THE ACTION OF G_K ON $\mathrm{Frac}(\mathcal{R})$ AND (φ, Γ)-MODULES

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Let K be a p-adic field, that is the fraction field of a complete d.v.r. of characteristic 0, with perfect residue field k of characteristic $p > 0$. We denote by $\mathcal{O}_{K_0} := W(k)$ the ring of Witt vectors over k, and by K_0 its fraction field, which we identify as a subfield of K.
As in the previous talk, $C := \hat{\bar{K}}$ with normalized valuation v, such that $v(p) = 1$.

For any subfield $L \subset C$, we can restrict the valuation on C to L, and have therefore, as usual, the notions of: ring of integers $\mathcal{O}_L$, maximal ideal $\mathfrak{M}_L$ and residue field k_L.

Throughout the lecture, we denote for any algebraic field extension L of K_0 $G_L := Gal(\bar{K}/L)$ and $H_L := Gal(\bar{K}/L^{\mathrm{cyc}})$.
Recall the ring $\mathcal{R} := R(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}) = R(\mathcal{O}_C/p\mathcal{O}_C)$ introduced in the last talk, with fraction field $\mathcal{C}$. The valuation $v_{\mathcal{R}}((x^{(n)})) := v_C(x^{(0)})$ defines a valuation on $\mathcal{C}$, which makes it a complete, non-archimedean, algebraically closed field of characteristic p (cf. last lecture).

We have seen so far in this seminar, how one can *explain* (p-adic or mod p) Galois representations of G_E (E of positive characteristic), in terms of certain "étale φ-modules". Our goal in this talk is to give an equivalence of categories between Galois representations of G_K (for K **a p-adic field**) and (φ, Γ)-modules.

1. The action of Galois on $\mathcal{R}$

Proposition 1.1. *Let L be an extension of K_0 contained in $\bar{K}$. Then $\mathcal{R}^{G_L} = R(\mathcal{O}_L/p\mathcal{O}_L)$ (and therefore also $\mathcal{C}^{G_L} = \mathrm{Frac}(R(\mathcal{O}_L/p\mathcal{O}_L))$). Moreover, the residue field of $\mathcal{R}^{G_L}$ is k_L, the residue field of L.*

Proof. Take $x \in \mathcal{R}^{G_L}$ and write it in *sequence representation*: $x = (x^{(n)})$, where $x^{(n)} \in \mathcal{O}_C$. The Galois group acts coordinatewise on $\mathcal{R}$, and therefore $x^{(n)^g} = x^{(n)}$, which means $x^{(n)} \in \mathcal{O}_C^{G_L}$, for all $n \in \mathbb{N}$.
Now, $\mathcal{O}_C^{G_L} = \mathcal{O}_{C^{G_L}} = \mathcal{O}_{\hat{L}} = \varprojlim \mathcal{O}_{\hat{L}}/p^n\mathcal{O}_{\hat{L}} = \varprojlim \mathcal{O}_L/p^n\mathcal{O}_L$; from which it follows $\mathcal{R}^{G_L} = R(\mathcal{O}_L/p\mathcal{O}_L)$.

Let $[\cdot] : \bar{k} \to \mathcal{R}$ be the Teichmüller lift. Since the residue field of R is $\bar{k}$, we get by taking G_L-invariants to $\bar{k} \overset{[\cdot]}{\hookrightarrow} R \twoheadrightarrow \bar{k}$ the identity map on k_L: $k^{G_L} = k_L \hookrightarrow R^{G_L} \twoheadrightarrow k_L$; hence k_L is the residue field of R. □

Corollary 1.2. *If $v(L^\times)$ is discrete, then $R^{G_L} = k_L$.*

Proof. We already know that $k_L \subset R^{G_L}$. Hence we only need to show that $x \in R^{G_L}$ with $v(x) > 0$ is only satisfied for the zero element.
For such an element, we have $v(x) = v_C(x^{(0)}) = p^n v(x^{(n)}) > 0$, from which follows that $v(x^{(n)}) \overset{n\to\infty}{\longrightarrow} 0$. Since the valuation is assumed to be discrete, there exists a non-negative integer N_0, such that for all $n \geq N_0$, $x^{(n)} = 0$, but then all coordinates happen to be zero, and so $x = 0$. □

2. $R(\mathcal{O}_{K_0^{\text{cyc}}}/p\mathcal{O}_{K_0^{\text{cyc}}})$, ε AND π

Take $\varepsilon := (\varepsilon^{(n)})_{n\geq 0} \in \mathcal{R}_0^{\text{cyc}} := R(\mathcal{O}_{K_0^{\text{cyc}}}/p\mathcal{O}_{K_0^{\text{cyc}}})$ a compatible system of primitive p^n-th roots of unity:

$$\varepsilon^{(0)} = 1,\ \varepsilon^{(1)} \neq 1 \text{ and } (\varepsilon^{(n+1)})^p = \varepsilon^{(n)}, \forall n \geq 1;$$

which we fix throughout this lecture.
Set $K_0^{\text{cyc}} := \varinjlim_n K_0(\varepsilon^{(n)})$ and $\pi := \varepsilon - 1 \in \mathcal{R}_0^{\text{cyc}}$.

Lemma 2.1. *The element π has valuation greater than 1. In particular, $\varepsilon := (\varepsilon^{(n)})_{n\in\mathbb{N}}$ is a unit of $\mathcal{R}_0^{cyc}$.*

Proof.

$$v(\pi) = v(\pi^{(0)}) = v(\lim_{m\to\infty}(\varepsilon^{(m)} - 1)^{p^m}) = \frac{p}{p-1} > 1,$$

since we know, from the classical cyclotomic theory, that

$$v(\varepsilon^{(m)} - 1) = \frac{1}{(p-1)p^{m-1}},\ \forall m \geq 1.$$

□

Remark 2.2. (1) *From Proposition 1.1 we know, that*

$$\mathcal{R}^{G_{K_0^{cyc}}} = \mathcal{R}_0^{cyc} \overset{dfn.}{=} R(\mathcal{O}_{K_0^{cyc}}/p\mathcal{O}_{K_0^{cyc}}).$$

(2) *Since the Galois action on $\mathcal{R}$ is continuous, $\mathcal{R}_0^{cyc}$ is still complete. The element π has valuation greater than one, and so $k[[\pi]] \subset \mathcal{R}_0^{cyc}$.*
Similarly, defining $\mathcal{C}_0^{cyc} := \mathrm{Frac}(\mathcal{R}_0^{cyc})$, we get that $k((\pi)) \subset \mathcal{C}_0^{cyc}$.

(3) *$\mathcal{R}_0^{cyc}$ is perfect; since for any $x = (x^{(n)})_{n\in\mathbb{N}} \in \mathcal{R}_0^{cyc}$ one can define $y := (y^{(n)})_{n\in\mathbb{N}}$ as $y^{(n)} := x^{(n+1)}$, and therefore $y \in \mathcal{R}_0^{cyc}$ and one easily checks that $x - y^p = 0$.*

(4) *Summing up the remarks above, we have the following inclusions*

$$\widehat{k[[\pi]]}^{rad} \subset \mathcal{R}_0^{cyc},\ \widehat{k((\pi))}^{rad} \subset \mathcal{C}_0^{cyc};$$

where the subscript rad denotes the radical completion in the algebraically closed field $\mathcal{C}$.

We set from now on $E_0 := k((\pi))$ and $\mathcal{O}_{E_0} := k[[\pi]]$.

Theorem 2.3. *We have indeed equalities:*

$$\widehat{k[[\pi]]}^{rad} = \mathcal{R}_0^{cyc},\ \widehat{k((\pi))}^{rad} = \mathcal{C}_0^{cyc}.$$

Proof. We need to prove that $\mathcal{O}_{E_0^{\text{rad}}} := k[[\pi]]^{\text{rad}}$ is dense in $\mathcal{R}_0^{\text{cyc}} = \varprojlim_{\varphi} \mathcal{O}_{K_0^{\text{cyc}}}/p\mathcal{O}_{K_0^{\text{cyc}}}$.

For this, it suffices to show that $\theta_m(\mathcal{O}_{E_0^{\text{rad}}}) = \mathcal{O}_{K_0^{\text{cyc}}}/p\mathcal{O}_{K_0^{\text{cyc}}},\ \forall m \in \mathbb{N}_0$. The inclusion $\subset$ is clear, so we prove $\supset$.

$\mathcal{O}_{K_0^{\text{cyc}}}$ equals the union (in $\mathcal{R}_0^{\text{cyc}}$) of the rings $\mathcal{O}_{K_0}[\pi_n]$, where $\pi_n := \varepsilon^{(n)} - 1$. Denote by $\overline{\pi}_n$ the image of π_n under the projection map $\mathcal{O}_{K_0^{\text{cyc}}} \to \mathcal{O}_{K_0^{\text{cyc}}}/p\mathcal{O}_{K_0^{\text{cyc}}}$; hence $\overline{\pi}_n = \varepsilon_n - 1$. Since $\mathcal{O}_{K_0^{\text{cyc}}}/p\mathcal{O}_{K_0^{\text{cyc}}}$ is a k-algebra generated by the $\overline{\pi}_n$ ($\mathcal{O}_{K_0} = W(k)$ and k is perfect), the claim follows, if we prove that

$$\overline{\pi}_n \in \theta_m(\mathcal{O}_{E_0^{\text{rad}}}) = \theta_m(k[[\pi]]^{\text{rad}})\ \forall m, n \in \mathbb{N}_0.$$

Since for any $s \in \mathbb{Z}$, $\pi^{p^s} \in k[[\pi]]^{\text{rad}}$ and $\pi^{p^s} = (\pi^{n-s})_{n\in\mathbb{N}_0} = \varepsilon^{p^s} - 1 = (\varepsilon^{(n-s)}) - 1 = (\varepsilon_{n-s} - 1)$; where $\varepsilon^{(n)} = 1$ for $n < 0$. Now, $\theta_m(\pi^{p^{m-n}}) = \varepsilon_{m-(m-n)} - 1 = \varepsilon_n - 1 = \overline{\pi}_n$. □

Remark 2.4. *At several stages (for example in the proof above), one can prove the statement using the $\epsilon^{(n)}$'s instead of the π_n's - something which may be easier in some cases.*[1]

3. A fundamental Theorem

Let E_0^{s} be the separable closure of $E_0 = k((\pi)) \subset \mathcal{C}_0^{\mathrm{cyc}}$ in $\mathcal{C}$.

Theorem 3.1. (1) *E_0^s is dense in $\mathcal{C}$, and stable under G_{K_0}.*
(2) *There exists an isomorphism $\mathrm{Gal}(\bar{K}/K_0^{cyc}) \xrightarrow{\sim} \mathrm{Gal}(E_0^s/E_0)$, given by restricting the natural action of $\mathrm{Gal}(\bar{K}/K_0^{cyc})$ on $\mathcal{C}$ to the subfield E_0^s.*

Proof. Since by Krasner's Lemma $\widehat{E_0^{\mathrm{s}}} = \widehat{\overline{E_0}}$, it suffices to show the density of $\overline{E_0}$ in $\mathcal{C}$, or what amounts to the same, the density of $\mathcal{O}_{\overline{E_0}}$ in $\mathcal{R}$.

As in the proof of Theorem 2.3, we need to show that $\theta_m(\mathcal{O}_{\overline{E_0}}) = \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$, for all $m \in \mathbb{N}_0$. Since $\overline{E_0}$ is algebraically closed, the general claim follows from the case $m = 0$; which we now prove.

Since

$$\mathcal{O}_{\overline{K}} = \varinjlim_{\substack{[L:K_0]<\infty \\ L/K_0 \text{ Galois}}} \mathcal{O}_L,$$

it suffices to check that $\theta_0(\mathcal{O}_{\overline{E_0}}) \supset \mathcal{O}_L/p\mathcal{O}_L$, for any such L.

Put $K_{0,n} := K_0(\varepsilon^{(n)})$, $L_n := L \cdot K_{0,n}$ and $J_n := \mathrm{Gal}(L_n/K_{0,n})$. The decreasing sequence of finite Galois groups J_n stabilizes to the finite group, say, J - we assume from now on, n to be big enough, so that $J_n = J$. Since $\bar{k} \subset \mathcal{O}_{\overline{E_0}}$, without loss of generality, we may replace K_0 by a finite algebraic extension K_0', so that all the extensions L_n/K_0' are totally ramified. By abuse of notation we keep using K_0 for K_0'.

Since $L_n/K_{0,n}$ is totally unramified, we can write $\mathcal{O}_{L_n} = \mathcal{O}_{K_{0,n}}[\nu_n]$, for nu_n a uniformizer of L_n. From Theorem 2.3 we have $\theta_0(\mathcal{O}_{\overline{E_0}}) \supset \mathcal{O}_{K_{0,n}}/p\mathcal{O}_{K_{0,n}}$, hence we need only to show that there exists an n, such that $\mathcal{O}_{L_n}/p\mathcal{O}_{L_n} \ni \overline{\nu_n}$ lies in $\theta_0(\mathcal{O}_{\overline{E_0}})$.

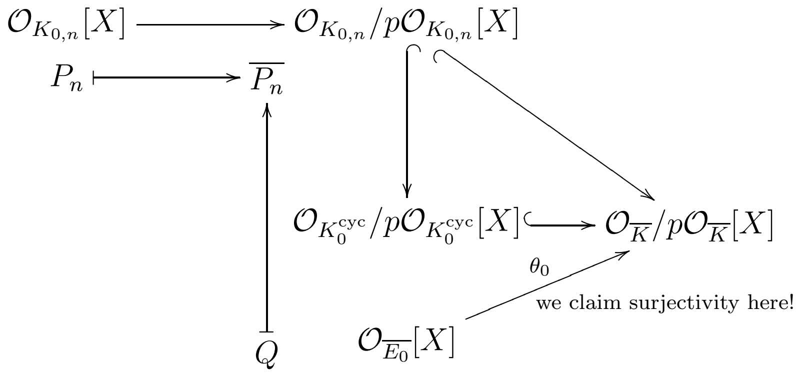
The case $J = \{\mathrm{Id}\}$ trivially holds; so we assume from now on $|J| := d > 1$. Let $P_n \in K_{0,n}[X]$ be the minimal (Eisenstein) polynomial of ν_n (of degree $d = absJ$):

$$P_n(X) = \prod_{g \in J} (X - g(\nu_n)).$$

By Lemma 3.2 below, we know that for any $1 \neq g \in J$, $\lim_{n\to\infty} v(g(\nu_n) - \nu_n) = 0$. Let n be large enough, so that furthermore ($J_n = J$ should still hold) $v(g(\nu_n) - \nu_n) < 1/d$ is fulfilled for all $g \in J \setminus \{\mathrm{Id}\}$.

We have the following diagram, where we choose Q to be a lift of $\overline{P_n}$ over $\mathcal{O}_{\overline{E_0}}[X]$ (monic and of degree d).

[1] This Remark was done by Gebhard during the talk.



Now choose a root $\alpha \in \mathcal{O}_{\overline{E_0}}$ of Q *closest to* ν_n. If we set $\beta := \theta_0(\alpha) \in \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$, by *closest to* ν_n we mean:

$$v(\beta - \overline{\nu_n}) \geq v(\beta - g(\overline{\nu_n})),\ \forall g \in J. \tag{1}$$

It may be necessary to make some comments on the meaning of v in the inequalities (1). For any $x \in \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$, pick a lift $\tilde{x} \in \mathcal{O}_{\bar{K}}$ and define

$$v(x) := \begin{cases} 1 & \text{if } 1 < v(\tilde{x}) < \infty, \\ v(\tilde{x}) & \text{else.} \end{cases}$$

This definition is independent of the lift $\tilde{x}$, and hence the meaning of (1) is explained.

Since α is a root of Q, $v(\overline{P_n}(\beta)) \geq 1$. Choose a lift $b \in \mathcal{O}_{\bar{K}}$ of β. Hence, we have $v(\overline{P_n(b)}) = v(\overline{P_n}(\beta)) \geq 1$, and so $v(P_n(b)) \geq 1$. This means, that there exists a $g_0 \in J$ such that $v(b - g_0(\nu_n)) \geq 1/d$. It follows then from (1) that $v(\beta - \overline{\nu_n}) \geq 1/d$, and since $v(\nu_n - g(\nu_n)) < 1/d$ for any $\mathrm{Id} \neq g \in J$, g_0 must be necessarily Id. Therefore $v(b - \nu_n) > v(b - g(\nu_n))$ for all $1 \neq g \in J$; which by Krasner's Lemma implies that $\nu_n \in K_{0,n}(b)$ (indeed $\nu_n \in \mathcal{O}_{K_{0,n}(b)}$, since it was assumed from the very beginning to be integral). The element ν_n can be represented by a polynomial in b with coefficients in $\mathcal{O}_{K_{0,n}}$, hence we need to lift b and the coefficients, i.e. the elements of $\mathcal{O}_{K_{0,n}}$. Since b reduces to β, we have $\nu_n \in \mathcal{O}_{K_{0,n}}/p\mathcal{O}_{K_{0,n}} = k[\overline{\epsilon^{(n)}}, \beta]$. Therefore $\overline{\nu_n} \in \theta_0(\mathcal{O}_{E_0(\pi^{1/p^n})}(\alpha))$, which proves the assertion.

Since $\widehat{E_0^{\mathrm{s}}} = \mathcal{C}$, and G_K acts naturally on $\mathcal{C}$, we get an action on $\widehat{E_0^{\mathrm{s}}}$. We prove now the stability of E_0^{s} with respect to the action of G_K, which gives an action of G_K on E_0^{s}.
Let $x \in E_0^{\mathrm{s}}$, with separable minimal polynomial $P_x(T) \in E_0[T]$, then for any $g \in G_K$, $g(x)$ is a root of the separable polynomial $(P_x)^g$. Therefore, for the stability of E_0^{s} we need only to show that the coefficients, i.e. the elements of E_0, are stable under the Galois action. But this is clear, since $g(\pi) = (1+\pi)^{\chi(g)} - 1 \in k((\pi)) = E_0$.

Summing up, we obtain a group homomorphism $G_K = \mathrm{Gal}(\overline{K}/K_0^{\mathrm{cyc}}) \to \mathrm{Gal}(E_0^{\mathrm{s}}/E_0)$, simply by restriction. The last claim of the Theorem is that this map is an isomorphism.

Injectivity: Let $g \in G_K$ be in the kernel. Since the action of Galois is continuous, this element acts also trivially on $\mathcal{C} = \widehat{E_0^{\mathrm{s}}}$: for any $x = (x^{(n)}) \in \mathcal{C}$, $g(x^{(n)}) = x^{(n)} \in C = \widehat{\overline{K}}$ for all n. But the map $\tilde{\theta}_0 : \mathcal{C} \to C$ is surjective (i.e. any element of C can be the zeroth coordinate of an element in $\mathcal{C}$), therefore g acts trivially on C and is consequently the identity map.

Surjectivity: By injectivity of the restriction map, we may identify G_K with a closed subgroup H of $\mathrm{Gal}(E_0^{\mathrm{s}}/E_0)$. If $H \neq \mathrm{Gal}(E_0^{\mathrm{s}}/E_0)$, then $(E_0^{\mathrm{s}})^H$ would define a non-trivial separable extension of E_0 inside $(\mathcal{C})^H = (\mathrm{Frac}(\mathcal{R}))^H = \widehat{E_0^{\mathrm{rad}}}$; which is impossible after Lemma 3.3 below.

□

We prove the two Lemmas used in the proof of the theorem.

Lemma 3.2. *With notation as in the proof of Theorem 3.1. We have for any* Id $\neq g \in J$ *that* $\lim_{n\to\infty} v(\nu_n - g(\nu_n)) = 0$.

Proof. Since $\mathcal{O}_{L_n} = \mathcal{O}_{K_{0,n}}[\nu_n]$, we know that the discriminant $\mathfrak{D}_{L_n/K_{0,n}}$ is generated by $P_n'(\nu_n) = \prod_{\mathrm{Id}\neq g\in J}(\nu_n - g(\nu_n))$. In Lecture 7, using *Herbrandt's integrals*, we see that $v(\mathfrak{D}_{L_n/K_{0,n}}) \to 0$ as n tends to infinity; which proves our claim, since

$$v(\mathfrak{D}_{L_n/K_{0,n}}) = \sum_{\mathrm{Id}\neq g\in J} v(\nu_n - g(\nu_n)).$$

□

Lemma 3.3. *Let E be a complete field of characteristic $p > 0$. There is no nontrivial separable extension of E inside $\widehat{E^{rad}}$.*

Proof. Let E' be a separable extension of E inside $\widehat{E^{\mathrm{rad}}}$. Denote by $\sigma_1, \ldots, \sigma_d$ the distinct embeddings of E' into E^{s} ($d = [E' : E]$). We extend each map σ_i to a map defined on E'^{rad} by setting $\sigma_i(a) := (\sigma_i(a^{p^n}))^{p^{-n}}$. By continuity, we get a map $\widehat{E'^{\mathrm{rad}}} = \widehat{E^{\mathrm{rad}}} \to \widehat{\overline{E}}$, which is the identity on E^{rad}, hence on the whole $\widehat{E^{\mathrm{rad}}} = \widehat{E'^{\mathrm{rad}}}$, and therefore σ_i must be the identity map, so $d = 1$. □

4. (φ, Γ)-MODULES

We assume here for simplicity $K = K_0$, and denote $E := E_0$ (see in the book, for the general case).
Let V be a $\mathbb{Z}_p$ (p-adic) representation of G_K. Since $H_K := \mathrm{Gal}(\overline{K}/K^{\mathrm{cyc}})$ is isomorphic to a Galois group $G_E := \mathrm{Gal}(E_0^{\mathrm{s}}/E)$ in characteristic p, the restricted action of G_K to H_K on V gives rise to a $\mathbb{Z}_p$ (p-adic) representation $V\mid_{H_K}$ of G_E. We know already from the representation theory of Galois groups of characteristic p-fields, that $V\mid_{H_K}$ corresponds to an étale φ-module over the Cohen ring $\mathcal{O}_{\mathcal{E}}$ of E given by

$$(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{unr}}}} \otimes_{\mathbb{Z}_p} V)^{H_K} \in \mathcal{M}_{\varphi}^{\text{ét}}(\mathcal{O}_{\mathcal{E}}).$$

From the exact sequence

$$(2)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & H_K & \longrightarrow & G_K & \xrightarrow{\chi} & \Gamma_K & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle\cong} & & & & & & \\ & & \mathrm{Gal}(\overline{E_0}/E) =: G_E & & & & & & \end{array}$$

we obtain an action of Γ_K on $(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{unr}}}} \otimes_{\mathbb{Z}_p} V)^{H_K}$ via the cyclotomic character χ.

Set $\tilde{B} := \mathrm{Frac}(W(\mathcal{C})) = W(\mathcal{C})[1/p] \supset \mathcal{E} := \mathrm{Frac}(\mathcal{O}_{\mathcal{E}})$. Denote by $[\cdot]$ the Teichmüller lift corresponding to the Cohen ring $\mathcal{O}_{\mathcal{E}}$, then $[\epsilon], \pi_\epsilon := [\epsilon] - 1 \in \mathcal{O}_{\mathcal{E}}$. We can now give an explicit description of a Cohen ring of E

$$\mathcal{O}_{\mathcal{E}} := \{\sum_{n=-\infty}^{+\infty} \lambda_n \pi_\epsilon^n \mid \lambda_n \in W(k),\ \lambda_n \overset{n\to-\infty}{\longrightarrow} 0\} = \widehat{W(k)((\pi_\epsilon))};$$

since one easily checks that this is a complete ring, whose maximal ideal is generated by p, and with residue field E.
The action of Frobenius on $[\epsilon] = (\epsilon, 0, 0, \ldots)$ is simply $\varphi([\epsilon]) = (\epsilon^p, 0, 0, \ldots)$, and $g([\epsilon]) = (\epsilon^{\chi(g)}, 0, 0, \ldots)$. These actions are commuting and semi-linear on $(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{unr}}}} \otimes_{\mathbb{Z}_p} V)^{H_K}$. This motivates the following

Definition 4.1. *An étale (φ, Γ)-module over $\mathcal{O}_{\mathcal{E}}$ is an étale φ-module over $\mathcal{O}_{\mathcal{E}}$ with a continuous semi-linear action of Γ_K which commutes with φ.*
The category of such modules will be denoted by $\mathcal{M}^{\text{ét}}_{\varphi,\Gamma}(\mathcal{O}_{\mathcal{E}})$.

Remark 4.2. *Similarly, one defines étale (φ, Γ)-modules for $\mathcal{E} := \mathrm{Frac}(\mathcal{O}_{\mathcal{E}})$.*

For any $\mathbb{Z}_p$-representation V of G_K, i.e. $V \in \mathrm{Rep}_{\mathbb{Z}_p}(G_K)$, we write

$$\mathbb{D}(V) := (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{unr}}}} \otimes_{\mathbb{Z}_p} V)^{H_k} \in \mathcal{M}^{\text{ét}}_{\varphi,\Gamma}(\mathcal{O}_{\mathcal{E}});$$

and for any $D \in \mathcal{M}^{\text{ét}}_{\varphi,\Gamma}(\mathcal{O}_{\mathcal{E}})$

$$\mathbb{V}(D) := (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{unr}}}} \otimes_{\mathcal{O}_{\mathcal{E}}} D)_{\varphi=1} \in \mathrm{Rep}_{\mathbb{Z}_p}(G_K).$$

Theorem 4.3. *The functors $\mathbb{D}$ and $\mathbb{V}$ are equivalences of (Tannakian) categories.*

Proof. Since the actions of φ and Γ_K commute, the equivalence of categories between $\mathrm{Rep}_{\mathbb{Z}_p}(H_K) = \mathrm{Rep}_{\mathbb{Z}_p}(G_E)$ and $\mathcal{M}^{\text{ét}}_{\varphi}(\mathcal{O}_{\mathcal{E}})$, which was proven in the last lecture, gives us the statement, by simply using the exact sequence (2). □

Remark 4.4. (1) *There is also a corresponding statement of the Theorem above for p-adic representations and étale (φ, Γ)-modules over $\mathcal{E}$ (where in the definition of the functors we have to tensorize over $\mathbb{Q}_p$, and over $\mathcal{E}$ respectively; cf. last lecture).*

(2) *An étale (φ, Γ)-module D over $\mathcal{E}$ can be explicitly given in the following way. Fix a topological generator γ_0 of Γ_K, and fix also a basis of D (which is of dimension $d < \infty$ by the étale assumption). Then, the action of γ_0 and the action of φ give rise to two matrices $M_{\gamma_0}, M_{\varphi} \in \mathrm{GL}_d(\mathcal{E})$. The fact that these two semi-linear action commute, is expressed through the following equation*

$$M_{\gamma_0}\gamma_0(M_{\varphi}) = M_{\varphi}\varphi(M_{\gamma_0}). \tag{3}$$

Therefore, an étale (φ, Γ)-module over $\mathcal{E}$ of rank d is nothing else than two matrices of $\mathrm{GL}_d(\mathcal{E})$, which satisfy the relation (3) above.