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Higher Derived Brackets for Arbitrary Derivations

by Theodore Th. Voronov

Abstract

We introduce and study a construction of higher derived brackets generated by a (not necessarily inner) derivation of a Lie superalgebra. Higher derived brackets generated by an element of a Lie superalgebra were introduced in our earlier work. Examples of higher derived brackets naturally appear in geometry and mathematical physics. From a totally different viewpoint, we show that higher derived brackets arise when one wants to turn the inclusion map of a subalgebra of a differential Lie superalgebra, with a given complementary subalgebra, into a fibration. (For a non-Abelian complementary subalgebra, this leads to a generalization of L_∞-algebras with dropped or weakened (anti)symmetry of the brackets.)

1 Introduction

Higher derived brackets were introduced by the author in [14], motivated by physical and differential-geometric examples. The starting point in the construction was an element Δ in a Lie superalgebra L provided with a direct sum decomposition $L = K \oplus V$ into two subalgebras, where V is Abelian. Then a sequence of symmetric brackets on V is 'derived' from Δ in the same way as the partial derivatives of a function:

$$\{\underbrace{\xi, \dots, \xi}_{n}\}_\Delta = P(-\operatorname{ad}\xi)^n \Delta,$$

for coinciding even arguments, P denoting the projector on V. (In particular examples this analogy becomes exact.) It was proved that the Jacobiators for the higher derived brackets of an odd Δ are equal to the higher derived brackets of Δ^2. In particular, this leads to L_∞-algebras and algebras related with them.

In this paper we introduce and study higher derived brackets generated by an arbitrary derivation $D\colon L \to L$, which does not have to be inner. See formula (2.1). (The case of non-inner derivations was touched on in the final version of [14] without proofs.) As in [14], we make use of the decomposition $L = K \oplus V$. The subalgebra V is still assumed to be Abelian, though at the end of the paper we briefly discuss how this condition can be relaxed.

Our first main result is Theorem 1, which we state and prove in Section 2. It says that the Jacobiators for the higher derived brackets generated by an odd derivation D are equal to the brackets generated by D^2. So it is an analog of a similar statement for Δ. However, the presently available proof of Theorem 1 is technically much harder. Notice also that strictly speaking, the theorem about brackets generated by Δ is not a corollary of the theorem for D because of the possible presence of a 'background' in the former.

Secondly, we establish a relation between higher derived brackets and homotopical algebra. This is done in Section 4. The main result is Theorem 2. The question of whether the higher derived brackets of Δ defined in our paper [14] can be interpreted in the framework of homotopical algebra was asked by the anonymous referee of [14]. In fact, he suggested linking them with the notion of a 'left cone' (i.e., a cocone, or a homotopy fiber in topologists' language). This question happened to be very fruitful. The proper framework for it is when the brackets are generated by an arbitrary derivation D. In Section 4 we show that such a homotopical-algebraic interpretation is indeed possible. We show that the higher derived brackets of D appear as part of the brackets in $\Pi L \oplus V$ that naturally arise from the condition that the canonical differential in $\Pi L \oplus V$ (viewed as a cone or a cocylinder) respects an algebra structure extended from L, and we prove that the latter brackets make $\Pi L \oplus V$ an L_∞-algebra if $D^2 = 0$. Thus we arrive at an alternative approach to higher derived brackets.

Behind Theorem 1 one can recognize a more general algebraic statement. If one considers the Lie superalgebra $\operatorname{Der} L$ of derivations of L and the Lie superalgebra $\operatorname{Vect} V$ of vector fields on V, both w.r.t. the commutator, then it is possible to see that the construction of higher derived brackets gives a homomorphism $\operatorname{Der} L \to \operatorname{Vect} V$. By identifying vector fields with multilinear operations on V specified by their Taylor expansion at zero, we arrive at the statement that V becomes a 'generalized' L_∞-algebra 'over' the Lie superalgebra $\operatorname{Der} L$ (that is, there is a family of brackets parametrized by elements of $\operatorname{Der} L$ obeying 'Jacobi type' relations following the relations in $\operatorname{Der} L$). This is a new algebraic notion. We discuss it briefly. As mentioned, we also briefly discuss the possibility of dropping the condition that V be Abelian. By doing so, we arrive at higher derived brackets that are not necessarily symmetric. This leads to another generalization of L_∞-algebras, which we hope to analyze further elsewhere.

Terminology and notation. We use the 'super' language and conventions; in particular, a vector space always means a $\mathbb{Z}_2$-graded vector space, and we freely identify it with the corresponding vector supermanifold; multilinearity, symmetry, antisymmetry, derivations, etc., are always understood in the $\mathbb{Z}_2$-graded sense. Π stands for the parity reversion functor, and the parity of homogeneous objects is denoted by a tilde, i.e. $\tilde{a} = 0$ or $\tilde{a} = 1$ if $a \in A_0$ or $a \in A_1$ respectively, for a in a $\mathbb{Z}_2$-graded module A.

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2 Construction of Higher Derived Brackets

The algebraic setup is as follows. We are given a Lie superalgebra L and a decomposition of L into a sum of two subalgebras:

$$L = K \oplus V.$$

Let $P\colon\ L \to V$ be the projector on V parallel to K, i.e., $V = \operatorname{Im} P$, $K = \operatorname{Ker} P$.

Consider an arbitrary derivation $D\colon\ L \to L$, either even or odd.

Definition 2.1. The *k-th (higher) derived bracket of D* is a multilinear operation

$$\underbrace{V \times \ldots \times V}_{k \text{ times}} \to V$$

given by the formula

$$\{a_1, \ldots, a_k\}_D := P[\ldots[[Da_1, a_2], a_3], \ldots, a_k] \tag{2.1}$$

where $a_i \in V$. Here $k = 1, 2, 3, \ldots$.

The derived brackets have the same parity as the parity of the derivation D.

In this paper we construct higher derived bracket for an arbitrary derivation D. Higher derived brackets were first defined in [14] in the case when D is an inner derivation, $D = \operatorname{ad} \Delta$ for some $\Delta \in L$.

Remark 2.1. The binary derived bracket when $P = \mathrm{id}$, i.e, $L = V$, $K = 0$, was introduced by Yvette Kosmann-Schwarzbach [6] following an idea of Koszul, and independently by the author [12] (unpublished). In [6] a slightly more general setting was considered, L being a Loday (Leibniz) algebra. Derived brackets have numerous applications, see [7] for a survey.

Lemma 2.1. *Suppose that V is an Abelian subalgebra. Then the derived brackets are symmetric (in the $\mathbb{Z}_2$-graded sense).*

From now on we assume that V is Abelian.
(Later we shall discuss whether this requirement can be relaxed.)

Remark 2.2. A symmetric multilinear operation is defined by its value on coinciding even arguments (to be more precise, this is true if extending scalars to include as many 'odd constants' as necessary, is allowed). For the higher derived brackets, if ξ is even, we have

$$\{\underbrace{\xi,\dots,\xi}_{n}\}_D = (-1)^{n-1} P\,(\operatorname{ad}\xi)^{n-1}\, D\xi \tag{2.2}$$

for any $n = 1, 2, \dots$, regardless of the parity of D.

We want to investigate if in addition to symmetry the derived brackets can satisfy other identities such as the Jacobi identity. Consider the binary bracket. Notice first that it is symmetric, not antisymmetric, compared to the bracket on a Lie algebra. To turn symmetry into antisymmetry we have to reverse parity and consider ΠV. The bracket induced on ΠV will be even (as a Lie bracket should be) if the bracket in V is odd. Therefore, to ask about (analogs of) the Jacobi identity for the higher derived brackets makes sense when the derivation D is odd.

Example 2.1. Consider a vector space V. Let $L = \operatorname{Der} S(V^*)$. Elements of L can be viewed as polynomial vector fields on V. Let ξ^i be the linear coordinates on V corresponding to a basis (e_i) in V. We can consider vectors in V as vector fields with constant coefficients, so $V \subset L$ will be an Abelian subalgebra. Then $L = K \oplus V$, where the subalgebra K consists of all vector fields vanishing at the origin. The projector P maps every vector field to its value at the origin (constant vector field). Consider an arbitrary vector field

$$X = X^j(\xi)\,\frac{\partial}{\partial \xi^j},$$

even or odd, and consider the derived brackets on V generated by the derivation $\operatorname{ad} X$,

$$\{u_1,\dots,u_k\}_X := \{u_1,\dots,u_k\}_{\operatorname{ad} X} = [\dots[X,u_1],\dots,u_k]\,(0)\,.$$

One can see that

$$\{e_{i_1},\dots,e_{i_k}\}_X = (\pm)\,X^j_{i_1\dots i_k}\,e_j$$

where

$$X^j_{i_1\dots i_k} = \frac{\partial^k X^j}{\partial\xi^{i_1}\dots\partial\xi^{i_k}}\,(0)\,.$$

In particular, consider a quadratic vector field:

$$Q = \frac{1}{2}\,\xi^i\xi^j Q^k_{ji}\frac{\partial}{\partial\xi^k}\,.$$

Then the k-th derived bracket of Q is zero unless $k = 2$, and the binary derived bracket is given by

$$\{e_i, e_j\}_Q = (\pm)\,Q^k_{ij} e_k\,. \tag{2.3}$$

Suppose that Q is odd. By a direct check one can see that *the Jacobi identity for the bracket* (2.3) *is equivalent to the condition* $Q^2 = 0$. (More precisely, the usual graded Jacobi identity is valid for the antisymmetric bracket in ΠV.) Q then can be identified with the Chevalley–Eilenberg differential in the standard cochain complex of the resulting Lie (super)algebra.

Odd vector fields with square zero are known as *homological*. Example 2.1 shows that the structure of a Lie superalgebra on a vector space corresponds to a quadratic homological vector field. If we drop the condition that the homological vector field be quadratic, we obtain 'L_∞-algebras' or 'strong homotopy Lie algebras' where the Jacobi identity for a binary bracket holds up to homotopy, which is a ternary bracket, and in its turn satisfies an analog of the Jacobi identity up to a homotopy, and so on. Describing algebraic structures by derivations of square zero is a very general principle dating back to Nijenhuis in the 1950's and used in recent works of Kontsevich (his "formality theorem" implying the existence of deformation quantization of Poisson manifolds, see [4, 5]).

Let V be a vector space endowed with a sequence of k-linear odd symmetric operations denoted by braces. Here $k = 0, 1, 2, 3, \ldots$.

Definition 2.2. The *n-th Jacobiator* is the following expression:

$$J^n(a_1, \ldots, a_n) = \sum_{k+l=n} \sum_{(k,l)\text{-shuffles}} (-1)^{\varepsilon} \{\{a_{\sigma(1)}, \ldots, a_{\sigma(k)}\}, a_{\sigma(k+1)}, \ldots, a_{\sigma(k+l)}\}$$

where the sign $(-1)^{\varepsilon}$ is given by the usual sign rule for permutations of homogeneous elements of V.

Let us recall the definition of L_∞-algebras, due to Lada and Stasheff [9].

Definition 2.3. An *L_∞-algebra*, or *strong(ly) homotopy Lie algebra*, is a vector space V endowed with a sequence of k-linear odd symmetric operations, $k = 0, 1, 2, 3, \ldots$, such that all the Jacobiators vanish.

Remark 2.3. We gave the definition in the form most convenient for our purposes. If one wishes to directly include the case of ordinary Lie algebras, the term L_∞-algebra should be applied to the structure on the opposite space, i.e., ΠV, where the corresponding operations are antisymmetric and are even for an even number of arguments and odd otherwise. Also, it is often assumed that the 0-bracket is zero. The 0-bracket is sometimes referred to as the 'background'.

Proposition 2.1. *There is a one-to-one correspondence between L_∞-algebra structures on V and formal homological vector fields on V:*

$$Q = Q^k(\xi)\,\frac{\partial}{\partial \xi^k} = \left(Q^k_0 + \xi^i Q^k_i + \frac{1}{2}\,\xi^i\xi^j Q^k_{ji} + \frac{1}{3!}\,\xi^i\xi^j\xi^l Q^k_{lji} + \ldots\right)\frac{\partial}{\partial \xi^k}\,.$$

This proposition is a well-known fact. What we can give here is an explicit invariant expression for the correspondence: *the brackets in an L_∞-algebra corresponding to a homological field Q are the higher derived brackets of Q*,

$$\{u_1, \dots, u_k\}_Q = [\dots[Q, u_1], \dots, u_k](0) \tag{2.4}$$

(this generalizes Example 2.1).

Let us return to our abstract setup. Consider the higher derived brackets (2.1) of an *odd* derivation D. We get a sequence of odd symmetric multilinear operations on V. By definition the 0-ary operation is zero. What about the Jacobiators?

Theorem 1. *Suppose that D preserves the subalgebra $K = \operatorname{Ker} P$. Then the n-th Jacobiator of the derived brackets of D equals the n-th derived bracket of D^2:*

$$J^n_D(a_1, \dots, a_n) = \{a_1, \dots, a_n\}_{D^2}\,, \tag{2.5}$$

for all $n = 1, 2, 3, \dots$.

Let us make two comments before giving the proof.

Firstly, notice that in our setup the condition that $L = K \oplus V$ is the sum of subalgebras where V is Abelian can be expressed by the identities

$$[Pa, Pb] = 0 \tag{2.6}$$

and

$$P[a, b] = P[Pa, b] + P[a, Pb], \tag{2.7}$$

for all a, b (a "distributivity law" for P). Notice that (2.7) is also equivalent to the vanishing of the Nijenhuis bracket of P with itself.

Secondly, the condition in the theorem that $D(K) \subset K$ can be written as the identity

$$PDP = PD\,. \tag{2.8}$$

Condition (2.8) already appears if we check the Jacobiator of order one:

$$J^1_D(a) = \{\{a\}_D\}_D = PDPDa = PDDa = \{a\}_{D^2},$$

if $PDP = PD$. (Notice that in general D does not have to preserve V. Indeed, if D preserves V, then all the derived brackets of D starting from the binary bracket, will vanish.)

Proof of Theorem 1. To simplify the notation, let us omit temporarily the subscript D from the brackets and Jacobiators. Since the Jacobiators are multilinear symmetric functions, it is sufficient to consider them for coinciding even arguments. Denote $J^n(\xi,\ldots,\xi)$ where ξ is even by $J^n(\xi)$. From Definition 2.2 and equation (2.2) we clearly obtain

$$J^n(\xi)=\sum_{l=0}^{n-1}C_n^l\{\{\underbrace{\xi,\ldots,\xi}_{n-l}\},\underbrace{\xi,\ldots,\xi}_{l}\},$$

where $C_n^l=\frac{n!}{l!(n-l)!}$ is the binomial coefficient, in our case appearing as the number of $(n-l,l)$-shuffles. It follows that

$$J^n(\xi)=\sum_{l=0}^{n-1}C_n^lP[\ldots[D\{\underbrace{\xi,\ldots,\xi}_{n-l}\},\underbrace{\xi],\ldots,\xi]}_{l}=$$
$$\sum_{l=0}^{n-1}C_n^lP[\ldots[DP(-1)^{n-l-1}(\operatorname{ad}\xi)^{n-l-1}D\xi,\underbrace{\xi],\ldots,\xi]}_{l}=$$
$$\sum_{l=0}^{n-1}C_n^l(-1)^{n-l-1}(-1)^lP(\operatorname{ad}\xi)^lDP(\operatorname{ad}\xi)^{n-l-1}D\xi=$$
$$\sum_{l=0}^{n-1}C_n^l(-1)^{n-1}P(\operatorname{ad}\xi)^lDP(\operatorname{ad}\xi)^{n-l-1}D\xi\,.$$

Consider $(\operatorname{ad}\xi)^lDP$. We want to move D to the left. Since

$$\operatorname{ad}\xi\cdot D-D\cdot\operatorname{ad}\xi=-\operatorname{ad}(D\xi)\,,$$

as one can easily check, it follows that for any $l\geqslant 1$

$$(\operatorname{ad}\xi)^lDP=(\operatorname{ad}\xi)^{l-1}(\operatorname{ad}\xi\cdot D-D\cdot\operatorname{ad}\xi+D\cdot\operatorname{ad}\xi)P=$$
$$(\operatorname{ad}\xi)^{l-1}(-\operatorname{ad}(D\xi)+D\cdot\operatorname{ad}\xi)P=-(\operatorname{ad}\xi)^{l-1}\operatorname{ad}(D\xi)P=-\left[(\operatorname{ad}\xi)^{l-1}D\xi,P(.)\right]$$

where we used $\operatorname{ad}\xi\cdot P=0$. Substituting this into the formula above we obtain

$$J^n(\xi)=\sum_{l=1}^{n-1}C_n^l(-1)^nP\left[(\operatorname{ad}\xi)^{l-1}D\xi,P(\operatorname{ad}\xi)^{n-l-1}D\xi\right]+(-1)^{n-1}PDP(\operatorname{ad}\xi)^{n-1}D\xi$$

or

$$(-1)^nJ^n(\xi)=\sum_{l=1}^{n-1}C_n^lP\left[(\operatorname{ad}\xi)^{l-1}D\xi,P(\operatorname{ad}\xi)^{n-l-1}D\xi\right]-PD(\operatorname{ad}\xi)^{n-1}D\xi$$

(where we also used (2.8)). We can re-arrange the first sum by adding it to itself in the reverse order and dividing by two:

$$\sum_{l=1}^{n-1} C_n^l P\left[(\operatorname{ad}\xi)^{l-1}D\xi, P(\operatorname{ad}\xi)^{n-l-1}D\xi\right] =$$
$$\frac{1}{2}\left(\sum_{l=1}^{n-1} C_n^l P\left[(\operatorname{ad}\xi)^{l-1}D\xi, P(\operatorname{ad}\xi)^{n-l-1}D\xi\right] + \sum_{l=1}^{n-1} C_n^l P\left[(\operatorname{ad}\xi)^{n-l-1}D\xi, P(\operatorname{ad}\xi)^{l-1}D\xi\right]\right) =$$
$$\frac{1}{2}\sum_{l=1}^{n-1} C_n^l \Big(P\left[(\operatorname{ad}\xi)^{l-1}D\xi, P(\operatorname{ad}\xi)^{n-l-1}D\xi\right] + P\left[(\operatorname{ad}\xi)^{n-l-1}D\xi, P(\operatorname{ad}\xi)^{l-1}D\xi\right]\Big).$$

Noticing that $\left[(\operatorname{ad}\xi)^{n-l-1}D\xi, P(\operatorname{ad}\xi)^{l-1}D\xi\right] = \left[P(\operatorname{ad}\xi)^{l-1}D\xi, (\operatorname{ad}\xi)^{n-l-1}D\xi\right]$, because ξ is even and $D\xi$ is odd, and using the distributivity relation (2.7), we find the following expression for the Jacobiator:

$$(-1)^n J^n(\xi) = \frac{1}{2}\sum_{l=1}^{n-1} C_n^l P\left[(\operatorname{ad}\xi)^{l-1}D\xi, (\operatorname{ad}\xi)^{n-l-1}D\xi\right] - PD(\operatorname{ad}\xi)^{n-1}D\xi$$

or

$$(2.9)\quad (-1)^n J^n(\xi) = \frac{1}{2}P\sum_{l=1}^{n-1} C_n^l \left[(\operatorname{ad}\xi)^{l-1}D\xi, (\operatorname{ad}\xi)^{n-l-1}D\xi\right] - P\left[D, (\operatorname{ad}\xi)^{n-1}\right] D\xi - P(\operatorname{ad}\xi)^{n-1}D^2\xi\,.$$

We shall now analyze the term $[D, (\operatorname{ad}\xi)^{n-1}]\,D\xi$. Using the formula for the commutator $[A, B^N]$ for arbitrary operators A, B, we get

$$\left[D, (\operatorname{ad}\xi)^{n-1}\right] D\xi = \sum_{i+j=n-2} (\operatorname{ad}\xi)^i [D, \operatorname{ad}\xi](\operatorname{ad}\xi)^j D\xi =$$
$$\sum_{i+j=n-2} (\operatorname{ad}\xi)^i \operatorname{ad}(D\xi)(\operatorname{ad}\xi)^j D\xi = \sum_{i+j=n-2} (\operatorname{ad}\xi)^i \left[D\xi, (\operatorname{ad}\xi)^j D\xi\right] =$$
$$\sum_{i+j=n-2}\ \sum_{r+s=i} C_i^r \left[(\operatorname{ad}\xi)^r D\xi, (\operatorname{ad}\xi)^{s+j} D\xi\right] = \sum_{i=0}^{n-2}\sum_{r=0}^{i} C_i^r \left[(\operatorname{ad}\xi)^r D\xi, (\operatorname{ad}\xi)^{n-2-r} D\xi\right] =$$
$$\sum_{r=0}^{n-2}\sum_{i=r}^{n-2} C_i^r \left[(\operatorname{ad}\xi)^r D\xi, (\operatorname{ad}\xi)^{n-2-r} D\xi\right]\,.$$

Since in the internal sum the commutators do not depend on the index of summation i, they can be taken out of the sum. It is possible to apply a well-known identity for sums of binomial coefficients (see, e.g. [1, p. 153]):

$$\sum_{i=r}^{m} C_i^r = C_r^r + C_{r+1}^r + \ldots + C_m^r = C_r^0 + C_{r+1}^1 + \ldots + C_m^{m-r} = C_{m+1}^{m-r},$$

where in our case $m = n-2$. Hence

$$\sum_{i=r}^{n-2} C_i^r = C_{n-1}^{n-2-r},$$

and we arrive at the equality

$$[D, (\operatorname{ad}\xi)^{n-1}]\, D\xi = \sum_{r=0}^{n-2} C_{n-1}^{n-2-r} \left[(\operatorname{ad}\xi)^r D\xi, (\operatorname{ad}\xi)^{n-2-r} D\xi\right] . \tag{2.10}$$

Notice that since $D\xi$ is odd and the bracket is symmetric, the left-hand side contains similar terms, with r and r', where $r = n-2-r'$. Hence, this sum can be re-arranged by writing it twice in opposite orders and dividing by two:

$$\begin{aligned}
&[D, (\operatorname{ad}\xi)^{n-1}]\, D\xi = \\
&\frac{1}{2}\left(\sum_{r=0}^{n-2} C_{n-1}^{n-2-r} \left[(\operatorname{ad}\xi)^r D\xi, (\operatorname{ad}\xi)^{n-2-r} D\xi\right] + \sum_{r=0}^{n-2} C_{n-1}^{r} \left[(\operatorname{ad}\xi)^{n-2-r} D\xi, (\operatorname{ad}\xi)^{r} D\xi\right]\right) \\
&\qquad = \frac{1}{2}\sum_{r=0}^{n-2} \left(C_{n-1}^{r+1} + C_{n-1}^{r}\right) \left[(\operatorname{ad}\xi)^r D\xi, (\operatorname{ad}\xi)^{n-2-r} D\xi\right] = \\
&\frac{1}{2}\sum_{r=0}^{n-2} C_{n}^{r+1} \left[(\operatorname{ad}\xi)^r D\xi, (\operatorname{ad}\xi)^{n-2-r} D\xi\right] = \frac{1}{2}\sum_{l=1}^{n-1} C_{n}^{l} \left[(\operatorname{ad}\xi)^{l-1} D\xi, (\operatorname{ad}\xi)^{n-1-l} D\xi\right] ,
\end{aligned}$$

which coincides with the first term in the formula for the Jacobiator (2.9). Substituting into (2.9), we see that the first two terms cancel, and we finally obtain

$$(-1)^n J^n(\xi) = -P(\operatorname{ad}\xi)^{n-1} D^2\xi$$

or

$$J_D^n(\xi) = (-1)^{n-1} P(\operatorname{ad}\xi)^{n-1} D^2\xi = \{\underbrace{\xi, \dots, \xi}_{n}\}_{D^2}$$

for an arbitrary even ξ. This implies identity (2.5) for all elements of V. □

We say that the derivation D is of *order* r with respect to the subalgebra V if for all $a_1, \dots, a_{r+1} \in V$

$$[\dots[Da_1, a_2], \dots, a_{r+1}] = 0.$$

Here $r = 0, 1, 2, \dots$

If D is of order r, all the derived k-brackets of D vanish for $k \geqslant r+1$.

Corollary 2.1. *For an odd derivation D, if the order of D^2 is r, then the higher derived brackets of D satisfy all the Jacobi identities with $r+1$ or more arguments,*

Corollary 2.2. *If the order of D^2 is zero, i.e., $D^2(V) = 0$, in particular if $D^2 = 0$, then the higher derived brackets of an odd derivation D define an L_∞-algebra.*

Proposition 2.1 shows that all L_∞-algebras are obtained in this way.

3 Examples

All examples of higher derived brackets naturally arising in applications are for the case when $D = \operatorname{ad}\Delta$ is an inner derivation given by some element Δ. This is the situation where higher derived brackets were first introduced in [14]. An analog of Theorem 1 was proved there for brackets generated by Δ. (That proof is simpler than the above proof of Theorem 1 for general D.) Let us clarify the relation between the higher derived brackets of an element $\Delta \in L$ as introduced in [14] and the higher derived brackets of a derivation $D\colon L \to L$ as in Definition 2.1.

Any element $\Delta \in L$, of course, gives an inner derivation $D = \operatorname{ad}\Delta\colon L \to L$, and the higher derived brackets of the derivation $D = \operatorname{ad}\Delta$

$$\{a_1, \dots, a_k\}_D = P\left[\dots\left[(\operatorname{ad}\Delta)a_1, a_2\right], \dots, a_k\right],$$

coincide with the brackets defined in [14],

$$\{a_1, \dots, a_k\}_\Delta = P\left[\dots\left[\left[\Delta, a_1\right], a_2\right], \dots, a_k\right],$$

where $k = 1, 2, 3, \dots$. However, for Δ there is a natural notion of a 0-bracket (no arguments, a distinguished element),

$$\{\varnothing\}_\Delta = P\Delta\,,$$

which is not defined for arbitrary derivations D. The Jacobiators for the higher derived brackets of Δ include this 0-bracket and start with the 0-th Jacobiator $\{\{\varnothing\}_\Delta\}_\Delta$. At the same time, the 0-ary bracket is assumed to be zero in all the Jacobiators for a general D and it does not appear in our Theorem 1. There is no obvious way of incorporating the 0-th bracket into the picture for a general derivation D. If $P\Delta \neq 0$, that means that $\Delta \notin K$, hence there is no guarantee that $(\operatorname{ad}\Delta)(K) \subset K$, which is a condition of Theorem 1. The calculation of $J^1_D(a)$ above shows that some sort of condition is needed (and at least a weaker condition $PD^2P = PDPDP$ is necessary). Therefore, Theorem 1 of [14], to which Theorem 1 is an analog, does not follow from Theorem 1, in general.

We can summarize by saying that the theory developed in [14] is a particular case of the theory developed here if $P(\Delta) = 0$, i.e., $\Delta \in K$. Then, in particular, $(\operatorname{ad}\Delta)(K) \subset K$ and Theorem 1 applies.

We shall leave open the question of how the theory for non-inner derivations can be modified to incorporate an 0-ary bracket.

With having this in mind, there are some examples of higher derived brackets, all coming from inner derivations. They are given for illustrative purposes only. More details can be found in [14]. See also [13], [3].

Example 3.1. The setup of Example 2.1. $L = \operatorname{Vect} V$, where V is a vector space, $P\colon X \mapsto X(0)$ is a projection onto the Abelian subalgebra of vector fields with constant coefficients. For an odd vector field Q such that $Q(0) = 0$ we get the higher derived brackets on V, $k = 1, 2, \ldots,$,

$$\{u_1, \ldots, u_k\}_Q = [\ldots [Q, u_1], \ldots, u_k] (0).$$

They define an L_∞-algebra with e zero background ('strict' in the terminology of [14]) if $Q^2 = 0$, and this is a canonical description of all (strict) L_∞-algebra structures on the space V.

Example 3.2. $L = \operatorname{End} A$ for a commutative associative algebra with unit A and $V = A$. The projector P maps an operator Δ to $\Delta 1 \in A \subset \operatorname{End} A$. The higher derived brackets of $\operatorname{ad} \Delta$ for an operator Δ such that $\Delta 1 = 0$ are the 'Koszul operations' (see [8])

$$\{a_1, \ldots, a_k\}_\Delta = [\ldots [\Delta, a_1], \ldots, a_k] 1,$$

$k = 1, 2, 3, \ldots,$. For a differential operator of order n the brackets with more than n arguments vanish and the top bracket is the symbol of Δ. An odd operator Δ satisfying $\Delta^2 = 0$ provides an example of a 'homotopy Batalin–Vilkovisky algebra'.

Example 3.3. $L = C^\infty(T^*M)$, $V = C^\infty(M)$, P is the restriction on M, and $i^*\colon C^\infty(T^*M) \to C^\infty(M)$, where $i\colon M \to T^*M$. For a Hamiltonian $S \in C^\infty(T^*M)$ such that $i^*S = 0$, on functions on M there are derived brackets

$$\{f_1, \ldots, f_k\}_S = i^* (\ldots (S, f_1), \ldots, f_k),$$

$k = 1, 2, 3, \ldots$, where in the right-hand side stand the canonical Poisson brackets on T^*M. If S is odd (for a nontrivial example M should be a supermanifold) and satisfies $(S, S) = 0$, we get 'higher Schouten brackets' on $C^\infty(M)$ giving an example of a 'homotopy Schouten algebra'.

Example 3.4. Similarly, let $L = C^\infty(\Pi T^*M)$, $V = C^\infty(M)$ and let P be the restriction on M. For a multivector field $\psi \in C^\infty(\Pi T^*M)$ such that $i^*\psi = 0$, on functions on M there are derived brackets

$$\{f_1, \ldots, f_k\}_\psi = i^* [\![\ldots [\![\psi, f_1]\!], \ldots, f_k]\!],$$

$k = 1, 2, 3, \ldots,$ where on the right-hand side we have the canonical Schouten brackets on ΠT^*M. Since the canonical Schouten brackets are odd, for an even ψ the derived brackets have alternating parity (even for an even number of arguments, odd for odd). If $[\![\psi, \psi]\!] = 0$, these 'higher Poisson brackets' on functions on M give an example of a 'homotopy Poisson algebra'.

Other examples of higher derived brackets which we shall not consider here, are 'homotopy Lie algebroids', which are an analog of L_∞-algebras in the world of algebroids, and the non-linear analogs in the world of graded manifolds [13] (see also [11]). We hope to be able to say more about such examples elsewhere.

It is a good question whether a genuinely non-inner derivation can naturally occur in examples of higher derived brackets coming from differential geometry or physics.

4 Relation with Homotopy Theory

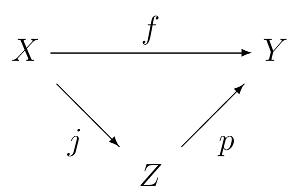
Now we shall show how our construction of the (higher) derived brackets arises naturally if one wishes to consider the homotopy theory of Lie superalgebras.

Let us re-formulate the setup in a way convenient for this purpose. We have a Lie superalgebra L with a decomposition $L = K \oplus V$, where K and V are subalgebras. Consider an odd derivation D such that $D(K) \subset K$, and from the start assume that D is of square zero. Hence we have an inclusion of differential Lie superalgebras

$$i\colon\ K \to L$$

and a given complement for the image of i, which is called V. (V is *not*, in general, a differential subalgebra.)

There is an idea, familiar to topologists, that every map can be made into a fibration by appropriately replacing a space by a homotopy equivalent one. More precisely, if we have a category where a "weak equivalence", "fibration" and "cofibration" make sense (i.e., a Quillen model category [10]), the following diagram is called a *cocylinder diagram*:



if j is a cofibration and weak equivalence, and p is a fibration. Then Z is also denoted by $\operatorname{Cocyl} f$. (To refresh the intuition, recall that for topological spaces that are cell complexes, cofibrations are just inclusions of subspaces, fibrations are 'Serre fibrations', i.e., maps satisfying the covering homotopy property, and weak equivalences are maps inducing isomorphisms of all homotopy groups. In this case, a cocylinder for any map $f\colon\ X \to Y$ may be obtained as a subspace in $X \times Y^I$ consisting of all pairs (x, γ) where $\gamma\colon\ I \to Y$ is a path such that $\gamma(0) = f(x)$.)

Can we do this (in an algebraic context) for the inclusion $K \to L$?

To begin with let us temporarily forget about the algebra structure. Consider just an arbitrary inclusion of complexes

$$i\colon\ K \to L$$

such that there is a given complementary subspace V (not a subcomplex!) and $L = K \oplus V$. In the context of this paper, a *complex* is simply a vector space with an odd operator of square zero.

Let P be the projector onto V parallel to K. The space V becomes a complex with the differential PD. Introduce into $L \oplus \Pi V$ an operator d as follows:

(4.1) $$d(x, \Pi a) := \big(Dx, -\Pi P(x + Da)\big),$$

for $x \in L$, $a \in V$. It is then straightforward to show that $d^2 = 0$. Consider the maps $j\colon K \to L \oplus \Pi V$ and $p\colon L \oplus \Pi V \to L$, where $j\colon x \mapsto (x, 0)$, $p\colon (x, \Pi a) \mapsto x$.

Lemma 4.1. *The following diagram*

$$\begin{array}{ccc} K & \xrightarrow{\quad i \quad} & L \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle p} & \\ & L \oplus \Pi V & \end{array}$$

is a cocylinder diagram in the category of complexes, i.e., the maps j and p are chain maps, $i = p \circ j$, the map $j\colon K \to L \oplus \Pi V$ is a monomorphism ('cofibration') *and a quasi-isomorphism* ('weak homotopy equivalence')*, and the map $p\colon L \oplus \Pi V \to L$ is an epimorphism* ('fibration')*.*

Proof. This is immediate. A quasi-inverse for j is the map

$$q\colon (x, \Pi a) \mapsto (1 - P)(x + Da).$$

□

Remark 4.1. As is well known, for maps of complexes there are canonical constructions of cylinders and cocylinders; they are modelled on the (co)chain complexes corresponding to the canonical topological cylinders and cocylinders. For a particular chain map it might be more convenient to consider a 'smaller' cylinder or cocylinder than the one featured by the standard construction. This is exactly what happens in our case. The standard cocylinder construction applied to the inclusion $i\colon K \to L$ would not yield the complex $L \oplus \Pi V$ as in Lemma 4.1, instead it would give a bigger complex $K \oplus L \oplus \Pi L = K \oplus K \oplus V \oplus \Pi K \oplus \Pi V$ that is homotopy equivalent to $L \oplus \Pi V$. One should also note that the complex $L \oplus \Pi V$ essentially coincides with the standard (co)cone of the projection $L \to V$. See the Appendix.

It follows from Lemma 4.1 that the space $\Pi V = \operatorname{Ker} p$, taken with the differential $-\Pi P D$, is a homotopy fiber of the inclusion of complexes $i\colon K \to L = K \oplus V$.

Now we want to 'turn the algebra structure on'. To this end, since we have been working with V rather than ΠV, let us first apply a parity shift to the cocylinder diagram above. Then we have the cocylinder diagram

$$\begin{array}{ccc} \Pi K & \xrightarrow{\quad i \quad} & \Pi L \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle p} & \\ & \Pi L \oplus V & \end{array}$$

for the inclusion of complexes $\Pi K \to \Pi L$. In particular, the differential in $\Pi L \oplus V$ is

$$d\colon\ (\Pi x, a) \mapsto (-\Pi Dx, P(x + Da)) \tag{4.2}$$

(which is the differential in the standard cone, see the Appendix, of the projection of complexes (L, D) onto (V, PD)).

Let us restore our framework. Assume as above that L is a Lie superalgebra with D being a derivation, and that V is an Abelian subalgebra. The Lie bracket in L induces an odd bracket in ΠL:

$$\{\Pi x, \Pi y\} = \Pi[x, y](-1)^{\tilde{x}}. \tag{4.3}$$

Is it possible to extend this to a bracket on the whole of $\Pi L \oplus V$?

Proposition 4.1. *There exists an odd binary bracket on $\Pi L \oplus V$ extending that on ΠL such that the operator* (4.2) *acts as a derivation. It is given by the formulae*

$$\{\Pi x, a\} = P[x, a], \tag{4.4}$$

$$\{a, b\} = P[Da, b] \tag{4.5}$$

for arbitrary $x \in L$ and $a, b \in V$.

Proof. As a starting point we use formula (4.3) for the bracket on ΠL, where Πx and Πy are considered as elements of $\Pi L \oplus V$. Apply d given by (4.2) to $\{\Pi x, \Pi y\}$ and require that the Leibniz rule be satisfied:

$$d\{\Pi x, \Pi y\} = -\{d\Pi x, \Pi y\} - (-1)^{\tilde{x}+1}\{\Pi x, d\Pi y\} \tag{4.6}$$

for all $x, y \in L$ (notice that the parity in (4.3) 'sits' at the opening bracket, hence the signs). Expanding d by (4.2), so that $d\Pi x = -\Pi Dx + Px$, and treating the brackets between elements of ΠL and V as unknown, we find that the failure of (4.6) for $x = y$ and $\tilde{x} = 1$ is the difference $\{Px, \Pi x\} - P[Px, x]$. Replacing Px by an arbitrary element of V, we arrive at the above definition (4.4). Now assume (4.4) and require the Leibniz rule for this new bracket:

$$d\{\Pi x, b\} = -\{d\Pi x, b\} + (-1)^{\tilde{x}}\{\Pi x, db\} \tag{4.7}$$

for all $x \in L$, $b \in V$. Here $d\Pi x = -\Pi Dx + Px$, $da = PDa$, and we treat the bracket in V as unknown. The failure of (4.7) equals $\{Px, b\} + (-1)^{\tilde{x}}P\,[Px, Db]$. Denoting $Px = a \in V$, we arrive at the formula $\{a, b\} = -(-1)^{\tilde{a}}P[a, Db]$ or, equivalently,

$$\{a, b\} = P[Da, b]$$

as the necessary and sufficient condition of (4.7). This is our derived bracket (2.1) for $k = 2$. The Leibniz rule for $\{a, b\}$ is now satisfied automatically and does not bring any new relations. □

Remark 4.2. Defining the bracket by formula (4.4) is a sufficient condition for (4.6). A more detailed analysis shows that (4.4) is also necessary at least when $x \in K$. Hence the condition that the operator (4.2) acts as a derivation defines the bracket in an essentially unique way.

One can see that a binary bracket defined in this way on $\Pi L \oplus V$ will not satisfy the Jacobi identity exactly, thus giving rise to a ternary bracket, and so on. Define the higher brackets on $\Pi L \oplus V$ as follows:

$$\{\Pi x, a_1, \ldots, a_n\} = P[\ldots[x, a_1], \ldots, a_n], \tag{4.8}$$

$$\{a_1, \ldots, a_n\} = P[\ldots[Da_1, a_2], \ldots, a_n], \tag{4.9}$$

where $n \geqslant 1$. As an unary bracket take the differential (4.2), and set the 0-ary bracket to zero. All the other brackets except those obtainable by symmetry, are defined to be zero. Formulae (4.8), (4.9) directly extend (4.4), (4.5) to many arguments, and formula (4.9) is the familiar higher derived bracket on V for all k.

Theorem 2. *The set of brackets* (4.3), (4.8) *and* (4.9)*, together with* (4.2)*, define on the space* $\Pi L \oplus V$ *the structure of an* L_∞*-algebra.*

Proof. We shall prove that all the brackets (4.2)–(4.9) satisfy all the generalized Jacobi identities. Consider the Jacobiator J^n in $\Pi L \oplus V$ with n arbitrary arguments. Without loss of generality we can assume that each of the arguments is either in ΠL or V. We claim that there can be no non-trivial Jacobiators with more than 3 arguments in ΠL. Indeed, J^n is a sum of terms of the form

$$\{\{\underbrace{\_,\_,\_}_{k}\},\underbrace{\_,\_,\_,\_}_{l}\}$$

where $k + l = n$ and $k \geqslant 1$. If there occur 4 elements of ΠL or more, then among those k or l arguments there must be at least 2 in ΠL, and it should be exactly $k = 2$ and $l = 2$, since there are no brackets involving 3 arguments in ΠL. Then the internal bracket also takes values in ΠL, hence we get 3 arguments in ΠL for the external bracket, so it must vanish. Consider the Jacobiators that contain exactly 3 arguments from ΠL. By a similar analysis one can see that the only potential non-vanishing Jacobiator is for $n = 3$, which is exactly the Jacobiator in ΠL and it vanishes since L is a Lie superalgebra. This leaves the Jacobiators with exactly 1 or 2 arguments in ΠL. (The Jacobiators with all arguments in V vanish by Theorem 1 applied to D such that $D^2 = 0$.) They are as follows:

(4.10) $J^{p+1}(\Pi x, a_1, \ldots, a_p) =$

$$\sum_{k=1}^{p} \sum_{(k,p-k)\text{-shuffles}} (-1)^{\tilde{x}+1}(-1)^{\varepsilon(\sigma;a_1,\ldots,a_p)} \left\{\Pi x, \{a_{\sigma(1)}, \ldots, a_{\sigma(k)}\}, a_{\sigma(k+1)}, \ldots, a_{\sigma(p)}\right\} +$$

$$\sum_{k=0}^{p} \sum_{(k,p-k)\text{-shuffles}} (-1)^{\varepsilon(\sigma;a_1,\dots,a_p)} \left\{\{\Pi x, a_{\sigma(1)},\dots,a_{\sigma(k)}\}, a_{\sigma(k+1)},\dots,a_{\sigma(p)}\right\}$$

and

$$(4.11)\quad J^{p+2}(\Pi x,\Pi y,a_1,\dots,a_p) = \{\{\Pi x,\Pi y\},a_1,\dots,a_p\} +$$
$$\sum_{k=0}^{p} \sum_{(k,p-k)\text{-shuffles}} (-1)^{\varepsilon(\sigma;a_1,\dots,a_p)} \Bigg((-1)^{\tilde x+1} \Big\{\Pi x, \{\Pi y, a_{\sigma(1)},\dots,a_{\sigma(k)}\}, a_{\sigma(k+1)},\dots,a_{\sigma(p)}\Big\}$$
$$+ (-1)^{(\tilde y+1)(\tilde x+\tilde a_{\sigma(1)}+\dots+\tilde a_{\sigma(k)})} \Big\{\Pi y, \{\Pi x, a_{\sigma(1)},\dots,a_{\sigma(k)}\}, a_{\sigma(k+1)},\dots,a_{\sigma(p)}\Big\}\Bigg).$$

Here $x,y\in L$, $a_i\in V$. By $(-1)^{\varepsilon(\sigma;a_1,\dots,a_p)}$ we denoted the sign arising from the action of a permutation σ on the product of p commuting homogeneous variables of parities $\tilde a_1,\dots,\tilde a_p$. The equalities $J^{p+1}=0$ and $J^{p+2}=0$ can be informally perceived, respectively, as expressing the fact that taking a bracket with Πx acts, in a sense, as a derivation, and that taking a bracket with $\{\Pi x,\Pi y\}$ acts, in a sense, as the commutator of brackets with Πx and with Πy. (All this in a generalized sense, involving partitions and shuffles). Hence these equalities are intuitively plausible. Let us prove them. For this sake consider $x=y$ and $a_i=\xi$ for all i, where $\tilde x=1$, $\tilde\xi=0$. Then (4.10) and (4.11) reduce to

$$(4.12)\quad J^{p+1}(\Pi x,\xi) := J^{p+1}(\Pi x,\underbrace{\xi,\dots,\xi}_{p}) =$$
$$\sum_{k=1}^{p} C_p^k \{\Pi x, \{\underbrace{\xi,\dots,\xi}_{k}\}, \underbrace{\xi,\dots,\xi}_{p-k}\} + \sum_{k=0}^{p} C_p^k \{\{\Pi x, \underbrace{\xi,\dots,\xi}_{k}\}, \underbrace{\xi,\dots,\xi}_{p-k}\}$$

and

$$(4.13)\quad J^{p+2}(\Pi x,\xi) := J^{p+2}(\Pi x,\Pi x,\underbrace{\xi,\dots,\xi}_{p}) =$$
$$\big\{\{\Pi x,\Pi x\},\underbrace{\xi,\dots,\xi}_{p}\big\} + 2\sum_{k=0}^{p} C_p^k \{\Pi x, \{\Pi x, \underbrace{\xi,\dots,\xi}_{k}\}, \underbrace{\xi,\dots,\xi}_{p-k}\},$$

respectively. Here C_p^k denotes the binomial coefficient. Substituting the definitions of the brackets in (4.12), we get after a simplification

$$J^{p+1}(\Pi x,\xi) = \{-\Pi Dx + Px, \underbrace{\xi,\dots,\xi}_{p}\} +$$
$$\sum_{k=1}^{p} C_p^k \Big(\big\{\Pi x, P[\dots[D\xi,\underbrace{\xi],\dots,\xi]}_{k-1}, \underbrace{\xi,\dots,\xi}_{p-k}\big\} + \big\{P[\dots[[x,\underbrace{\xi],\xi],\dots,\xi]}_{k}, \underbrace{\xi,\dots,\xi}_{p-k}\big\}\Big) =$$

$$(-1)^{p+1}P(\operatorname{ad}\xi)^p Dx + (-1)^p P(\operatorname{ad}\xi)^p DPx +$$
$$\sum_{k=1}^{p} C_p^k \Big(P(-1)^{p-1}(\operatorname{ad}\xi)^{p-k}\left[x, P(\operatorname{ad}\xi)^{k-1}D\xi\right] + P(-1)^p(\operatorname{ad}\xi)^{p-k}DP(\operatorname{ad}\xi)^k x\Big)$$

or

$$(-1)^p J^{p+1}(\Pi x,\xi) = -P(\operatorname{ad}\xi)^p Dx + P(\operatorname{ad}\xi)^p DPx +$$
$$\sum_{k=1}^{p} C_p^k \Big(-P(\operatorname{ad}\xi)^{p-k}\left[x, P(\operatorname{ad}\xi)^{k-1}D\xi\right] + P(\operatorname{ad}\xi)^{p-k}DP(\operatorname{ad}\xi)^k x\Big).$$

Using the identity $(\operatorname{ad}\xi)^k DP = -[(\operatorname{ad}\xi)^{k-1}D\xi, P(\,.\,)]$, for $k \geqslant 1$ (see the proof of Theorem 1), we can re-write this as

$$(4.14)\quad (-1)^p J^{p+1}(\Pi x,\xi) = P\Bigg(-(\operatorname{ad}\xi)^p Dx - [(\operatorname{ad}\xi)^{p-1}D\xi, Px] +$$
$$\sum_{k=1}^{p-1} C_p^k \Big(-\left[(\operatorname{ad}\xi)^{p-k}x, P(\operatorname{ad}\xi)^{k-1}D\xi\right] - \left[(\operatorname{ad}\xi)^{p-k-1}D\xi, P(\operatorname{ad}\xi)^k x\right]\Big)$$
$$- \left[x, P(\operatorname{ad}\xi)^{p-1}D\xi\right] + DP(\operatorname{ad}\xi)^p x \Bigg) =$$
$$-P(\operatorname{ad}\xi)^p Dx + PD(\operatorname{ad}\xi)^p x - P[(\operatorname{ad}\xi)^{p-1}D\xi, x] - \sum_{k=1}^{p-1} C_p^k P\left[(\operatorname{ad}\xi)^{p-k}x, (\operatorname{ad}\xi)^{k-1}D\xi\right] =$$
$$P\left[D, (\operatorname{ad}\xi)^p\right]x - \sum_{k=1}^{p} C_p^k P\left[(\operatorname{ad}\xi)^{p-k}x, (\operatorname{ad}\xi)^{k-1}D\xi\right]$$

where we used identities (2.7) and (2.8). Now, by arguing in the same way as we did when deducing the expression (2.10) for the commutator of D and $(\operatorname{ad}\xi)^N$ acting on $D\xi$ in the proof of Theorem 1, we can deduce the equality

$$[D, (\operatorname{ad}\xi)^p]\,x = \sum_{r=0}^{p-1} C_p^{p-1-r}\left[(\operatorname{ad}\xi)^r D\xi, (\operatorname{ad}\xi)^{p-1-r}x\right] =$$
$$\sum_{k=1}^{p} C_p^k \left[(\operatorname{ad}\xi)^{k-1}D\xi, (\operatorname{ad}\xi)^{p-k}x\right].$$

Notice that since x is odd, ξ is even, and D is odd, in the Lie bracket above both arguments are odd, so the order is irrelevant. We immediately see that the two terms in the last line of (4.14) cancel, and thus for all x and ξ

$$J^{p+1}(\Pi x,\xi) = 0,$$

as desired. Now consider $J^{p+2}(\Pi x, \xi)$. Substituting the definitions of the brackets into (4.13), we get

$$J^{p+2}(\Pi x, x) = -\big\{\Pi[x,x], \underbrace{\xi, \dots, \xi}_{p}\big\} + 2\sum_{k=0}^{p} C_p^k \big\{\Pi x, P(-\operatorname{ad}\xi)^k x, \underbrace{\xi, \dots, \xi}_{p-k}\big\} =$$

$$-(-1)^p P(\operatorname{ad}\xi)^p [x,x] + 2(-1)^p \sum_{k=0}^{p} C_p^k P(\operatorname{ad}\xi)^{p-k}\left[x, P(\operatorname{ad}\xi)^k x\right],$$

or

$$(-1)^{p+1} J^{p+2}(\Pi x, x) = P(\operatorname{ad}\xi)^p [x,x] - 2\sum_{k=0}^{p} C_p^k P(\operatorname{ad}\xi)^{p-k}\left[x, P(\operatorname{ad}\xi)^k x\right] =$$

$$P(\operatorname{ad}\xi)^p [x,x] - 2\sum_{k=0}^{p} C_p^k P\left[(\operatorname{ad}\xi)^{p-k} x, P(\operatorname{ad}\xi)^k x\right] =$$

$$P(\operatorname{ad}\xi)^p [x,x] - P\sum_{k=0}^{p} C_p^k \left[(\operatorname{ad}\xi)^{p-k} x, (\operatorname{ad}\xi)^k x\right] =$$

$$P(\operatorname{ad}\xi)^p [x,x] - P(\operatorname{ad}\xi)^p [x,x] = 0,$$

where we used the commutativity of V and identity (2.7). Thus for all x and ξ

$$J^{p+2}(\Pi x, \xi) = 0,$$

as desired. This completes the proof of the theorem. □

A remarkable fact about the formulae for the brackets in $\Pi L \oplus V$ is that they arise naturally if one wants to extend the bracket in ΠL keeping the differential (4.2) a derivation. Of course, the crucial and much harder thing is to prove that they indeed give the structure of an L_∞-algebra as stated by Theorem 2. The subspace V is a subalgebra (even an ideal) with respect to this structure, and the induced brackets are exactly the higher derived brackets.

Corollary 4.1. *The complex $\Pi L \oplus V$, with operations defined as above, is a cocylinder for $i\colon \Pi K \to \Pi L$ in the category of L_∞-algebras, and V with the higher derived brackets of D is a homotopy fiber (or a cocone), in this category, for the inclusion i of differential Lie superalgebras.*

Remark 4.3. The idea of relating the higher derived brackets of Δ with homotopical algebra was proposed by the referee of the first version of [14]. He conjectured, for the $\mathbb{Z}$-graded case, an interpretation of these brackets in terms of a 'homotopy left cone' (cocone, in our terminology) and suggested a formula of type (4.8) for the extended brackets. In this section we showed that the conjecture about a homotopical-algebraic interpretation of higher derived brackets is correct, in the natural setup where the brackets are generated by an arbitrary odd derivation D. Corollary 4.1 gives the precise statement.

The considerations of this section give an alternative and quite unexpected, viewpoint of higher derived brackets. For a given derivation D, which is assumed to be a differential, the construction of the complex $\Pi L \oplus V$, viewed as a cone (for $L \to V$) or a cocylinder (for $\Pi K \to \Pi L$) is canonical. The higher derived brackets of D appear as an answer to the question of how to extend the algebra structure to $\Pi L \oplus V$ from L.

Notice also that although homological or homotopical algebra requires $D^2 = 0$ from the start, we never directly used this identity in the proof of Theorem 2, except where we referred to Theorem 1 in the particular case when $D^2 = 0$; hence it seems reasonable that the homotopical-algebraic picture can be rephrased in a way allowing to incorporate a possibly non-zero D^2.

5 Generalizations and Discussion

Let us return to Theorem 1 and see what information can be extracted from it if one does not immediately set D^2 equal to zero. To be able to make a precise statement, notice that our construction of higher derived brackets allows extension of scalars, in the following sense.

Consider an arbitrary commutative superalgebra Λ with unit (a good example is the Grassmann algebra with N generators, $\Lambda = \Lambda_N$) and the tensor product $L \otimes \Lambda$. It is a Lie superalgebra over Λ, and $\mathrm{Der}_\Lambda(L \otimes \Lambda) = (\mathrm{Der}\, L) \otimes \Lambda$. Thus the higher derived brackets can be constructed from $D \in \mathrm{Der}_\Lambda(L \otimes \Lambda)$, i.e., a derivation with coefficients in Λ. They will be operations on $V \otimes \Lambda$. (In particular, brackets generated by $D \in \mathrm{Der}\, L$ can be considered on $V \otimes \Lambda$ for any Λ and this explains why it is sufficient to check the Jacobiators only on even arguments.) Clearly, Theorem 1 remains valid. Now, the map which assigns to a derivation D all its higher derived brackets is a linear operation in the sense that it commutes with sums and with multiplication by scalars. Now we shall make use of the following obvious algebraic statement: *if a linear map of Lie superalgebras maps the squares of odd elements to squares, for all extensions of scalars by various Λ, then it is a Lie algebra homomorphism.* (Indeed, by polarization, it maps all brackets of odd elements to the brackets; then by using suitable odd constants, even elements can be turned into odd, and after that the constants can be eliminated.)

An arbitrary sequence of multilinear symmetric operations on V can be encoded in a (formal) vector field X, which serves as their generating function, so that the operations are obtained as the higher derived brackets of X:

$$\{u_1, \ldots, u_k\}_X = [\ldots [X, u_1], \ldots, u_k]\,(0)$$

where $u_i \in V$, $X \in \mathrm{Vect}\, V$, as in (2.4). If we restrict ourselves to formal vector fields, this correspondence will be one-to-one. The sequence of the Jacobiators of the brackets derived from X has the vector field X^2 as the generating function (this is a very special case of Theorem 1, but can be seen directly).

Consider now an arbitrary derivation $D\colon L \to L$. Denote the vector field on V corresponding to the higher derived brackets of D, by Q_D. Theorem 1 then can be re-formulated as the equality

$$(Q_D)^2 = Q_{D^2} \tag{5.1}$$

for all odd D. Having in mind the above remarks, we see that Theorem 1 is equivalent to the following.

Theorem 3. *The correspondence $D \mapsto Q_D$ is a homomorphism of Lie superalgebras* $\operatorname{Der} L \to \operatorname{Vect} V$, *i.e.,*

$$[Q_{D_1}, Q_{D_2}] = Q_{[D_1,D_2]} \tag{5.2}$$

for all $D_1, D_2 \in \operatorname{Der} L$.

(It is an interesting question whether there is a more direct way of constructing a vector field on V from the following data: the homological field specifying the Lie bracket in L and a derivation D.)

Let $\mathfrak{g}$ be a Lie superalgebra and V a vector space. We call the space V a *generalized L_∞-algebra over* $\mathfrak{g}$ (or: a $\mathfrak{g}$-*parametric L_∞-algebra*) if there is given a homomorphism $\mathfrak{g} \to \operatorname{Vect} V$. We can visualize this as (sequences of) brackets in V parametrized by elements of $\mathfrak{g}$. Relations between elements of $\mathfrak{g}$ give rise to 'generalized Jacobi identities' in V between the corresponding brackets.

Example 5.1. If $\mathfrak{g}$ has dimension $0|1$, with a single odd basis element Q satisfying $Q^2 = 0$, then we get a usual L_∞-algebra structure.

Example 5.2. If $\mathfrak{g}$ has dimension $1|1$, with a basis H, Q with H even, Q odd, satisfying $Q^2 = H$, then a generalized L_∞-algebra over $\mathfrak{g}$ is the same as an arbitrary sequence of odd symmetric brackets that a priori are not subject to any relations. (In fact, there are some relations that are always satisfied, they are the 'mixed' Jacobi identities for odd brackets and their Jacobiators, corresponding to the identity $[H, Q] = 0$.)

Apart from these two opposite extremes there should be other interesting examples.

Another attractive direction is to study the higher derived brackets where V in the decomposition $L = K \oplus V$ is not assumed Abelian. Notice that this is exactly the case in the original definition of a (binary) derived bracket: given a Lie superalgebra L and an odd derivation $D\colon L \to L$, then for arbitrary $a, b \in L$

$$[a, b]_D := [Da, b] \tag{5.3}$$

(we use a sign convention convenient for the comparison with (2.1)). This is a particular case of (2.1) for $k = 2$ if $L = V$ and $K = 0$. It is known that the derived bracket (5.3) is not, in general, symmetric:

$$[a, b]_D - (-1)^{\tilde{a}\tilde{b}}[b, a]_D = D[a, b] \tag{5.4}$$

(in typical applications it is possible to restrict to an Abelian subalgebra, thus restoring symmetry and making it into a different special case of (2.1) for $k = 2$ with a 'hidden' P).

Proposition 5.1. *In the context of $L = K \oplus V$ where V is not necessarily Abelian, the k-th derived brackets defined by* (2.1) *satisfy the identity*

$$\{a_1, \dots, a_i, a_{i+1}, \dots, a_k\}_D - (-1)^{\tilde{a}_i \tilde{a}_{i+1}} \{a_1, \dots, a_{i+1}, a_i, \dots, a_k\}_D = \{a_1, \dots, [a_i, a_{i+1}], \dots, a_k\}_D \tag{5.5}$$

for the transposition of two adjacent arguments, for all $a_1, \dots, a_k \in V$ and all $i = 1, \dots, k-1$. Here on the right-hand side we have the $(k-1)$-th derived bracket with the Lie bracket of the arguments a_i and a_{i+1} inserted at the i-th position.

The proof is not hard and we omit it. Formula (5.5) generalizes (5.4).

It is known that the classical derived bracket, though not symmetric, satisfies the Jacobi identity, defining an odd Loday algebra if $D^2 = 0$. What about analogs for higher derived brackets? What is the precise list of relations in an algebraic structure defined by the higher derived brackets if V is non-Abelian? (It includes an even Lie bracket as well as a sequence of odd brackets and may be called an 'L_∞-algebra on a Lie algebra background'.) It may be possible to make use of a homotopic-algebraic approach such as in Section 4. We hope to consider these questions elsewhere.

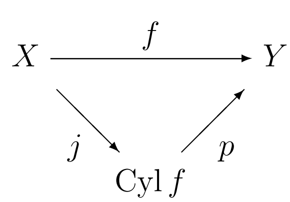
A Appendix. Standard cylinders and cocylinders

Here we collect, for reference purposes, the formulae for the standard constructions of cylinders and cocylinders of chain maps (compare, e.g., [2]). They all originate in topological constructions of the cylinder $X \times I$ and cocylinder X^I.

A *complex* is a ($\mathbb{Z}_2$-graded) vector space equipped with an odd operator d such that $d^2 = 0$. A *map* or a 'chain map' is an even linear map commuting with d.

Let $f\colon X \to Y$ be a map of complexes.

The standard *cylinder* diagram for $f\colon X \to Y$ is the commutative diagram



where

$$\operatorname{Cyl} f = X \oplus \Pi X \oplus Y$$

with the differential given by

$$d(x_1, x_2, \Pi y) = (dx_1 - x_2, \Pi(-dx_2), dy + f(x_2)).$$

The maps j and p are given by the formulae

$$j(x) = (x, 0, 0)$$
$$p(x_1, \Pi x_2, y) = f(x_1) + y,$$

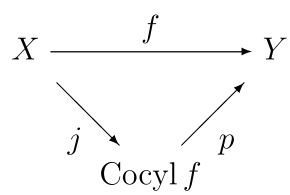
and p is a quasi-isomorphism with a quasi-inverse map $i\colon Y \to \operatorname{Cyl} f$, $i(y) = (0, 0, y)$. The *cone* of f is the cofiber of j, i.e., $\operatorname{Cyl} f/j(X)$. Hence

$$\operatorname{Con} f = \Pi X \oplus Y$$

with the differential

$$d(\Pi x, y) = (\Pi(-dx), dy + f(x)).$$

In a similar way, the standard *cocylinder* diagram for $f\colon X \to Y$ is the commutative diagram



where

$$\operatorname{Cocyl} f = X \oplus Y \oplus \Pi Y$$

with the differential given by

$$d(x, y_1, \Pi y_2) = (dx, dy_1, \Pi(f(x) - y_1 - dy_2)).$$

The maps j and p are given by the formulae

$$j(x) = (x, f(x), 0)$$
$$p(x, y_1, \Pi y_2) = y_1,$$

and j is a quasi-isomorphism with a quasi-inverse map $q\colon \operatorname{Cocyl} f \to X$, $q(x, y_1, \Pi y_2) = x$. The *cocone* of f is the fiber (kernel) of p. Hence

$$\operatorname{Cocon} f = X \oplus \Pi Y$$

with the differential

$$d(x, \Pi y) = (dx, \Pi(f(x) - dy)).$$

It follows that $\Pi \operatorname{Con} f = \operatorname{Con} f^{\Pi} = \operatorname{Cocon}(-f)$; i.e., up to a sign, the cone and cocone of a chain map f are related by the parity shift functor. In the main text, the complex $L \oplus \Pi V$ appearing there as a cocylinder of the inclusion of complexes $i\colon K \to L$, can be alternatively viewed as the canonical $\operatorname{Cocon}(-P)$ or as $\Pi \operatorname{Con} P$ where the projector P is treated as a map $L \to V$, so V with the differential PD is considered as a quotient complex of L (rather than a subspace of L).

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Dr Theodore Voronov
School of Mathematics
University of Manchester
Sackville Street
Manchester M60 1QD
United Kingdom
`theodore.voronov@manchester.ac.uk`