Computation of Siegel Modular Forms of Genus 2

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When compared with classical (elliptic) modular forms, Siegel modular forms of degree n are

- multivariate modular forms
- harder to compute
- not as well-studied

Definition

Let $\mathfrak{H}_n = \{Z = X + iY \in M_{n\times n}(\mathbb{C}) : {}^tZ = Z, Y > 0\}$ be the Siegel upper half space of degree n.

Definition

Let $\mathrm{Sp}_{2n}(\mathbb{R})$ be the symplectic group; i.e., the subgroup of $\mathrm{SL}_{2n}(\mathbb{R})$ preserving a fixed nondegenerate alternating bilinear form on $\mathbb{R}^{2n}$.

Proposition

Write $M \in SL_{2n}(\mathbb{R})$ as $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where $A, B, C, D \in M_n(\mathbb{R})$. Then $M \in Sp_{2n}(\mathbb{R})$ iff ${}^tAD - {}^tBC = 1$ and ${}^tBD$ and ${}^tAC$ are symmetric.

A holomorphic function $F : \mathfrak{H}^n \to \mathbb{C}$ is a Siegel modular form of degree $n \in \mathbb{Z}$, $n > 0$, and weight $k \in \mathbb{Z}$, $k > 0$, if for all $\alpha = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n$ it satisfies the transformation property

$$\begin{aligned} F(Z) &= (F|_k\alpha)(Z) \\ &:= \det(CZ + D)^{-k} F((AZ + B)(CZ + D)^{-1}). \end{aligned}$$

If $n = 1$ then F must satisfy an additional growth condition.

$$F(Z) = \sum_{\substack{r,n,m \in \mathbf{Z} \\ r^2 - 4mn \leq 0 \\ n,m \geq 0}} a_F(n,r,m) q^n \zeta^r q'^m$$

where

- $[n, r, m]$ is the positive semidefinite binary quadratic form $nX^2 + rXY + mY^2$ of discriminant $r^2 - 4mn$ and
- $q = e^{2\pi iz}$ $(z \in \mathfrak{H}^1)$,$q' = e^{2\pi i\omega}$ $(\omega \in \mathfrak{H}^1)$, and $\zeta = e^{2\pi i\tau}$ $(\tau \in \mathbb{C})$.

Definition

If the Fourier expansion of F is supported on positive definite quadratic forms, then F is a cusp form.

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sage: M4 = ModularForms(1,4)
sage: M6 = ModularForms(1,6)
sage: E4 = M4.basis()[0]
sage: F = siegel_modular_form (E4, M6(0),16); F
Siegel modular form None on Sp(2,Z) of weight 4.
sage: F.coeffs()

{(0, 0, 0): 1/60, (0, 0, 1): 4, (0, 0, 2): 36,
(0, 0, 3): 112, (0, 0, 4): 292, (1, 0, 1): 504,
(1, 0, 2): 3024, (1, 0, 3): 8288, (1, 1, 1): 224,
(1, 1, 2): 2304, (1, 1, 3): 6048, (1, 1, 4): 16128,
(2, 1, 2): 16128, (2, 2, 2): 10080}
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Standard approaches to computing modular forms (and Hecke data)

1. ... find the generators of the ring
2. ... compute enough Θ-series
3. ... use modular symbols
4. ... counting points on some geometric object
5. ... apply the trace formula

Standard approaches to computing modular forms (and Hecke data)

1. ... find the generators of the ring – feels very *ad-hoc* but can be fast
2. ... compute enough Θ-series – computing number of ways of representing one quadratic form by another is slow
3. ... use modular symbols – current machinery only sees the top dimension, not the cohomological dimension
4. ... counting points on some geometric object – not clear how to do this for scalar valued Siegel modular forms; can be done for vector valued Siegel modular forms and for paramodular forms
5. ... apply the trace formula – at best would only give Hecke data but this isn't enough to characterize a form

Theorem (Igusa)

The ring of Siegel modular forms of degree 2, level 1 and even weight are generated by the unique modular forms of weight 4 and 6 (Eisenstein series) and the unique cusp forms of weight 10 and 12.

Theorem (Saito-Kurokawa)

To every cusp form of degree 1 and weight $2k-2$ one can associate a cusp form F of degree 2 so that

$$L(F,s)=\zeta(\cdot)\zeta(\cdot)L(f,s)$$

Skoruppa made the lift effective:

- explicit formula for a lift from cusp forms of degree 1 and weight $2k-2$ to Jacobi forms of weight k and index 1
- explicit formula for a lift from Jacobi forms of weight k and index 1 to Siegel forms of weight k and degree 2

Lifts vs Nonlifts

- lifts violate Ramanujan-Petersson; nonlifts may or may not
- lifts violate the Riemann hypothesis; nonlifts may or may not
- lifts have L-functions that factor; nonlifts may or may not
- lifts satisfy multiplicity one; nonlifts may or may not

An L-function attached to a Siegel modular form of degree 2 is determined by two data:

- first, an automorphic representation π of adelic points of PGSp(4)
- second, a d-dimensional representation ρ of the dual group of PGSp(4):

$$\rho : \mathrm{Sp}(4, \mathbb{C}) \longrightarrow \mathrm{GL}(d, \mathbb{C}).$$

We blur the distinction between π and F and write an L-functions as $L(s, F, \rho)$.

Two best-known *L*-functions:

4: $\rho = \mathrm{id} := \mathrm{spin}$: Tate's thesis determines the Γ-factors and the non-archimedean factors:

$$[(1-\alpha_1 X)(1-\alpha_2 X)(1-X/\alpha_1)(1-X/\alpha_2)]^{-1}$$

5: $\rho = \mathrm{stan}$: Tate's thesis determines the Γ-factors and the nonarchimedean factors:

$$[(1-X)(1-\alpha_0 X)(1-\alpha_0\alpha_1 X)(1-\alpha_0\alpha_2 X)(1-\alpha_0\alpha_1\alpha_2 X)]^{-1}$$

Possess standard properties: functional equation, analytic continuations, etc.

The adjoint L-function is a degree 10 L-function determined by

$$\rho := \mathrm{Ad} : \mathrm{Sp}(4, \mathbb{C}) \to \mathrm{GL}(10, \mathbb{C})(\cong \mathrm{GL}(\mathfrak{sp}(4, \mathbb{C})))$$

where

$$g \mapsto (X \mapsto gXg^{-1})$$

Non-archimedean factors:

$$L(s,\pi_p,\mathrm{Ad})^{-1} = (1-p^{-s})^2(1-\alpha_1 p^{-s})(1-\alpha_1^{-1}p^{-s})$$
$$(1-\alpha_2 p^{-s})(1-\alpha_2^{-1}p^{-s})(1-\alpha_1\alpha_2 p^{-s})$$
$$(1-\alpha_1^{-1}\alpha_2 p^{-s})(1-\alpha_1\alpha_2^{-1}p^{-s})(1-\alpha_1^{-1}\alpha_2^{-1}p^{-s})$$

Archimedean factors:

$$2^{-4s-3k+9}\pi^{-5s-3k+4}\Gamma(s+1)\Gamma(s+k-2)\Gamma(s+k-1)$$
$$\times\,\Gamma(s+2k-3)\Gamma\left(\frac{s+1}{2}\right)^2$$

Theorem (Farmer-Garthwaite-Schmidt-R., 2008)

Using the smoothed approximate functional equation, we verify the functional equation for the adjoint L-function to a reasonable degree of accuracy. We also find the first 4 zeroes of the L-function.

- a similar result is expected for the “next” L-function
- we only know prime indexed Dirichlet series coefficients up to $p = 79$ and use some clever averaging techniques to evaluate the L-function

Conjectures in degree 1

- Maeda's conjecture
- Riemann hypothesis for L-functions attached to modular forms
- Sato-Tate conjecture for modular forms

Conjectures in degree 2

- Maeda's conjecture – shown to be false by Skoruppa (CAP representation?)
- Riemann hypothesis for L-functions attached to modular forms – false for lifts, in joint work with an undergraduate have verified it for nonlifts (spinor and standard L-functions)
- Sato-Tate conjecture for modular forms – same distribution for lifts, ongoing for nonlifts