Trace formula and Brandt matrices
Talk in the Forschungsseminar on Quaternion Algebras

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Contents

1 Introduction

Let H/K be an quaternion algebra, $K = \text{Quot}(R)$, L/K a separable extention of degree 2. B an R-order of L, O an R-order of H.

Definition 1.1 *An embedding $f : L \to H$ is a maximal embedding of B in O if $f(L) \cap B = f(B)$.*

As an example, let $L = K(h), h \in H$. Then by Skolem-Noether $C(h) = \{xhx^{-1} \mid x \in H^\star\}$ is in bijection with the set of embeddings of L in H and $C(h, B) = \{xhx^{-1} \mid x \in H^\star, K(xhx^{-1}) \cap O = xBx^{-1}\}$ is in bijection with the set of maximal embeddings of B in O.
Let $N(O) = \{x \in H^\star \mid xOx^{-1} = O\}$ be the normalizer of O and $G \subseteq N(O)$ a subgroup. For $x \in H$ let $\tilde{x} : y \mapsto xyx^{-1}$ be the inner automorphism associated to x and let $\tilde{G} := \{\tilde{x} \mid x \in G\}$.
Then $C(h, B)$ is stable under the left action of $\tilde{G}$. This justifies the second definition:

Definition 1.2 *A class of maximal embeddings of B in O modulo G is a $\tilde{G}$-orbit of maximal embeddings of B in O.*

The goal of this talk is to proof a trace formula for the number of classes of maximal embeddings where K is a number field and O is an Eichler Order.

2 Local Case

Let K be a local field. We have to distinguish between two different cases: The case where H is a division quaternion algebra and the case where H is a matrix algebra. In the global setting this will correspond to the places p where H_p is ramified (first case) and those where H_p is unramified (second case).

2.1 First Case: H a division quaternion algebra

From Adam's talk we know that $H \cong (L', \pi)$ where L' is the unramified seperable quadratic extention of K. Let w be the map $w : H^\star \to \mathbb{Z}, h \mapsto w(h) := v(n(h))$. Then $O := \{h \in H^\star \mid w(h) \geqslant 0\} \cup \{0\}$ is the unique maximal order, hence the unique Eichler order of H.
Let

$$\left(\frac{L}{\pi}\right) := \begin{cases} -1 & \text{if } L/K \text{ is unramified,} \\ 0 & \text{if } L/K \text{ is ramified} \end{cases}$$

be the local Artin symbol.

Theorem 2.1 *If B is a maximal order, then if $m(B,G)$ denotes the number of maximal embeddings of B in O modulo G we have*

$$m(B,G) = \begin{cases} 1 & \textit{if } G = N(O), \\ 1 - \left(\frac{L}{\pi}\right) & \textit{if } G = O^\star. \end{cases}$$

If B is not maximal, than it can't be maximal embedded in O.

2.2 Second Case: $H \cong M_2(K)$ a matrix algebra

Let O be a maximal order of H and O' an Eichler order of level (π) of H.
Let

$$\left(\frac{B}{\pi}\right) := \begin{cases} \left(\frac{L}{\pi}\right) & \text{if } B \text{ is maximal,} \\ 1 & \text{otherwise} \end{cases}$$

be the local Eichler symbol.

Theorem 2.2 *The number of maximal embeddings of B in O modulo $O^\star$ is* 1. *The number of maximal embeddings of B in O' modulo G is*

$$\begin{cases} 1 + \left(\frac{B}{\pi}\right) & \textit{if } G = O'^\star, \\ 0 \textit{ or } 1 & \textit{if } G = N(O') \end{cases}$$

Remark 2.3 This theorem showes that B can't be maximal embedded in O' if and only if B is maximal and L/K is unramified.

3 Global Case: The Trace formula

Let K be a number field, $R = O_K$, O an Eichler order of H of level N, S a finite set of places satisfying

(a) $\{\text{infinite places}\} \subset S$

(b) S satisfies the Eichler condition (There is a place $p \in S$ where H_v/K_v is unramified).

(c) $\{p \mid N\} \cap S = \varnothing$

Let X denote the set of all places of K.
The discriminant of O can be writen as DN with

$$D := \prod_{p \notin S,\ p \in \mathrm{Ram}(H)} (p).$$

The main goal of my talk is the following theorem:

Theorem 3.1 (Trace formula) *Let $m_p := m_p(D, N, B, O^\star)$ be the number of maximal embeddings of B_p in O_p modulo $O_p^\star$ for all $p \notin S$. $(I_i), 1 \leqslant i \leqslant h$ a system of representatives of classes of left ideals of O and $O^{(i)} = O_r(I_i)$ the right order of I_i. Let $m_{O^\star}^{(i)}$ be the number of maximal embeddings of B in $O^{(i)}$ modulo $0^{(i)\star}$. Then*

$$\sum_{i=1}^{h} m_{O^\star}^{(i)} = h(B) \prod_{p \notin S} m_p$$

where $h(B)$ is the class number of B.

Remark 3.2 The product on the right hand side is finite. We have

$$\prod_{p \notin S} m_p = \prod_{p \mid D} m_p \prod_{p \mid N} m_p = \prod_{p \mid D} \left(1 - \left(\frac{B}{p}\right)\right) \prod_{p \mid N} \left(1 + \left(\frac{B}{p}\right)\right).$$

where $\left(\frac{B}{p}\right)$ is the global Eichler Symbol defined as:

$$\left(\frac{L}{p}\right) = \begin{cases} 1 & \text{if } p \text{ splits in } L, \\ -1 & \text{if } p \text{ is inert in } L, \\ 0 & \text{if } p \text{ ramifies in } L \end{cases}$$

and

$$\left(\frac{B}{p}\right) = \begin{cases} \left(\frac{L}{p}\right) & \text{for } p \in S \text{ or } B_p \text{ maximal}, \\ 1 & \text{otherwise.} \end{cases}$$

We will proof the trace formula in a more general setting:
Let G_p be groups with $O_p^\star \subseteq G_p \subseteq N(O_p)$ for all $p \notin S$ and let $G_p := H_p^\star$ for $p \in S$. We suppose that $G_p = O_p^\star$ for all but finitly many p. Let $G_{\mathbb{A}} := \prod_{p \in X} G_p \subseteq H_{\mathbb{A}}^\star$ be the adelic version of the G_p's and let $G := G_{\mathbb{A}} \cap H^\star$.
Let $O^{(i)}, 1 \leqslant i \leqslant t$ be a system of representatives of Eichler orders of level N. In Björns talk we had a disjoint union decomposition:

$$H_{\mathbb{A}}^\star = \coprod_{i=1}^{t} N(O_{\mathbb{A}}) x_i H^\star$$

and with $O_{\mathbb{A}}^{(i)} := x_i^{-1} O_{\mathbb{A}} x_i$ we have $O^{(i)} = H \cap O_{\mathbb{A}}^{(i)}$. Let $G_{\mathbb{A}}^{(i)} := x_i^{-1} G_{\mathbb{A}} x_i$ and $G^{(i)} := H \cap G_{\mathbb{A}}^{(i)}$. Let $H^{(i)} := N(O_{\mathbb{A}}^{(i)}) \cap H^\star$, $n_G^{(i)} := \mathrm{card}(G_{\mathbb{A}}^{(i)} \backslash N(O_{\mathbb{A}}^{(i)}) / H^{(i)})$ and $h_G(B) := \mathrm{card}(B'_{\mathbb{A}} \backslash L_{\mathbb{A}}^\star / L^\star)$ with $B'_{\mathbb{A}} = B_{\mathbb{A}} \cap G_{\mathbb{A}}$.
Then we can proof the following theorem:

Theorem 3.3 *Let $m_p = m_p(D, N, B, G)$ be the number of maximal embeddings of B_p in O_p modulo G_p for all $p \notin S$. Let $m_G^{(i)}$ be the numer of maximal embeddings of B in $O^{(i)}$ modulo $G^{(i)}$. Then*

$$\sum_{i=1}^{t} n_G^{(i)} m_G^{(i)} = h_G(B) \prod_{p \notin S} m_p.$$

It is not so hard to see that Theorem 3.3 implies Theorem 3.1.

4 An application: Brandt matrices

Let N be a rational prime and H the quaternion algebra over $\mathbb{Q}$ wich is ramified at N and ∞. Let O be a fixed maximal order of H and let $\{I_1, \ldots, I_n\}$ be a system of representatives of classes of left ideals of O, $O^{(i)} = O_r(I_i)$ the right order of I_i. Then $\Gamma_i = O^{(i)\star}/\mathbb{Z}^\star$ is a discrete subgroup of $(H \otimes \mathbb{R})^\star/\mathbb{R}^\star \cong \mathrm{SO}_3(\mathbb{R})$, wich is compact, so Γ_i is finite. We let $w_i = |\Gamma_i|$. Then Eichlers mass formula states

$$\sum_{i=1}^{n} \frac{1}{w_i} = \frac{N-1}{12}.$$

Let $M_{ij} := I_j^{-1} I_i$ be the product ideal. It is a left ideal of $O^{(j)}$ with right order $O^{(i)}$. Let $n(M_{ij})$ denote the unique rational number such that $\frac{n(b)}{n(M_{ij})}$ are integers with no common factor for all $b \in M_{ij}$. We let

$$f_{ij} := \frac{1}{w_j} = \sum_{b \in M_{ij}} e^{2\pi i (\frac{n(b)}{n(M_{ij})})\tau} = \sum_{m \geqslant 0} B_{ij}(m) q^m.$$

This is a modular form of weight 2 for $\Gamma_0(N)$.

Definition 4.1 *The m-th Brandt-Matrix is $B(m) := (B_{ij}(m))_{1 \leqslant i,j \leqslant n}$.*

Using the trace formula we can compute $\mathrm{Trace}(B(m))$ in terms of the Hurwitz class numer:
For B an order of discriminant $-d$ and rank 2 over $\mathbb{Z}$ we set $h(d) = |\operatorname{Pic}(B)|$ and $u(d) = |B^\star/\mathbb{Z}^\star|$. For $D > 0$ we set

$$H(D) := \sum_{df^2 = -D} \frac{h(d)}{u(d)}$$

and

$$H_N(D) = \begin{cases} 0 & \text{if } N \text{ splits in } O_{-D} =: O, \\ H(D) & \text{if } N \text{ is inert in } O, \\ \frac{1}{2} H(D) & \text{if } N \text{ ramifies in } O \text{ and } N \text{ doesn't divide the conductor of } O, \\ H_N(\frac{D}{N^2}) & \text{if } N \text{ divides the conductor of } O. \end{cases}$$

Further let $H_N(0) := \frac{N-1}{24}$.
Then we can show the following theorem:

Theorem 4.2

$$\mathrm{Trace}(B(m)) = \sum_{s \in \mathbb{Z}, s^2 \leqslant 4m} H_N(4m - s^2)$$

For a proof, see chapter 1 of Gross' paper.

References

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