Computing Hecke Operators On Drinfeld Cusp Forms

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Notation

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- $k := \mathbb{F}_q$ with $q = p^r$
- $K := \mathrm{Quot}(k[T])$
- v_∞ valuation of K at the place ∞, i.e. $v_\infty(\frac{f}{g}) = \deg(g) - \deg(f), v_\infty(0) = \infty$
- K_∞ the completition of K at v_∞, i.e. $K_\infty = k((\pi_\infty))$ the laurent series ring where π_∞ is the uniformizer T^{-1}.
- $\mathcal{O}_\infty := \{x \in K_\infty \mid v_\infty(x) \geqslant 0\}$

The Bruhat-Tits-Tree $\mathcal{T}$

Definition of $\mathcal{T}$

- Let $X(\mathcal{T})$ be the equivalence classes of $\mathcal{O}_\infty$-lattices in K_∞^2. Each such equivalence class defines a vertex of $\mathcal{T}$.
- Let $\Lambda, \Lambda' \in X(\mathcal{T})$ and choose a lattice $L \in \Lambda$. Λ and Λ' are connected in $\mathcal{T}$ iff there exists a $L' \in \Lambda'$ such that $L' \subseteq L$ and $L/L' \simeq \mathcal{O}_\infty/\pi_\infty\mathcal{O}_\infty$. The set of directed edges of $\mathcal{T}$ is called $Y(\mathcal{T})$.

Theorem about the structure of $\mathcal{T}$

$\mathcal{T}$ is a $q+1$-regular tree, i.e. $\mathcal{T}$ is a connected, cycle-free tree, where every vertix has $q+1$ neighbours.

Example for $q = 3$

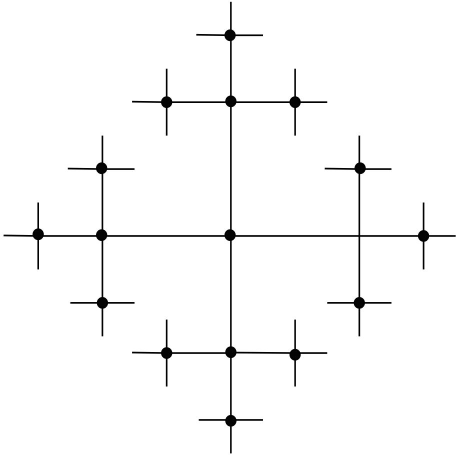

Figure: The Bruhat-Tits-Tree for $k = \mathbb{F}_3$

Operation of $\mathrm{GL}_2(k[T])$ on $\mathcal{T}$

- There is a bijection

$$X(\mathcal{T}) \longrightarrow \mathrm{GL}_2(K_\infty)/\,\mathrm{GL}_2(\mathcal{O}_\infty)K_\infty^\star$$

- There is a bijection

$$Y(\mathcal{T}) \longrightarrow \mathrm{GL}_2(K_\infty)/\Gamma_\infty K_\infty^\star$$

with $\Gamma_\infty := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\infty) \mid v_\infty(c) > 0\}$

- $\mathrm{GL}_2(k[T])\backslash\mathcal{T}$ is just a half-line.
  Reason: $\mathrm{GL}_2(k[T])\backslash\,\mathrm{GL}_2(K_\infty)/\,\mathrm{GL}_2(\mathcal{O}_\infty)K_\infty^\star \cong \{\begin{pmatrix} 1 & 0 \\ 0 & \pi^n \end{pmatrix} \mid n \in \mathbb{N}\}$
- We write Λ_n for the class of the lattice $\mathcal{O}_\infty \oplus \pi_\infty^n \mathcal{O}_\infty$

Congruence subgroups

Let $N \in \mathbb{F}_q[T]$ be normalized.

- $\Gamma(N) := \{\gamma \in \mathrm{GL}_2(k[T]) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N\}$
- A subgroup of $\mathrm{GL}_2(k[T])$ containig $\Gamma(N)$ for any $N \in k[T]$ is called a congruence subgroup.
- $\Gamma_0(N) := \{\gamma \in \mathrm{GL}_2(k[T]) \mid \gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mod N\}$
- $\Gamma_1(N) := \{\gamma \in \mathrm{GL}_2(kT]) \mid \gamma \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod N\}$
- Congruence subgroups are of finite index in $\mathrm{GL}_2(\mathbb{F}_q[T])$, since $\Gamma(N)\backslash \mathrm{GL}_2(k[T]) \cong \begin{pmatrix} k^\star & 0 \\ 0 & 1 \end{pmatrix} \mathrm{SL}_2(k[T]/N)$.

Calculation of $\Gamma\backslash\mathcal{T}$ with Γ a congruence subgroup, Idea

- $\Gamma\backslash\mathcal{T}$ is a covering of $\mathrm{GL}_2(k[T])\backslash\mathcal{T}$
- $\mathrm{GL}_2(k[T])\backslash\mathcal{T}$ is a simple half line.
- $\mathrm{GL}_2(k[T])\backslash\mathcal{T} : \Lambda_0 \to \Lambda_1 \to \Lambda_2 \to \dots$
- Elements of $\Gamma\backslash\mathcal{T}$ are Γ-orbits of $\mathcal{T}$. Every $\mathrm{GL}_2(k[T])$-orbit of $\mathcal{T}$ decomposes into finitly many Γ-orbits, since $\Gamma\backslash\mathrm{GL}_2(k[T])$ is finite.
- We need to know $\mathrm{Stab}_{\mathrm{GL}_2(k[T])}(\Lambda_i)$ to see how an $\mathrm{GL}_2(k[T])$-orbit decomposes.

Algorithm for the calculation of $\Gamma\backslash\mathcal{T}$

- Let $G_i := \mathrm{Stab}_{\mathrm{GL}_2(k[T])}(\Lambda_i)$. A simple calculation shows, that $G_0 = \mathrm{GL}_2(k)$ and $G_i = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in k^\star, b \in k[T], \deg(b) \leqslant i\}$.
- Let $S = \{s_1, \ldots, s_n\}$ be a set of representativs of $\Gamma\backslash\mathrm{GL}_2(k[T])$.
- Let Υ be the standard half line $\Lambda_0 \to \Lambda_1 \to \Lambda_2 \to \ldots$ and $s_i(\Upsilon)$ the halfline $s_i(\Lambda_0) \to s_i(\Lambda_1) \to s_i(\Lambda_2) \to \ldots$
- Then $\Gamma\backslash\mathcal{T}$ can be obtained by taking the halflines $s_1(\Upsilon), \ldots, s_m(\Upsilon)$ and identify vertices and edges using the following rules:
  1. Only identify vertices and edges of the same level.
  2. $s_i(\Lambda_n) \sim s_j(\Lambda_n)$ iff there exists a $g \in G_n$ such that $s_i g s_j^{-1} \in \Gamma$.
  3. $s_i((\Lambda_0, \Lambda_1)) \sim s_j((\Lambda_0, \Lambda_1))$ iff there exists a $g \in G_0 \cap G_1$ such that $s_i g s_j^{-1} \in \Gamma$.
  4. $s_i((\Lambda_n, \Lambda_{n+1})) \sim s_j((\Lambda_n, \Lambda_{n+1}))$ iff there exists a $g \in G_n$ such that $s_i g s_j^{-1} \in \Gamma$ for $n \geqslant 1$.

Example: $q = 2$, $\Gamma_1(T^2)\backslash\mathcal{T}$

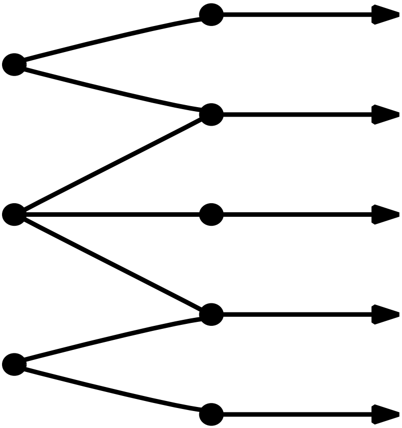

Figure: The Quotient $\Gamma_1(T^2)\backslash\mathcal{T}$ for $k = \mathbb{F}_2$

Example: $q = 3$, $\Gamma_1(T^2)\backslash\mathcal{T}$

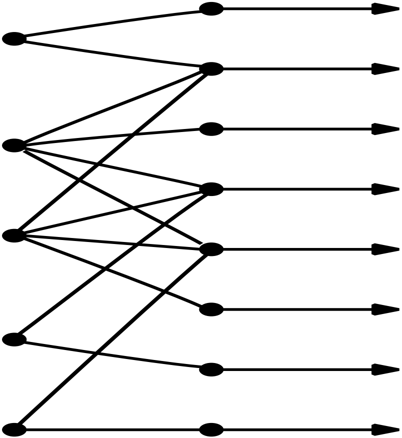

Figure: The Quotient $\Gamma_1(T^2)\backslash\mathcal{T}$ for $k = \mathbb{F}_3$

Definition of harmonic cocycles

- For an edge $e \in Y(\mathcal{T})$ let $e^\star$ denote the same edge with orientation reversed.
- For an vertex $v \in X(\mathcal{T})$ we write $e \mapsto v$ if v is the target of e
- Let M be any vector space with a $\mathrm{GL}_2(k[T])$-operation and $\Gamma \subseteq \mathrm{GL}_2(k[T])$ a congruence subgroup.
  1. A function $c : Y(\mathcal{T}) \longrightarrow M$ is called an M-valued harmonic cocycle, if
     1. for all vertices $v \in X(\mathcal{T})$ we have $\sum_{e \mapsto v} c(e) = 0$.
     2. $c(e^\star) = -c(e)$ for all edges $e \in Y(\mathcal{T})$.
  2. A function $c : Y(\mathcal{T}) \longrightarrow M$ is called Γ-equivariant, if for all $\gamma \in \Gamma$ we have $c(\gamma e) = \gamma c(e)$
- A Γ-equivariant harmonic cocycle c is called cuspidal, if there exist a finite subgraph $Z \subset \Gamma \backslash \mathcal{T}$ with $c(e) = 0$ for all $e \notin Y(\pi^{-1}(Z))$.

Theorem: Automatic cuspidality

Let M be a finite-dimensional vector-space over a field of characteristic p with a $\mathrm{GL}_2(k[T])$-operation. Then every M-valued Γ-equivariant harmonic cocycle is cuspidal.

Connection with Drinfeld cusp forms

Theorem (Teitelbaum, 1990)

There is an explicit $k[\Gamma]$-module V_m (with $dim_k V_m = m - 1$ and independent of Γ), such the following holds: Let Γ be a congruence subgroup of $GL_2(k[T])$ and let $S_m(\Gamma)$ be the space of Drinfeld cusp forms of level $m \geqslant 2$ for Γ. Then there is a (Hecke-equivariant) isomorphism from $S_m(\Gamma)$ to $C_{har}(\Gamma, V_m)$.

Stable Edges

- From now on let Γ be one of $\Gamma_1(N)$ or $\Gamma(N)$, i.e. Γ is p'-torsion-free for $p' \neq p$.
- An edge $e \in Y(\mathcal{T})$ (or a vertex $v \in X(\mathcal{T})$) is called Γ-stable, if $\mathrm{Stab}_\Gamma(e) = \{1\}$ (or $\mathrm{Stab}_\Gamma(v) = \{1\}$). (So, i.e. there are no $\mathrm{GL}_2(k[T])$-stable edges!)
- Fact: The stable part of the tree is connected in $\Gamma\backslash\mathcal{T}$.
- Fact: A vertex $v \in X(\mathcal{T})$ is stable if and only if its image in $\Gamma\backslash\mathcal{T}$ has exactly $q+1$ neighbours. An edge $v \in Y(\mathcal{T})$ is stable, if and only if one of the adjacent vertices is stable (except for the case $\Gamma_1(T)$).
- Fact: For every unstable edge $e \in Y(\mathcal{T})$ there is a finite and easy to compute set $\mathrm{Source}(e)$ of stable edges of $\mathcal{T}$ such that

$$c(e) = \sum_{e' \in \mathrm{Source}(e)} c(e')$$

Stable Edges, cont.

- So a harmonic cocycle c is determined by the values of c on the stable part of $\Gamma\backslash\mathcal{T}$
- Let $n = \deg(N)$. Then an edge in the covering over $(\Lambda_i, \Lambda_{i+1})$ with $i \geqslant n$ is unstable.
- In fact a harmonic cocycle is determined by the values of c on the stable edges over the edge (Λ_0, Λ_1), and for every stable vertix over Λ_0 we get one relation between these edges.

Example: $q = 2$, $\Gamma_1(T^2)\backslash\mathcal{T}$

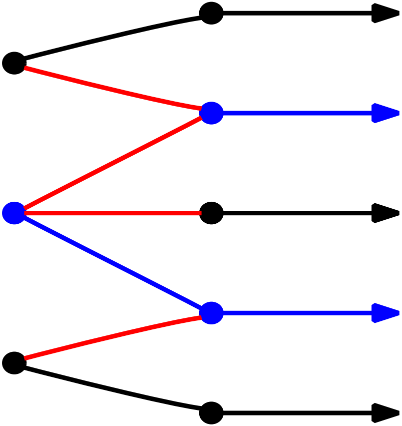

Figure: Colored: Stable edges and vertices. Red: Minimal set of edges, that determine a harmonic cocycle.

Hecke Operators on $C_{har}(\Gamma, V_m)$

- Translating the Hecke-action to the tree gives:

$$T_p(c)(e) = \sum_{\delta \in (\Gamma \cap \Gamma_0(p)) \backslash \Gamma} \delta^{-1} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} c(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \delta e).$$

- Let $e = (\gamma\Lambda_0, \gamma\Lambda_1)$ be given. To evaluate $c(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \delta e)$ we consider the matrices $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \delta\gamma$ and $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \delta\gamma \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$.
- Writing both these matrices in the form $\gamma' \begin{pmatrix} 1 & 0 \\ 0 & \pi^k \end{pmatrix} \alpha$ with $\alpha \in K_\infty^\star \mathrm{GL}_2(\mathcal{O}_\infty)$ and $\gamma' \in \mathrm{GL}_2(k[T])$ we find the new edge $\gamma'(\Lambda_k, \Lambda_{k+1})$.
- Write $\gamma' = \gamma_0 s_j$ with $\gamma_0 \in \Gamma$ and $s_j \in S$ and use the Γ-equivariance of c to obtain an edge in the pre-stored quotient graph $\Gamma\backslash\mathcal{T}$.
- If this edge is stable, than we know the value of c at this edge. If not, than we have to sum over the source of the edge.

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- **J.T. Teitelbaum:** Rigid Analytic Modular Forms: An Integral Transform Approach, In: *The Arithmetic of Function Fields, edited by D. Goss, D. R. Hayes, and M. I. Rosen, deGruyter Press* (1992)
- my master's thesis (Diplomarbeit)