poisson bivector fields of the leaves.

Remark 4.1. If P is $\mathcal{F}$-leaf-tangent, $\mathcal{F}$ may be seen as a *regularizing foliation* of the symplectic foliation $\mathcal{S}$ of P. There exists a global, numerical invariant of a Poisson structure on a manifold M^n, the *regularizing dimension*, which is the smallest possible dimension $p \leq n$ of a regularizing foliation. The role of this invariant is yet to be studied.

Example 4.1. Put $\mathbb{R}^n = \mathbb{R}^3 \times \mathbb{R}^{n-3}$ and take the Poisson structure P defined by the Lie-Poisson structure of the factor $\mathbb{R}^3$ seen as the dual of the Lie algebra $so(3)$. P is leaf-tangent to the foliation defined by the factor $\mathbb{R}^3$ of $\mathbb{R}^n$. The regularizing dimension of P is 3.

Example 4.2. [10] Let $p : \mathbb{G}^* \to B$ be a bundle of Lie coalgebras (i.e., the dual of a bundle $p : \mathbb{G} \to B$ of Lie algebras) over a manifold B. Then the Lie-Poisson structures of the fibers yield a leaf-tangent Poisson structure $\mathbb{L}$ of $\mathbb{G}^*$. If (x^i) are local coordinates on B and (y_a) are linear coordinates along the fibers of $\mathbb{G}^*$, one has

$$\mathbb{L} = \frac{1}{2}\alpha_{ab}^c(x^i) y_c \frac{\partial}{\partial y_a} \wedge \frac{\partial}{\partial y_b}, \tag{4.2}$$

where $\alpha_{ab}^c(x^i)$ are the structural constants of the corresponding fibers of $\mathbb{G}$. Equivalently, $\forall z \in \mathbb{G}^*$ and for fiber-wise linear functions on $\mathbb{G}^*$ seen as elements $X, Y \in \mathbb{G}_{p(z)}$,

$$\mathbb{L}_z(X, Y) = < z, C_{p(z)}(X, Y) >, \tag{4.3}$$

where the "tensor field" $C \in \Gamma[(\wedge^2\mathbb{G}^*) \otimes (\mathbb{G})]$ is defined by

$$C_{p(z)}(X, Y) = [X, Y]_{\mathbb{G}_{p(z)}}. \tag{4.4}$$

The properties of a leaf-tangent Poisson structure reflect corresponding properties along the leaves. We discuss Poisson cohomology as an example.

Lemma 4.1. *Let P be a leaf-tangent Poisson structure on $(M, \mathcal{F})$ and let σ be the Lichnerowicz coboundary operator of P. Then, for every normal bundle H of $\mathcal{F}$, one has a decomposition*

$$\sigma = \sigma'_{-1,2} + \sigma''_{01}, \tag{4.5}$$

where the indices denote the bidegree, and

$$\sigma'^2 = 0,\ \sigma''^2 = 0,\ \sigma' \circ \sigma'' + \sigma'' \circ \sigma' = 0. \tag{4.6}$$

Proof. Since $\sigma Q = -[P, Q]$, $Q \in \mathcal{V}^{hk}(M)$ (e.g., [7]), (2.9) shows that σ may have only components of bidegree $(-1, 2)$, $(0, 1)$, $(-2, 3)$. We will see that the $(-2, 3)$-component vanishes.

Indeed, $\sigma_{-2,3}$ is obtained by computing $i([P, Q])\varphi$, $Q \in \mathcal{V}^{h,k}(M)$, $\varphi \in \Omega^{h-2,k+3}(M)$ via (2.8). Since in this case $i(Q)\varphi = 0$ (it should be of bidegree $(-2, 3)$), we get

$$i([P, Q])\varphi = i(Q)[di(P)\varphi - i(P)d\varphi], \tag{4.7}$$

where only the component ∂ of d may bring a non zero contribution.

To continue, first look at the evaluation of a Lie derivative $L_Y\psi$, where $Y \in \Gamma F$ and $\psi \in \Omega^{st}(M)$ for arguments $X_1, ..., X_u \in \mathcal{V}^{10}_{pr}(M)$, $Y_1, ..., Y_v \in \mathcal{V}^{01}(M)$. We see that $L_Y\psi$ may have components of bidegree equal either to (s, t) or to $(s-1, t+1)$ only.

Therefore,

$$i(Q)L_Y\psi = i(Q)[di(Y) + i(Y)d]\psi = 0, \tag{4.8}$$

$\forall Q \in \mathcal{V}^{hk}(M)$, $\forall \psi \in \Omega^{h-2,k+3} \oplus \Omega^{h-2,k+2}$, $\forall Y \in \Gamma F$.

Furthermore, from (4.8), using the the fact that

$$i(P \wedge Q) = i(Q)i(P) \qquad (P, Q \in \mathcal{V}^*(M))$$

and a form $\varphi \in \Omega^{h-2,k+3}$, we get

$$i(Q)[i(Y_1 \wedge Y_2)d - di(Y_1 \wedge Y_2)]\varphi = i(Y_2 \wedge Q)i(Y_1)d\varphi - i(Q)di(Y_2)i(Y_1)\varphi$$

$$= -i(Y_2 \wedge Q)di(Y_1)\varphi + i(Q)i(Y_2)di(Y_1)\varphi = 0.$$

Since P is spanned over $\mathbb{R}$ by wedge products $Y_1 \wedge Y_2$ of tangent vector fields of $\mathcal{F}$, the result of (4.7) is zero.

Properties (4.6) follow from $\sigma^2 = 0$. □

Properties (4.6) show that $(\mathcal{W}^{hk} = \mathcal{V}^{kh}(M), \sigma)$ is a double, semipositive, cochain complex, and the cohomology of such a complex is the limit of a spectral sequence [6].

Proposition 4.1. *Let P be a leaf-tangent Poisson structure on $(M, \mathcal{F})$ and denote by $H^h_P(\mathcal{F}, P)$ the cohomology spaces of the cochain complex $(\mathcal{V}^{0*}, \sigma'')$. Then, the Poisson cohomology of P is the limit of a spectral sequence (E^{hk}_r, d_r) where*

$$E^{hk}_2 = \mathcal{V}^{k0}(M) \otimes_{\mathbb{R}} H^h_P(\mathcal{F}, P) \tag{4.9}$$

and d_2 is induced by the operator σ'.

Proof. The spaces

$$\mathcal{W}_l(M) = \oplus_{k \geq l} \oplus_h \mathcal{V}^{hk}(M)$$

yield a regular filtration of the Lichnerowicz-Poisson complex $(\mathcal{V}(M), \sigma)$. The required spectral sequence is the spectral sequence (E^{hk}_r, d_r) defined by this filtration. ¿From the definition of a spectral sequence we get

$$E^{kh}_0 = \mathcal{V}^{hk}(M), \quad d_0 = 0.$$

The cohomology of E_0 yields

$$E^{kh}_1 = \mathcal{V}^{hk}(M), \quad d_1 = \sigma''.$$

Then, the cohomology spaces of the complex E_1 are the spaces given by formula (4.9), and d_2 is induced by σ'. □

Remark 4.2. The leaf-tangent Poisson bivector field $P \in \Gamma \wedge^2 F$ may be seen as a Poisson bivector of the Lie algebroid F defined by the foliation. As such, it induces a Lie algebroid structure on the dual bundle F^*, and $H^h_P(\mathcal{F}, P)$ are the cohomology spaces of this Lie algebroid.

5 Coupling Poisson structures

Now, we proceed with a discussion of certain aspects of the geometry of general Poisson structures on a foliated manifold. More precisely, we will discuss the *coupling* situation, a generalization of the symplectic structure that describes the coupling of a particle and a field discovered by S. Sternberg [5]. The notion of a coupling Poisson structure on a fiber bundle was defined and studied by Y. Vorobiev [10], who used it in order to get information about Poisson structures in the neighborhood of a symplectic leaf.

In what follows we present general results on coupling Poisson structures on foliated manifolds [9]. (In [9] the coupling property is also extended to Jacobi structures.) The notation is that of Section 1, we consider the manifold M, the foliation $\mathcal{F}$, the normal bundle H, and the bivector field P written under the form (2.16).

Definition 5.1. The bivector field P is *$\mathcal{F}$-almost coupling via H* if

$$\sharp_P(ann\,F) \subseteq H. \tag{5.1}$$

The bivector field P is *$\mathcal{F}$-coupling* if $\sharp_P(ann\,F)$ is a normal bundle H of $\mathcal{F}$.

With (2.16), the almost coupling condition is equivalent to $\bar{P}_{11} = 0$, hence, with the condition $\sharp_P(ann\,H) \subseteq F$. The coupling condition is equivalent with

$$dim(\sharp_P(ann\,F)) = q, \tag{5.2}$$

where q is the codimension of $\mathcal{F}$. The first term of (5.2) is the rank of the term P' of (2.16) for any choice of H, hence, coupling may exist only if q is even. In the case of a coupling field P, we use the normal bundle $H = \sharp_P(ann\,F)$.

Example 5.1. Let P be an arbitrary Poisson bivector field on the foliated manifold $(M, \mathcal{F})$. Let S be a symplectic leaf of P, embedded in M and transversal to $\mathcal{F}$. Then condition (5.2) holds on S, hence, on an open neighborhood V of S, and P is $\mathcal{F}$-coupling on V. In particular [10], for P and S as above, there exists a tubular neighborhood V of S where P is coupling for the fibers of the tubular structure of V.

Proposition 5.1. *An almost coupling bivector field P is Poisson iff*

$$\begin{aligned}
&d'\gamma(\sharp_{P'}\alpha, \sharp_{P'}\beta) - (L_{\sharp_{P'}\gamma}P')(\alpha, \beta) = 0,\\
&(L_{\sharp_{P''}\lambda}P')(\alpha, \beta) + \lambda([\sharp_{P'}\alpha, \sharp_{P'}\beta]) = 0,\\
&(L_{\sharp_{P'}\alpha}P'')(\lambda, \mu) = 0,\\
&d''\nu(\sharp_{P''}\lambda, \sharp_{P''}\mu) - (L_{\sharp_{P''}\nu}P'')(\lambda, \mu) = 0,
\end{aligned} \tag{5.3}$$

$\forall \alpha, \beta, \gamma \in \Omega^{10}(M)$, $\forall \lambda, \mu, \nu \in \Omega^{01}(M)$.

Proof. Use (2.17)-(2.22) to express $[P, P] = 0$ for

$$P = P' + P''. \tag{5.4}$$

□

The last condition (5.3) means that the component P'' is an $\mathcal{F}$-leaf-tangent Poisson bivector field.

Proposition 5.2. [10]. *A coupling Poisson structure of a foliated manifold $(M, \mathcal{F})$ is equivalent with a triple (P'', H, σ), where P'' is a leaf-tangent Poisson structure, H is a normal bundle of $\mathcal{F}$ and σ is a non-degenerate cross section of $\wedge^2(ann\,F)$ such that*

$$d'\sigma = 0, \tag{5.5}$$

$$\sharp_{P''}\{d[\sigma(X, Y)]\} = -p_F[X, Y], \quad \forall X, Y \in \mathcal{V}^{10}_{pr}(M), \tag{5.6}$$

$$L_X P'' = 0, \qquad \forall X \in \mathcal{V}^{10}_{pr}(M). \tag{5.7}$$

Proof. The coupling condition defines H,which enables us to write (5.4) and get P'' such that the last condition (5.3) holds. Furthermore, P' is a regular bivector field with $im\,\sharp_{P'} = H$, and we may write $P' = \sharp_{P'}(\sigma) = \sharp_P(\sigma)$, where $\sigma \in \wedge^2(ann\,F)$ is equivalent with P' mod. F and has the maximal rank. Then, for the 1-forms of bidegree (10) of (5.3) we may write

$$\alpha = \flat_\sigma X,\ \beta = \flat_\sigma Y,\ \gamma = \flat_\sigma Z, \quad X, Y, Z \in \mathcal{V}^{10}(M), \tag{5.8}$$

where X, Y, Z are uniquely defined.

By (2.14), the first condition (5.3) becomes (5.5).

Furthermore, conditions (5.3) are tensorial. In particular, the second condition (5.3) holds iff it holds for $X, Y \in \mathcal{V}^{10}_{pr}(M)$ in (5.8), and the definition of the Lie derivative shows the equivalence of the second condition (5.3) with (5.6).

Finally, the third condition (5.3) is equivalent to (5.7) because, for a projectable vector field X, $L_X P''$ vanishes if at least one of its arguments is of bidegree (10).

Now, notice that we may define a triple (P'', H, σ) for any coupling bivector field P, and one has an isomorphism $\flat_\sigma : H \to ann\,F$. Conversely, given such a triple, we get

$$P(\xi, \eta) = \sigma(\flat_\sigma^{-1}\xi', \flat_\sigma^{-1}\eta') + P''(\xi'', \eta''), \tag{5.9}$$

where

$$\xi = \xi' + \xi'',\ \eta = \eta' + \eta'', \quad \xi', \eta' \in ann\,F,\ \xi'', \eta'' \in ann\,H,$$

and the conditions stated in Proposition 5.2 imply (5.3). □

Remark 5.1. If the triple (P'', H, σ) satisfies all the conditions of Proposition 5.2, these conditions are also satisfied by any triple $(P'', H, \sigma + \epsilon\tau)$ where $\tau \in \wedge^2(ann\,F)$ is closed and $\epsilon \in \mathbb{R}$. If ϵ is small enough, $\sigma + \epsilon\tau$ is non degenerate on H, and the new triple also provides a coupling Poisson structure.

Remark 5.2. If the coupling Poisson tensor field P of (5.4) is defined by a symplectic form ω, P'' is equivalent with a closed 2-form θ of bidegree (02) that defines a symplectic structure on each leaf of $\mathcal{F}$. Then, if $\mathcal{F}$ is a fibration, we are in the case discovered by Sternberg [5].

The following result is about projectable coupling Poisson structures (5.4). Projectability holds iff $(L_Y P')(\alpha, \beta) = 0$, $\forall Y \in \Gamma F$, $\forall \alpha, \beta \in \Gamma_{pr}(ann\,F)$.

Proposition 5.3. *P given by* (5.4) *is a projectable coupling Poisson bivector field iff i) P'' is Poisson, ii) H is integrable; iii) the mod. F equivalent* 2*-form σ of P' is a transversal symplectic form of $\mathcal{F}$; iv)* (5.7) *holds.*

Proof. If P' is projectable, the same holds for the equivalent 2-form σ, and we see that (5.5) implies iii) and that (5.6) implies the integrability of H. Conversely, iii), iv) imply (5.5), (5.7) and the projectability of σ and P', which, together with ii), shows that the two sides of (5.6) are zero. □

Proposition 5.3 tells that projectable, $\mathcal{F}$-coupling Poisson tensors exist only on locally product manifolds M with structural foliations $\mathcal{F}, \mathcal{H}$ where $\mathcal{H}$ is a leaf-wise symplectic foliation.

Remark 5.3. If P given by (5.4) is Poisson, almost coupling and projectable, P' is a Poisson bivector field too, since by (2.17) the first two conditions (5.3) imply $[P', P'] = 0$. In the terminology of Section 2, P' defines a *tame Hamiltonian structure* of $\mathcal{F}$. Conversely, if we have such a tame structure, with a corresponding Poisson structure P' and a leaf-tangent Poisson structure P'', and if the third condition (5.3) holds, we get a projectable, almost coupling, Poisson structure on $(M, \mathcal{F})$. Of course, we may always take $P'' = 0$, hence, the tame Hamiltonian structures of $\mathcal{F}$ and the projectable, almost coupling, Poisson structures of $(M, \mathcal{F})$ are equivalent objects (not in a one-to-one correspondence, however).

6 Vorobiev-Poisson structures

In this section we give a presentation of a class of coupling Poisson structures defined by Vorobiev [10].

Let $p : \mathbb{G}^* \to B$ be a bundle of Lie coalgebras such that: i) B is a symplectic manifold with the symplectic form ω; ii) the dual Lie algebras bundle $\mathbb{G} \to B$ is the kernel of the (surjective) anchor $\rho : A \to TB$ of a *transitive Lie algebroid* $p : A \to B$ [3].

Let

$$A = Q \oplus \mathbb{G} \tag{6.1}$$

be a splitting of the vector bundle A, and $p_Q, p_{\mathbb{G}}$ the corresponding natural projections. Then $\rho|_Q : Q \to TB$ is an isomorphism and we denote by $\gamma : TB \to Q$ its inverse i.e.,

$$\gamma(\rho(s)) = p_Q(s), \qquad \forall s \in A. \tag{6.2}$$

Let $U \subseteq B$ be an open local-trivialization neighborhood of all the vector bundles above and (x^i) $(i = 1, ..., n = dim\, B)$ local coordinates on U. Then $\mathbf{q}_i = \gamma(\partial/\partial x^i)$ is a local basis of Q, and we may complete it by a local basis $(\mathbf{g}_a)$ of $\mathbb{G}$ $(a = 1, ..., k = rank\, \mathbb{G})$ to a local basis of A. Let (θ^a) be the dual basis of $(\mathbf{g}_a)$ for the vector bundle $\mathbb{G}^*$. These bases define fiber-wise local coordinates (y^a) and (y_a) on $\mathbb{G}$, $\mathbb{G}^*$, respectively. Since $\mathbb{G} = ker\, \rho$ and ρ is a morphism of Lie algebroids, the local expressions of the Lie bracket of A must be of the form

$$\begin{gathered}[\mathbf{g}_a, \mathbf{g}_b]_A = \alpha^c_{ab}(x)\mathbf{g}_c, \;\; [\mathbf{g}_a, \mathbf{q}_i]_A = \beta^c_{ai}(x)\mathbf{g}_c, \\ [\mathbf{q}_i, \mathbf{q}_j]_A = \gamma^c_{ij}(x)\mathbf{g}_c + \gamma^h_{ij}(x)\mathbf{q}_h.\end{gathered} \tag{6.3}$$

(The Einstein summation convention is in use.) We will freely use the following identifications:

$$\eta^a \frac{\partial}{\partial y^a} \Leftrightarrow \eta^a \mathbf{g}_a, \;\; \mu_a \frac{\partial}{\partial y_a} \Leftrightarrow \mu_a \theta_a. \tag{6.4}$$

Since $\mathbb{G} = ker\, \rho$, the formula

$$\nabla_X \eta = [\gamma(X), \eta]_A, \quad X \in \mathcal{V}^1(B),\; \eta \in \Gamma\mathbb{G}, \tag{6.5}$$

defines a connection of the vector bundle $\mathbb{G}$, which yields a dual connection on $\mathbb{G}^*$ and connections on all the associated tensor bundles of $\mathbb{G}$. All are denoted again by ∇, except for situations where we want to emphasize the connection ∇^* on $\mathbb{G}^*$. With (6.3), the local components of ∇ are given by

$$\nabla_{\frac{\partial}{\partial x^i}} \mathbf{g}_a = \Gamma^b_{ai} \mathbf{g}_b, \quad \Gamma^b_{ai} = -\beta^b_{ai}. \tag{6.6}$$

The Jacobi identity of the A-bracket yields

$$\nabla C = 0, \tag{6.7}$$

where the tensor C is defined by (4.4). Furthermore, since

$$\rho([\gamma(X), \gamma(Y)]_A) = [X, Y],$$

we get

$$p_Q[\gamma(X), \gamma(Y)]_A = \gamma([X, Y]). \tag{6.8}$$

Formula (6.8) yields

$$R_\nabla(X, Y)\eta = [p_{\mathbb{G}}[\gamma(X), \gamma(Y)]_A, \eta]_A, \tag{6.9}$$

where R_∇ is the curvature of the connection ∇.

Finally, we write

$$T\mathbb{G}^* = \mathcal{H} \oplus \mathcal{V}, \tag{6.10}$$

with the projections $p_\mathcal{H}, p_\mathcal{V}$, where $\mathcal{V}$ is tangent to the fibers and $\mathcal{H}$ is the horizontal distribution of ∇^*.

On $\mathbb{G}^*$, we have a triple (P'', H, σ) as in Proposition 5.2, where $P'' = \mathbb{L}$, $\mathbb{L}$ being the leaf-tangent Poisson bivector field defined by (4.3), $H = \mathcal{H}$ and

$$\sigma_z(\mathcal{X}, \mathcal{Y}) = \omega_{p(z)}(X, Y) - z(p_{\mathbb{G}}[\gamma(X), \gamma(Y)]_A), \tag{6.11}$$

where $z \in \mathbb{G}^*$, $\mathcal{X}, \mathcal{Y} \in \Gamma\mathcal{H}$ are the horizontal lifts of $X = p_*\mathcal{X}, Y = p_*\mathcal{Y}$. The definition of σ is completed by asking it to have bidegree (20) with respect to (6.10).

The distribution $\mathcal{H}$ and the form σ are defined by the local formulas

$$\mathcal{H} = span\{\mathcal{X}_i = \frac{\partial}{\partial x^i} + \Gamma^b_{ai} y_b \frac{\partial}{\partial y_a}\}, \tag{6.12}$$

$$\sigma_z(\mathcal{X}_i, \mathcal{X}_j) = \omega_{ij}(x) - \gamma^c_{ij} y_c, \tag{6.13}$$

where $\mathcal{X}_i$ is the horizontal lift of $\partial/\partial x^i$, the coefficients γ, Γ are those of (6.3) and (6.6), and ω_{ij} are the natural local components of ω on B. We see that the horizontal lifts of vector fields of B are projectable with respect to the vertical foliation $\mathcal{V}$ of $\mathbb{G}^*$.

Proposition 6.1. [10] *There exists a neighborhood $\mathcal{U}$ of B, as the zero section of $\mathbb{G}^*$, where the triple $(\mathbb{L}, \mathcal{H}, \sigma)$ defines a coupling Poisson bivector field.*

Proof. Using (6.12), one gets

$$\mathcal{X}(< z, \eta >) =< z, \nabla_X \eta >, \tag{6.14}$$

where $z \in \Gamma\mathbb{G}^*, \eta \in \Gamma\mathbb{G},\ p(z(x)) = p(\eta(x)), x \in B$, and then

$$d\sigma(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = d\omega(X, Y, Z) - \sum_{Cycl(X,Y,Z)} < z, p_{\mathbb{G}}[\gamma(X), [\gamma(Y), \gamma(Z)]_A)]_A >= 0,$$

since $d\omega = 0$ and $[\,,\,]_A$ satisfies the Jacobi identity. Thus, (5.5) holds.

Furthermore, by using the identification (6.4), (6.12) and (6.14), we get

$$(L_{\mathcal{X}}\mathbb{L})_z(\eta, \nu) =< z, (\nabla_X C)(\eta, \nu) >, \tag{6.15}$$

and (5.7) follows from (6.7).

Finally, (4.2), (4.4), and (6.9) yield

$$\mathbb{L}_z(p_{\mathbb{G}}[\gamma(X), \gamma(Y)]_A, \eta) =< z, [p_{\mathbb{G}}[\gamma(X), \gamma(Y)]_A, \eta]_A >$$

$$=< z, R_\nabla(X, Y)\eta >= - < R_{\nabla^*}(X, Y)z, \eta >,$$

which is equivalent to

$$\sharp_{\mathbb{L}_z}(p_{\mathbb{G}}[\gamma(X), \gamma(Y)]_A) = -R_{\nabla^*}(X, Y)z = p_{\mathcal{V}}[\mathcal{X}, \mathcal{Y}](z).$$

On the other hand, $\forall \mu \in \Gamma\mathbb{G}$ the (01)-component of $d < z, \mu_{p(z)} >$ with respect to (6.10) identifies with μ, and (6.11) yields

$$\sharp_{\mathbb{L}_z}\{d[\sigma(\mathcal{X}, \mathcal{Y})]\} = p_{\mathcal{V}}[\mathcal{X}, \mathcal{Y}],$$

which is (5.6) in our case.

Therefore, since σ is non degenerate on a neighborhood $\mathcal{U}$ of the zero section of $\mathbb{G}^*$, a corresponding coupling Poisson structure on $\mathcal{U}$ exists. □

Definition 6.1. The coupling Poisson structure defined by Proposition 6.1 on the neighborhood $\mathcal{U}$ of B will be called a *Vorobiev-Poisson structure*.

The Vorobiev-Poisson structure of a given bundle $\mathbb{G}^*$ is unique up to equivalence. Indeed, one has:

Proposition 6.2. [10] *The Vorobiev-Poisson structures defined by two splittings*

$$A = Q \oplus \mathbb{G}, \quad A = \tilde{Q} \oplus \mathbb{G} \tag{6.16}$$

on neighborhoods $\mathcal{U}_1, \mathcal{U}_2$ *of* B *in* $\mathbb{G}^*$ *are Poisson-equivalent in a neighborhood* $V \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$.

Proof. The notation below is that of Proposition 6.1 with a tilde for everything related to the second splitting.

The difference $\phi = \tilde{\gamma} - \gamma$ is a $\mathbb{G}$-valued 1-form on B and, if $s \in \Gamma A$ has the two decompositions

$$s = p_{\mathbb{G}}(s) + p_Q(s) = \tilde{p}_{\mathbb{G}}(s) + p_{\tilde{Q}}(s),$$

and (6.2) implies

$$\tilde{p}_{\mathbb{G}}(s) = p_{\mathbb{G}}(s) - \phi(\rho(s)), \quad p_{\tilde{Q}}(s) = p_Q(s) + \phi(\rho(s)). \tag{6.17}$$

Furthermore, on $\mathbb{G}^*$ we get a scalar 1-form $\psi \in ann\,\mathcal{V}$ defined by

$$\psi_z(\mathcal{X}) = < z, \phi_{p(z)}(X) >, \qquad z \in \mathbb{G}^*. \tag{6.18}$$

We define a family of splittings $A = Q_t \oplus \mathbb{G}$ by the projectors

$$p_{Q_t} = p_Q + t(\phi \circ \rho), \qquad t \in \mathbb{R}, \tag{6.19}$$

which is such that $Q_0 = Q$, $Q_1 = \tilde{Q}$, and the corresponding family P_t of Vorobiev-Poisson bivector fields defined by the triples $(\mathbb{L}, \mathcal{H}_t, \sigma_t)$ of the connection ∇^t defined by (6.5) for Q_t.

It is easy to compute the connection coefficients of ∇^t and we get

$$\mathcal{H}_t = span\{\mathcal{X}_{t,i} = \mathcal{X}_i + t\alpha^b_{ac}\phi^c_i y_b \frac{\partial}{\partial y_a}\}, \tag{6.20}$$

where the components ϕ^c_i are given by

$$\phi(\frac{\partial}{\partial x^i}) = \phi^c_i \mathbf{g}_c. \tag{6.21}$$

The corresponding basis of $ann\,\mathcal{H}_t = \mathcal{V}^*$ consists of the forms

$$\mu_{t,a} = \mu_a - t\alpha^b_{ac}\phi^c_i y_b dx^i, \tag{6.22}$$

where $\mu_a = \mu_{0,a}$. From (6.20) we get the horizontal lift of $X \in TB$ to $\mathcal{H}_t$

$$\mathcal{X}_t(z) = \mathcal{X}(z) - t\, coad_{\phi(X)}(\mathbb{E}(z)), \quad z \in \mathbb{G}^*,\, \mathbb{E}(z) = y_a \frac{\partial}{\partial y_a} \Leftrightarrow y_a\theta^a \tag{6.23}$$

($\mathbb{E}$ is the infinitesimal homothety of $\mathbb{G}^*$). Then, (6.22) define a bijection $\lambda \leftrightarrow \lambda_t$ between $ann\,\mathcal{H}$ and $ann\,\mathcal{H}_t$ given by:

$$\lambda_t = \lambda - tL_{\sharp_{\mathbb{L}}\lambda}\psi, \tag{6.24}$$

where ψ is the 1-form (6.18).

Now, since the differences $\tilde{\mathcal{X}} - \mathcal{X}, \tilde{\mathcal{Y}} - \mathcal{Y}$ are vertical, and using (6.14) and (6.17), we get

$$\sigma_t(\mathcal{X}_t, \mathcal{Y}_t) = \sigma(\mathcal{X}, \mathcal{Y}) - td\psi(\mathcal{X}, \mathcal{Y}) - t^2\mathbb{L}(\phi(X), \phi(Y)). \tag{6.25}$$

We define a time-dependent vector field $\Xi_t \in \Gamma(\mathcal{H}_t)$ by

$$\Xi_t = \sharp_{P_t}\psi, \;\Leftrightarrow\; \flat_{\sigma_t}\Xi_t = i(\Xi_t)\sigma_t = -\psi. \tag{6.26}$$

The vector field Ξ_t yields the autonomous vector field $\tilde{\Xi} = \Xi_t + \partial/\partial t$ on $\mathbb{G}^* \times \mathbb{R}$, and we prove that the flow Φ_t of $\tilde{\Xi}$ preserves the lift of the tensor field P_t to $\mathbb{G}^* \times \mathbb{R}$. Then, the projection of the diffeomorphism Φ_1 onto $\mathbb{G}^*$ will be the required equivalence of coupling Poisson structures. The proof will be accomplished by showing that

$$L_{\tilde{\Xi}}P_t = 0 \tag{6.27}$$

holds on $\mathbb{G}^* \times \mathbb{R}$. Condition (6.27) obviously holds if one of the arguments is dt. In the other cases, careful computations show that (6.27) is a consequence of conditions (5.3) for the bivector fields P_t, $\forall t \in \mathbb{R}$ [9]. $\square$

Remark 6.1. The equivalence of Poisson structures given by Proposition 6.2 does not preserve the foliation $\mathcal{V}$.

The Vorobiev-Poisson structures provide geometric information on the embedded leaves S of a Poisson manifold (M, P). The cotangent Lie algebroid structure $T^*M|_S \to S$ and the kernel of the anchor of this algebroid is the *conormal bundle* of S, i.e., the annihilator of TS in $T^*M|_S$.

Let NS be a normal bundle of S (i.e., $TM|_S = TS \oplus NS$) and U a tubular neighborhood of S with the fibers tangent to NS. At the points of S there exist local adapted coordinates (x^α, x^κ) $(\alpha = 1, ..., codim(S); \kappa = codim(S) + 1, ..., dim(M))$ such that the local equations of S are $x^\alpha = 0$ and

$$N^*S = ann(TS) = span\{dx^\alpha|_S\}, \;\; T^*S = ann(NS) = span\{dx^\kappa|_S\}. \tag{6.28}$$

The vector bundles N^*S and T^*S may play the role of $\mathbb{G}$ and Q of Vorobiev's construction and the bases (6.28) may play the role of the bases $(\mathbf{g}_a)$, $(\mathbf{q}_i)$ of (6.3). If $P^{\alpha\beta}, P^{\alpha\kappa}, P^{\kappa\nu}$ are the local components of P with respect to the local coordinates defined above, one has $P^{\alpha\beta}|_S = P^{\alpha\nu}|_S = 0$, whence

$$\frac{\partial P^{\alpha\beta}}{\partial x^\kappa}|_S = 0, \quad \frac{\partial P^{\alpha\nu}}{\partial x^\kappa}|_S = 0.$$

The brackets (6.3) of the present case will be

$$\begin{array}{c} \{dx^\alpha|_S, dx^\beta|_S\} = \frac{\partial P^{\alpha\beta}}{\partial x^\gamma}|_S dx^\gamma|_S, \; \{dx^\alpha|_S, dx^\kappa|_S\} = \frac{\partial P^{\alpha\kappa}}{\partial x^\gamma}|_S dx^\gamma|_S, \\ \{dx^\kappa|_S, dx^\nu|_S\} = \frac{\partial P^{\kappa\nu}}{\partial x^\gamma}|_S dx^\gamma|_S + \frac{\partial P^{\kappa\nu}}{\partial x^\theta}|_S dx^\theta|_S, \end{array} \tag{6.29}$$

where β, γ have the same domain as α and ν, θ have the same domain as κ.

Therefore, one has a Vorobiev-Poisson structure on a neighborhood of S and, in view of (4.2), (6.12) and (6.13), the associated triple is given by

$$\begin{array}{l} P'' = \frac{1}{2}\xi^\gamma \frac{\partial P^{\alpha\beta}}{\partial x^\gamma}|_{x^\gamma=0} \frac{\partial}{\partial \xi^\alpha} \wedge \frac{\partial}{\partial \xi^\beta}, \\ H = span\left\{ \frac{\partial}{\partial x^\kappa} - \xi^\gamma \frac{\partial P^{\alpha\kappa}}{\partial x^\gamma}|_{x^\gamma=0} \frac{\partial}{\partial \xi^\alpha} \right\}, \\ \sigma = \frac{1}{2}(p_{\kappa\nu} - \xi^\gamma \frac{\partial P^{\kappa\nu}}{\partial x^\gamma}|_{x^\gamma=0}) dx^\kappa \wedge dx^\nu, \end{array} \tag{6.30}$$

where ξ^α are fiber coordinates in the normal bundle NS and $p_{\kappa\nu}P^{\nu\theta} = \delta^\theta_\kappa$.

The Vorobiev-Poisson structure (6.30) is defined up to Poisson equivalence and its S-transversal part may be seen as a linear approximation of the S-transversal part of the original Poisson structure P.

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