Eta Products and Models for Modular Curves

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"Are we not drawn onward, we few, drawn onward to new era?"

Cusps, as Families of Tate Curves

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Definition: For $d|N$, the canonical families of width d on $X_0(N)$ are given by $(\mathbb{C}_p^*/q^d, \langle \zeta q \rangle)$, where ζ is any primitive r^{th} root of unity such that $\text{lcm}(r, d) = N$.

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Each family defines a rigid map from the disk $|q| < \epsilon$ into $X_0(N)$ which takes $q = 0$ to one of the cusps ($z \to z^{\text{gcd}(d,N/d)}$ followed by an injection). Note: it is easy to check when two families represent the same cusp (there are $\phi(\text{gcd}(d, N/d))$ different ones of width d).

Lemma: $(\mathbb{C}_p^*/q^d, \langle \zeta q \rangle) \sim (\mathbb{C}_p^*/q_1^d, \langle \zeta q_1 \rangle)$ if and only if $q^{\gcd(d,N/d)} = q_1^{\gcd(d,N/d)}$.

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First suppose $\langle \lambda^{jN/r} q \rangle = \langle \lambda^{jN/r} \lambda^{kN/d} q \rangle$. Then

$$jN/r = (jN/r + kN/d)(1 + ld) + mN$$

$$0 = kN/d + ldjN/r + ldkN/d + mN.$$

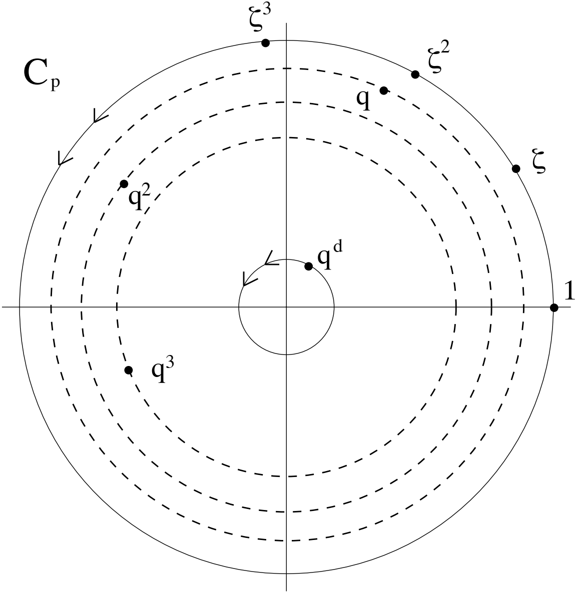
Thus, we see that $d \Big| (kN/d)$. So

$$\frac{d}{\gcd(d,N/d)} \Bigg| k.$$

Therefore $q_1^{\gcd(d,N/d)} = \lambda^{kN\gcd(d,N/d)/d} q^d = q^d$.

The other direction is similar.

The Tate Curve $\mathbb{C}_p^*/q^d$ with Some of its N-torsion



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So at this point, we've defined q-expansions associated to families of Tate curves, based on the moduli-theoretic definition of modular forms. **Why?**

Pullbacks by Level-Lowering Maps

Suppose that $M\ell \big| N$. It is well-known that we have maps, $\pi_\ell : X_0(N) \to X_0(M)$, defined by

$$\pi_\ell(E, C) = (E/C[\ell], C[M\ell]/C[\ell]).$$

We can also define the π_ℓ pullback of a modular form f for $\Gamma_0(M)$.

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Definition: Let f be a weight k modular form for $\Gamma_0(M)$, with $M\ell \big| N$ as above. Then we define

$$(\pi_\ell^* f)(E, C) = \iota^*(f(E/C[\ell], C[M\ell]/C[\ell])),$$

where $\iota : E \to E/C[\ell]$ is the canonical isogeny and

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Big Idea: We can use Tate curve calculations to compute the q-expansions of $\pi_\ell^* f$ in terms of the q-expansions of f.

Theorem 1: Suppose r, d, M, and ℓ are positive divisors of N, such that $M\ell \Big| N$ (so $\pi_\ell : X_0(N) \to X_0(M)$) and $\operatorname{lcm}(d, r) = N$. Suppose ζ is a primitive r^{th} root of unity. Suppose f is any weight k modular form for $\Gamma_0(M)$.

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If

$$f(\mathbb{C}_p^*/(\zeta^{bNg/d} q^{\ell g^2/d}), \langle(\zeta q)^{\ell g/d}\rangle[M]) = f(q)\left(\tfrac{\mathrm{d}z}{z}\right)^{\otimes k},$$

then

$$\pi_\ell^* f(\mathbb{C}_p^*/q^d, \langle\zeta q\rangle) = f(q)\left(\frac{\ell g}{d}\right)^k \left(\tfrac{\mathrm{d}z}{z}\right)^{\otimes k}.$$

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pf/ We want to compute $\pi_\ell^* f$ on $(\mathbb{C}_p^*/q^d, \langle\zeta q\rangle)$ using the definition of π_ℓ^*, and in this case $E = \mathbb{C}_p^*/q^d$ and $C = \langle\zeta q\rangle$. So first we apply $\iota : E \to E/C[\ell]$ to get $\mathbb{C}_p^*/\langle q^d, (\zeta q)^{N/\ell}\rangle$. Now use

$$\mathbb{C}_p^*/\langle q^d, (\zeta q)^{N/\ell}\rangle \xrightarrow{\cong} \mathbb{C}_p^*/(\zeta^{bNg/d} q^{\ell g^2/d}) \qquad z \mapsto z^{\ell g/d}$$

Although we will only use the theorem to pull back forms of level 1 (in particular Δ), the theorem does tell us explicitly how to obtain the q-expansions of $\pi_\ell^* f$ at any cusp, given the q-expansions at the image cusp. (It's just a substitution, since q is just a number here!)

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Corollary 1.1: Suppose f is a form for $\Gamma_0(1)$ with q-expansion $f(q)$. Then the expansion at $(\mathbb{C}_p^*/q^d, \langle \zeta q \rangle)$ of $\pi_\ell^* f$ is given by

$$f(\zeta^{bNg/d} q^{\ell g^2/d}) \left(\frac{\ell g}{d}\right)^k .$$

Part II

Eta Products

Ligozat's Criteria

Definition: Let Δ be the usual weight 12 level 1 cusp form, with q-expansion $\Delta(q)$. Let $\eta(q) = (\Delta(q))^{1/24}$. So

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

An eta product is an expression of the form $\prod_{\ell \mid N} (\eta(q^{\ell}))^{r_{\ell}}$.

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Theorem 2: (Ligozat) An eta product is a weight 0 modular form, i.e. a modular function, on $X_0(N)$, if and only if

(i) $\sum r_d = 0$
(ii) $\sum d \cdot r_d \equiv 0 \pmod{24}$
(iii) $\sum \frac{N}{d} \cdot r_d \equiv 0 \pmod{24}$
(iv) $\prod \left(\frac{N}{d}\right)^{r_d} \in \mathbb{Q}^2$.

q-expansions of Δ pullbacks

Theorem 3: For any $\ell \big| N$, $\Delta(q^\ell)$ is the q-expansion at infinity of a weight 12 modular form for $\Gamma_0(N)$. Its q-expansion associated to the family, $(\mathbb{C}_p^*/q^d, \langle \zeta q \rangle)$ is given by

$$\Delta(\zeta^{bNg/d} q^{\ell g^2/d}) \left(\frac{g}{d}\right)^{12}.$$

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pf/ This is a direct application of Corollary 1.1. The form in question is actually $\ell^{-12}\pi_\ell^*(\Delta)$. So from the corollary, its expansion is

$$\ell^{-12}\Delta(\zeta^{bNg/d} q^{\ell g^2/d}) \left(\frac{\ell g}{d}\right)^{12}.$$

Corollary 3.1: The leading term of the q-expansion of any delta product (associated to a fixed family) is given by

$$\prod_{\ell | N} \left(\zeta^{bNg/d} q^{\ell g^2/d} \right)^{r_\ell} \left(\tfrac{g}{d} \right)^{12 r_\ell} .$$

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In particular, the ord at the corresponding cusp of the corresponding **eta** product is

$$\frac{1}{24\,\gcd(d,N/d)}\sum_{\ell|N}\frac{\ell\cdot r_\ell}{d}(\gcd(d,N/\ell))^2.$$

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Moreover, when the ord is 0 at a particular cusp, the **value** of the corresponding delta product is given (up to an r^{th} root of unity) by

$$\prod_{\ell|N} (\tfrac{1}{d}\,\mathrm{gcd}(d,N/\ell))^{12r_\ell}.$$

Eta products on $X_0(18)$

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Applying the corollary about ords, we find:

$$\begin{bmatrix} 1 & 2 & 3 & 6 & 9 & 18 \\ 2 & 1 & 6 & 3 & 18 & 9 \\ 1 & 2 & 3 & 6 & 1 & 2 \\ 2 & 1 & 6 & 3 & 2 & 1 \\ 9 & 18 & 3 & 6 & 1 & 2 \\ 18 & 9 & 6 & 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_6 \\ r_9 \\ r_{18} \end{bmatrix} = \begin{bmatrix} \text{ord}(d=1) \\ \text{ord}(d=2) \\ \text{ord}(d=3) \\ \text{ord}(d=6) \\ \text{ord}(d=9) \\ \text{ord}(d=18) \end{bmatrix}$$

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Let $f = \frac{\eta_2^2 \eta_9}{\eta_1 \eta_{18}^2}$.

By Ligozat, it is a legitimate function, and by above it has divisor $(1/2) - (\infty)$. Hence it is a parameter on this genus 0 curve.

Moreover, the values of f^{24} at the $d = 18, 6, 3$, and 2 cusps are 3^{24}, 3^{12}, 3^{12}, and 1 (respectively, up to a root of unity). These values can be easily verified (in *this* case) by choosing eta products which vanish at the other cusps, and then comparing with f.

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Theorem 4: (Shimura) Let f be a form of even weight k. Let $\nu = f(z)(\mathrm{d}z)^{k/2}$. Then

$$\mathrm{Div}(f) = \mathrm{Div}(\nu) + (k/2) \cdot \left(\sum (1 - e_i^{-1}) P_i + cusps \right),$$

where $\{P_i\}$ are the elliptic points of order e_i.

A Nice Example Involving $X_0(11)$

Doing the usual thing, we find that $t = \eta_1^{12}/\eta_{11}^{12}$ is a legitimate function with divisor $5(0) - 5(\infty)$. We also get a weight *two* cusp form by taking $(\eta_1\eta_{11})^2$. Using the preceding theorem, its divisor (as a form) is $(0) + (\infty)$.

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There aren't any elliptic points. So the function,

$$x := \frac{\mathrm{d}t/t}{(\eta_1\eta_{11})^2},$$

has degree 2, with a simple pole at each cusp. This implies that x is a parameter on $X_0(11)^+ := X_0(11)/w_{11}$. By comparing q-expansions, we find:

$$t^2 + \frac{1}{5^5}(x^5 + 170x^4 + 9345x^3 + 167320x^2 - 7903458)t + 11^6 = 0.$$

Remark:

This reduces mod 11 to

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By letting

$$y = \frac{2 \cdot 5^5 t + (x^5 + 170x^4 + 9345x^3 + 167320x^2 - 7903458)}{(x+47)(x^2+89x+1424)}$$

we arrive at the "nicer" model:

$$y^2 = (x-8)(x^3 + 76x^2 - 8x + 188).$$

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Note: $\pi_1^* j = \dfrac{(60y + 61x^2 + 864x - 2016)^3}{5^6 t}$.

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(2) Using (1), we have nice formulas for everything you'd want to know about eta products.

(3) Eta products can be used to get really nice explicit models for $X_0(N)$, even if $N = p$.

Part III

Implementation

Eta Product Package Wish List

Easy: (just derived the formulas)

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(4) Basis for eta products (as a $\mathbb{Z}$-module).
(5) Eta products of minimal degree.
(6) Equations relating choice of finitely many eta products.

Basis Calculation for $X_0(18)$

Ligozat condition is equivalent to a system of $(2 + p(N))$ homogeneous linear congruences mod 24, in $(d(N) - 1)$ variables (where $p(N)$ is the number of primes dividing N). Simple Gaussian elimination should give a basis over $\mathbb{Z}/24\mathbb{Z}$ and then over $\mathbb{Z}$.

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$$\begin{array}{cccccccl}
r_1 & +2r_2 & +3r_3 & +6r_6 & +9r_9 & +18r_{18} & \equiv 0 & \pmod{24} \\
18r_1 & +9r_2 & +6r_3 & +3r_6 & +2r_9 & +r_{18} & \equiv 0 & \pmod{24} \\
12r_1 & & +12r_3 & & +12r_9 & & \equiv 0 & \pmod{24} \\
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\end{array}$$

Basis Calculation for $X_0(18)$

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\end{array}
$$

$$
\begin{bmatrix}
-17 & -16 & -15 & -12 & -9 \\
17 & 8 & 5 & 2 & 1 \\
12 & 0 & 12 & 0 & 12 \\
0 & 0 & 12 & 12 & 0
\end{bmatrix}
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$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_6 \\ r_9 \\ r_{18} \end{bmatrix} = a \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ -3 \end{bmatrix} + b \begin{bmatrix} 4 \\ -2 \\ 4 \\ 0 \\ 0 \\ -6 \end{bmatrix} + c \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \\ 0 \\ -3 \end{bmatrix} + d \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Using eta products to find an equation on $X_0(26)$

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On $X_0(26)$, the eta product divisor matrix is:

$$\begin{bmatrix} 1 & 2 & 13 & 26 \\ 2 & 1 & 26 & 13 \\ 13 & 26 & 1 & 2 \\ 26 & 13 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ r_2 \\ r_{13} \\ r_{26} \end{bmatrix} = \begin{bmatrix} \operatorname{ord}(d=1) \\ \operatorname{ord}(d=2) \\ \operatorname{ord}(d=13) \\ \operatorname{ord}(d=26) \end{bmatrix}$$

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Initially, we choose the following two eta products:

$$t = \frac{\eta_2^2 \eta_{13}^2}{\eta_{26}^2 \eta_1^2} \qquad (t) = (1/2) + (1/13) - (0) - (\infty)$$

$$u = \frac{\eta_2^4 \eta_{13}^2}{\eta_{26}^4 \eta_1^2} \qquad (u) = 3(1/2) - 3(\infty)$$

Since $u = 13$ at the cusp 0, we let $v = t(u - 13)$ and must have:

$$a_1 u^4 + a_2 v^3 + a_3 v^2 u + a_4 v u^2 + a_5 u^3 + \\ a_6 v^2 + a_7 uv + a_8 u^2 + a_9 v + a_{10} u + a_{11} = 0.$$

By comparing q-expansions, we find:

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$$y^2 = x^6 - 8x^5 + 8x^4 - 18x^3 + 8x^2 - 8x + 1.$$

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Remark: $\pi_1^*(\eta_1^2/\eta_{13}^2) = u/t^2$ and $\pi_2^*(\eta_1^2/\eta_{13}^2) = u/t$.