RESEARCH SEMINAR: THE EIGENCURVE

WS 2009/10

The main topics of this seminar are p-adic modular forms and the eigencurve. The study of p-adic modular forms was started by Serre, Dwork and Katz, one of the goals being to explain congruences between classical modular forms.

Serre took the approach to define p-adic modular forms as limits of classical modular forms. Katz, on the other hand, started from the observation that modular forms are sections of line bundles on quotients $\Gamma\backslash\mathbb{H}$ of the complex upper half plane $\mathbb{H}$ by a congruence subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$. Interpreting this quotient as a modular curve, i.e., as a moduli space of elliptic curves, we can generalize this notion to other base rings. To obtain a truly p-adic notion of modular forms, one considers functions on the complement of small disks around lifts of the points corresponding to supersingular elliptic curves.

Although this reasoning was important as motivation, Katz [K] tried to avoid rigid analytic techniques. To the contrary, Coleman, who took up the subject in the 1990's made rigid analytic geometry (à la Tate) one of the foundations of the subject and in this way succeeded to explain many things in a more conceptual way, and in particular to give a definition of (overconvergent) p-adic modular forms of more general weights than just integers.

One of the most interesting phenomena that occur with p-adic modular forms is that they vary in families. The simplest example is the family of Eisenstein series. Recall that for an even integer $k \geq 4$, the Eisenstein series

$$E_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n\geq 1} \sigma_{k-1}(n)q^n$$

is a modular form of weight k. Here $q = \exp(2\pi i z)$, and $\sigma_d(n)$ denotes the sum of the d-th powers of the positive divisors of n. Although in terms of usual modular forms E_k makes sense only for k an even integer, the coefficients of its Fourier expansion above are defined for any $k \in \mathbb{C}$. Similarly, we can obtain a "p-adic family", where p is an odd prime, in the following way:

$$E_k^*(z) = E_k(z) - p^{k-1}E_k(pz) = \frac{(1-p^{k-1})\zeta(1-k)}{2} + \sum_{n\geq 1} \sigma_{k-1}^*(n)q^n,$$

where $\sigma_d^*(n)$ is the sum of the d-th powers of the positive divisors of n that are coprime to p. (The modification amounts to omitting the Euler factor at p.) Then $E_k^*(z)$ is again a modular form (an old-form of level p). Let S be the set of positive even integers $k \geq 4$, $k \equiv 0(p-1)$. This is a dense subset of $\mathbb{Z}_p$, the ring of p-adic integers with the p-adic topology. Then the Fourier coefficients of $E_k^*(z)$ are continuous as k varies in S (with respect to the topology on S induced by the embedding $S \subset \mathbb{Z}_p$).

Generalizing work of Hida (concerned with “ordinary” modular forms) and previous work of Coleman, Coleman and Mazur [CM] constructed a rigid analytic space, the eigencurve C, which parameterizes all p-adic eigenforms (of tame level N, finite slope, and that are overconvergent). Instead of integers, we allow more general weights here. As is to be expected, if the modular forms vary p-adically, this also has to be true for the weights, and the natural weight space is the set of continuous $\mathbb{C}_p^\times$-valued characters of $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$, which as a rigid analytic space is just a disjoint union of unit disks. The paper [CM] will be the main focus of our seminar.

The eigencurve is described as a closed subspace of $X \times \mathbb{A}^1$, the product of the affine line and the rigid analytic space associated with the universal deformation ring of certain Galois (pseudo-)representations. An eigenform f corresponds to the point in $X \times \mathbb{A}^1$ given by the associated Galois pseudorepresentation and the eigenvalue of f under the Atkin-Lehner operator U_p.

Although this definition is fairly easy to write down, it is very hard to prove any properties from it. For instance, it is not obvious that C is a curve. In the final chapter of their paper, Coleman and Mazur give a different construction of C (more precisely, of the underlying reduced rigid subspace), which works by gluing local pieces. In this way, they prove that C is in fact a curve, as the terminology suggests, and also show some further properties. Nevertheless, there remain many open questions.

The work of Coleman and Mazur was further generalized by Buzzard [B2] who axiomatized the construction of the eigencurve and developed an “eigenvariety machine”. There is also a different approach by Emerton [E], and related work of Bellaïche, Calegari, Chenevier, Kassaei, Kisin, and others. We will not have time to discuss these more recent developments in the seminar, though.

See Kassaei's survey [Ka] for a more thorough survey, and Buzzard's lecture notes for a more informal account of a large part of the relevant topics with many pictures.

Technical note: To avoid additional technicalities, we always allow ourselves to assume that $p > 5$ and that $N = 1$ (with the notation of [CM]). The assumption that $N = 1$ is often made there anyway (but see [B2]).

Level of the talks: The talks 2, 3, and 6 are the simplest ones, and are practically self-contained. For talk 4, some knowledge about p-adic L-functions is useful. Talks 5, 8, and 9 are probably a little harder. Talk 7 with the definition of the Atkin-Lehner operator U is probably the most difficult talk (but for the later talks one can mostly use the U-operator as a black box). In talks 10 and particularly 11, basically everything studied before comes into the play.

Program

1. **Overview and motivation.** See [CM], in particular the introduction and §1.5, and also [Ka], [B3], etc.

2. **Rigid analytic geometry (2 talks).** Explain the basic notions of rigid analytic geometry à la Tate: affinoid algebras, affinoid spaces, the Grothendieck topology, rigid analytic spaces. For a brief account see Schneider's survey [Sch], for more detailed accounts see the lecture notes [Bo] by Bosch, the book [FP] of Fresnel and van der Put, and the encyclopedic volume [BGR] by Bosch, Güntzer, and Remmert.

Explain how to construct the rigid space associated with a scheme, and with a formal scheme, resp., also in the non-adic case ([CM] 1.1, cf. also [Ber]). Explain the specialization map.

Discuss the following examples in detail:

- $\mathbb{A}^1_K$
- $B_K(0,1)$, the wide open unit disk
- The Tate elliptic curve
- Modular curves over $\mathbb{C}_p$

Explain the notion of strict neighborhood, see [CM] 2.1, [C2] A5.

3. **Modular forms à la Katz.** Explain Katz' method of defining modular forms over an arbitrary base, [K] Chapter 1 and Appendix A1, see also [G]. Discuss the q-expansion principle. (In this talk, we do not yet deal with p-adic modular forms.)

4. **The Eisenstein family.** The Eisenstein family is of crucial importance in Coleman's definition of overconvergent modular forms. See the remarks at the end of [CM] 2.2.

Introduce weight space ([CM] 1.4), and define the Eisenstein family (see [C2] B1, [CM] 2.2, and the references given there). Note the close relation to p-adic L-functions, the topic of the previous research seminar. Discuss the Eisenstein series from the point of view of the preceding talk (in particular its reduction modulo p, [K] 2.0, 2.1).

5. **p-adic modular forms à la Coleman.** Explain Coleman's definition of overconvergent modular forms, [CM] 2.4. (For integral weight this is easy, because one can define overconvergent modular forms in terms of sections of a line bundle on a suitable open of the (rigid analytic) modular curve, see loc. cit 2.1. It is not so clear, how to define overconvergent modular forms of arbitrary weight. Coleman's strategy is to say that a power series is (the q-expansion of) an overconvergent modular form of weight κ, if the quotient F/E_κ is a modular function (where $E_\bullet$ is the Eisenstein family).) See also [C2] B, in particular B4.

Compare this notion with classical and Katz' p-adic modular forms, [K] Ch. 2, [CM] 2.3.

6. **p-adic Banach spaces.** Discuss [C2] A1–4, see also [Se]. In particular: Orthonormalizable p-adic Banach spaces. Completely continuous operators. Characteristic series/Fredholm determinants.

7. **The Atkin-Lehner operator U_p and the Hecke algebra (2 talks).** The operator U and its spectral theory on the spaces of overconvergent modular forms is a crucial ingredient for all that follows. Define U following [C1] §1–3, see also [K] Ch. 3, [G] II.3, [C2] B2, B3, B4, [B3] 6th lecture.

Explain the effect of U on q-expansions. Explain that for U as an operator on the space of *overconvergent* modular forms we have an interesting spectral theory ([G] II.3.3, II.3.4).

The other Hecke operators are easier to deal with, see [CM] §3.

8. **Fredholm theory.** Cover [CM] §1.3, §4, and in particular, explain [CM] Theorem 4.3.1 (and all terms occurring there). Define and discuss the spectral curves as in loc. cit. 4.4.

(See also [C2], App. I, for "explicit" formulas.)

9. **Pseudo-representations (2 talks).** Since universal deformation rings of Galois representations (with coefficients in a finite field) exist only under certain assumptions, for us the notion of pseudo-representation is more suitable to parameterize modular forms.

Go through [CM] §5: Define the notion of pseudo-representations. Compare the notions of pseudo-representation and representation, in particular loc. cit. Theorem 5.1.2. Define and discuss the rigid space X_p as defined in loc. cit. State the theorems about the pseudo-representation attached to a modular form.

10. **The eigencurve.** Now we can define the eigencurve, as a rigid analytic subspace of $X_p \times \mathbb{A}^1$, cut out by certain Fredholm determinants. The basic idea is to view the set of all overconvergent (normalized ...) eigenforms as the subset $\{(\text{pseudo-repr. attached to the eigenform}, U_p - \text{eigenvalue})\} \subset X_p \times \mathbb{A}^1$.

Discuss [CM] §6, and in particular Theorem 6.2.1: the points of the eigencurve are (certain) overconvergent modular eigenforms with non-zero U_p-eigenvalue. Cf. also loc. cit. 1.5.

11. **Properties of the eigencurve.** (A selection of) [CM] §7. Explain the construction of the curve D by gluing local pieces (7.1–7.3). State Theorem 7.5.1, and discuss as much of it, and of section 7.6, as possible.

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