Semi-stable representations and filtered (φ, N)-modules

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Essen, 22 January 2009

Abstract

These are notes from my talk in the Forschungsseminar on p-adic Galois representations, which mainly follows the Fontaine-Ouyang book project. Mistakes are likely, so, please beware.

1 Notation

We fix the following data througout the talk:

- p a prime.
- $\mathbb{Q}_p^{\mathrm{unr}}$ the maximal unramified extension of $\mathbb{Q}_p$ (inside some fixed algebraic closure $\overline{\mathbb{Q}}_p$).
- $K/\mathbb{Q}_p$ a finite extension, a "p-adic field" (inside $\overline{\mathbb{Q}}_p$).
- $K_0 = \mathbb{Q}_p^{\mathrm{nr}} \cap K =$ the maximal absolutely unramified extension of $\mathbb{Q}_p$ contained in K.
- k the residue field of K_0 and K.
- $\sigma : \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{unr}}/\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{unr}}/\mathbb{Q}_p)$ the absolute arithmetic Frobenius, coming from $x \mapsto x^p$ on $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.
- $G_K = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group.

2 Filtered vector spaces - Hodge numbers and polygons

Let V be a K-vector space together with a filtration $\mathrm{Fil}^\bullet V$ which is decreasing, separated and exhaustive. This means that for every $i \in \mathbb{Z}$ the sub-K-vector spaces $\mathrm{Fil}^i V$ of V satisfy

- $\mathrm{Fil}^i V \supseteq \mathrm{Fil}^{i+1} V$ for all $i \in \mathbb{Z}$ (decreasing),
- $\bigcap_{i \in \mathbb{Z}} \mathrm{Fil}^i V = (0)$ (separated) and
- $\bigcup_{i \in \mathbb{Z}} \mathrm{Fil}^i V = D_K$ (exhaustive).

A *homomorphism of filtered vector spaces* $\varphi : V \to W$ is a K-linear map compatible with the filtration, i.e. $\varphi(\mathrm{Fil}^i V) \subseteq \mathrm{Fil}^i W$. In particular, the filtration on sub-K-vector spaces $V' \leq V$ is such that $\mathrm{Fil}^i V' \leq \mathrm{Fil}^i V$.

We define the *i-th graded piece* as

$$\mathrm{gr}^i V := \mathrm{Fil}^i V / \mathrm{Fil}^{i+1} V.$$

We say that j is a *jump* if $\mathrm{gr}^j V \neq 0$.

Let $V_1, V_2, \ldots, V_r$ be filtered K-vector spaces. The *tensor product* $V_1 \otimes_K V_2 \otimes_K \cdots \otimes_K V_r$ is equipped with the filtration

$$\mathrm{Fil}^i V := \sum_{i_1+i_2+\cdots+i_r=i} \mathrm{Fil}^{i_1} V_1 \otimes \mathrm{Fil}^{i_2} V_2 \otimes \cdots \otimes \mathrm{Fil}^{i_r} V_r.$$

As this definition is symmetric in the V_i, it descends to a filtration on $\mathrm{Sym}^r V$ and $\bigwedge^r V$.

We first define the Hodge number abstractly.

Definition 2.1 *(i) Let V be 1-dimensional with only jump in the filtration at j. The* Hodge number *is defined as*

$$t_H(V) := t_H(V, \mathrm{Fil}) := j.$$

(ii) If $\dim_K V = h > 1$, the Hodge number *is defined as*

$$t_H(V) := t_H(\bigwedge^h V)$$

with the induced filtration on the right.

More concretely, we have:

Proposition 2.2 *We have $t_H(V) = \sum_{i \in \mathbb{Z}} i \cdot \dim_K \mathrm{gr}^i V$.*

Proof. Let $j_1 < \cdots < j_s$ be the jumps of the filtration of V. We know that there is only a single jump in the filtration of the h-th exterior product, as it is of dimension 1. Hence, we are looking for the biggest possible choice of $i_1 \leq i_2 \leq \cdots \leq i_h$ such that

$$\mathrm{Fil}^{i_1} V \otimes \mathrm{Fil}^{i_2} V \otimes \cdots \otimes \mathrm{Fil}^{i_h} V \neq (0).$$

- Choose j_s as often as possible so that there is

$$0 \neq v_{1,s} \wedge v_{2,s} \wedge \cdots \wedge v_{h_s,s} \in \mathrm{gr}^{j_s} V = \mathrm{Fil}^{j_s} V.$$

  Necessarily, h_s equals the dimension of $\mathrm{gr}^{j_s} V$.

- Choose j_{s-1} as often as possible, so that there is

  $$0 \neq \bar{v}_{1,s-1} \wedge \bar{v}_{2,s-1} \wedge \cdots \wedge \bar{v}_{h_{s-1},s-1} \in \operatorname{gr}^{j_s} V.$$

  Necessarily, h_{s-1} equals the dimension of $\operatorname{gr}^{j_{s-1}} V$. Note that by taking representatives, we so far have

  $$0 \neq v_{1,s-1} \wedge v_{2,s-1} \wedge \cdots \wedge v_{h_{s-1},s-1} \wedge v_{1,s} \wedge v_{2,s} \wedge \cdots \wedge v_{h_s,s} \in \operatorname{Fil}^{j_{s-1}} V.$$

- Continue like this down to j_1.

From

$$\sum_{i \in \mathbb{Z}} i \cdot \dim_K \operatorname{gr}^i V = \sum_{k=1}^{s} j_k \cdot \dim_K \operatorname{gr}^{j_k} V$$

we obtain the claimed formula. □

Proposition 2.3 *(a) If $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of filtered K-vector spaces (the maps must be compatible with the filtration, all filtrations are separated, exhaustive and desending), then we have*

$$t_H(V) = t_H(V') + t_H(V'').$$

(b) Let V_1 and V_2 be two filtered K-vector spaces. Then we have

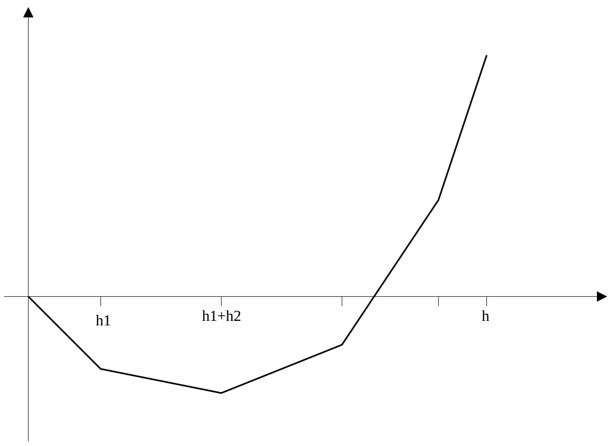
$$t_H(V_1 \otimes_K V_2) = t_H(V_1)\dim_K(V_2) + t_H(V_2)\dim_K(V_1).$$

We now associate the *Hodge polygon* to a filtered K-vector space with jumps $j_1 < \cdots < j_s$ and $h_i = \dim_K \operatorname{gr}^{j_i} V$. It is the polygon with vertices

$$(0,0), (h_1, j_1 h_1), (h_1 + h_2, j_1 h_1 + j_2 h_2), \ldots, (h, \sum_{i=1}^{s} j_i h_i = t_H(V)).$$

The slope of the r-th line segment is the position of the r-th jump, i.e. equal to j_r, since the slope is

$$\frac{\sum_{i=1}^{r} j_i h_i - \sum_{i=1}^{r-1} j_i h_i}{\sum_{i=1}^{r} h_i - \sum_{i=1}^{r-1} h_i} = \frac{j_r h_r}{h_r} = j_r.$$



3 Semi-linear algebra - Newton numbers and polygons

The beginning of this section is very basic. However, in the end we need to quote a theorem of Dieudonné's (or Manin's). Thanks to Kay and Andre for telling me about it! I had - in vain - tried to prove it over the week-end. It would still be interesting to find an elementary proof.

The integral theory of what we treat here is that of isocrystals (see, for instance, Katz: Slope filtration of F-crystals). We will, however, not go into this theory and use an ad-hoc approach, just as in the book (the book hides important concepts in Remark 6.47 without giving any citation or any appreciation of the depth of the statements).

Definition 3.1 *Let D be a K_0-vector space. A map $\varphi : D \to D$ is called* semi-linear *if it is $\mathbb{Q}_p$-linear and satisfies $\varphi(ad) = \sigma(a)\varphi(d)$ for all $a \in K_0$ and all $d \in D$.*

Conceptually speaking, this is a very bad definition because the composition of two semi-linear maps is not semi-linear any more! One would have to weaken the concept to the existence of i such that $\varphi(ad) = \sigma^i(a)\varphi(d)$. Or, more generally, one could allow any $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, even if the Galois group is non-cyclic (and even non-abelian).

The ad-hoc approach is to still use matrices over K_0 for describing semi-linear maps. Let us fix a basis $\{e_1, \ldots, e_r\}$ of D as K_0-vector space and say that φ is represented by the matrix $A = (a_{i,j})$ with respect to the chosen basis, i.e. $\varphi(e_i) = \sum_{j=1}^n a_{j,i} e_j$. Let $\{e'_1, \ldots, e'_r\}$ be another basis. Write

$$e'_k = \sum_{i=1}^n c_{i,k} e_i \text{ and } e_j = \sum_{\ell=1}^n d_{\ell,j} e'_\ell$$

such that $DC = I = CD$ with $C = (c_{i,j})$ and $D = (d_{i,j})$.

We compute:

$$\begin{aligned}\varphi(e'_k) = \varphi(\sum_{i=1}^n c_{i,k} e_i) &= \sum_{i=1}^n \sigma(c_{i_k})\varphi(e_i) = \sum_{i=1}^n \sum_{j=1}^n \sigma(c_{i,k}) a_{j,i} e_j \\ &= \sum_{i=1}^n \sum_{j=1}^n \sum_{\ell=1}^n \sigma(c_{i,k}) a_{j,i} d_{\ell,j} e'_\ell = \sum_{\ell=1}^n \big(DA\sigma(C)\big)_{\ell,k} e'_\ell.\end{aligned}$$

Hence, the matrix representing φ with respect to the basis $\{e'_1, \ldots, e'_r\}$ is $C^{-1}A\sigma(C)$. For some reason, this formula differs from the one in the book (maybe: column vectors vs. row vectors?).

Corollary 3.2 *Let $\varphi : D \to D$ be a semi-linear map on the finite dimensional K_0-vector space D. Let A be the matrix of φ with respect to some basis. Then the* Newton number

$$t_N(D) := t_N(D, \varphi) := v_p(\det(A))$$

is well-defined, i.e. does not depend on the choice of basis.

Proof. We have $\det(C^{-1}A\sigma(C)) = \frac{\sigma(\det(C))}{\det(C)}\det(A)$ and $\frac{\sigma(s)}{s}$ is a unit in $\mathcal{O}_{K_0}$ for all $s \neq 0$. □

The semi-linear map $\varphi : D \to D$ gives a semi-linear map on tensor powers, symmetric powers and on $\bigwedge^h D$. If h is the dimension of D, let (a) the 1×1-matrix representing φ on $\bigwedge^h D$. We have the equality:

$$t_N(D) = t_N(\bigwedge^r D) = v_p(a).$$

This is due to the definition of the determinant.

Proposition 3.3 *(a) If $0 \to D' \to D \to D'' \to 0$ is a short exact sequence of finite-dimensional K_0-vector spaces compatible with semi-linear maps φ', φ, φ'', then we have*

$$t_N(D) = t_N(D') + t_N(D'').$$

(b) Let D_1 and D_2 be two finite dimensional K_0-vector spaces with semi-linear φ_i. Then we have

$$t_N(D_1 \otimes D_2) = t_N(D_1)\dim_{K_0}(D_2) + t_N(D_2)\dim_{K_0}(D_1)$$

for $\varphi(d_1 \otimes d_2) = \varphi(d_1) \otimes \varphi(d_2)$.

We are now going to introduce the *Newton polygon*. The case $K_0 = \mathbb{Q}_p$ is elementary and we start by it. In this case, we define the Newton polygon of D as the usual Newton polygon for the characteristic polynomial f of φ. The slopes of the Newton polygon are the valuations of the eigenvalues of φ: We factor f into irreducibles: $f = \prod_{i=1}^r f_i$. The valuations of the zeros of an irreducible polynomial are equal: we call that valuation the *slope* of the irreducible polynomial or of its roots. More precisely, possibly after base change, for every occuring slope α there is $d \in D$ and λ such that $\varphi(d) = \lambda d$ and $v_p(\lambda) = \alpha \in \mathbb{Q}$. We order the slopes in size: $\alpha_1 < \alpha_2 < \cdots < \alpha_s$ (we may have $s < r$, since slopes can appear more than once). We can decompose D as

$$D = \bigoplus_{i=1}^s D_{\alpha_i}$$

(generalised Jordan normal form). Now the Newton polygon is the polygon with vertices

$$(0,0), (h_1, \alpha_1 h_1), (h_1 + h_2, \alpha_1 h_1 + \alpha_2 h_2), \ldots, (h, \sum_{i=1}^s \alpha_i h_i = t_N(D))$$

with $h_i = \dim_{K_0} D_{\alpha_i}$. The slope of the k-th line segment is α_k.

We now go back to general $K_0 \subset \mathbb{Q}_p^{\mathrm{unr}}$. The miracle is that in the semi-linear world something even stronger holds, which one could call *diagonalisability of every semi-linear map*. We first have to introduce base change for semi-linear maps to the maximal unramified extension. Given D and φ, we define

$$\varphi : \mathbb{Q}_p^{\mathrm{unr}} \otimes_{K_0} D \to \mathbb{Q}_p^{\mathrm{unr}} \otimes_{K_0} D, \quad x \otimes d \mapsto \sigma(x) \otimes \varphi(d).$$

Note that this is a well-defined semi-linear map on $\mathbb{Q}_p^{\mathrm{unr}} \otimes_{K_0} D$.

The following short calculation illustrates (part of) the difficulty of handling semi-linear maps (compare with part (b) below). Let $d \in \mathbb{Q}_p^{\mathrm{unr}} \otimes_{K_0} D$ and $\lambda \in \overline{\mathbb{Q}}_p$ such that $\varphi(d) = \lambda d$ and let $x \in K_0$. Then

$$\varphi(xd) = \sigma(x)\varphi(d) = \frac{\sigma(x)}{x} xd.$$

The eigenvalue changed, but its valuation did not, as $\frac{\sigma(x)}{x}$ has valuation 0.

Theorem 3.4 (Dieudonné, Manin) *Let D and φ as above.*

(a) There exist rational numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_s$, called the slopes *of φ, and φ-stable sub-K_0-vector spaces D_{α_j} for $j = 1, \dots, s$ of D such that*

$$D = \bigoplus_{i=1}^{s} D_{\alpha_i}$$

and each $\mathbb{Q}_p^{\mathrm{unr}} \otimes_{K_0} D_{\alpha_j}$ has a basis $\{e_1, \dots, e_m\}$ such that for all $i = 1, \dots, m$ there is $\lambda_i \in \overline{\mathbb{Q}}_p$ with $v_p(\lambda_i) = \alpha$ and $\varphi(e_i) = \lambda_i e_i$.

(b) If there is $d \in \mathbb{Q}_p^{\mathrm{unr}} \otimes_{K_0} D_\alpha$ and $\lambda \in \overline{\mathbb{Q}}_p$ such that $\varphi(d) = \lambda d$, then $v_p(\lambda) = \alpha$.

(c) $\sum_{j=1}^{s} \alpha_j \dim_{K_0} D_{\alpha_j} = t_N(D)$.

(d) $\alpha_j \dim_{K_0} D_{\alpha_j} \in \mathbb{Z}$ for all $j = 1, \dots, s$.

In the general case, we define the Newton polygon as before, i.e. as the polygon with vertices

$$(0,0), (h_1, \alpha_1 h_1), (h_1 + h_2, \alpha_1 h_1 + \alpha_2 h_2), \dots, (h, \sum_{i=1}^{s} \alpha_i h_i = t_N(D))$$

with $h_i = \dim_{K_0} D_{\alpha_i}$. The slope of the k-th line segment is α_k.

4 Semi-stable p-adic Galois representations

In this section, we will define semi-stable and crystalline representations by using the rings B_{st} and B_{cris} in the way that we are meanwhile used to (e.g. Christian Liedtke's talk).

In the previous talk, Stefan Kukulies introduced the rings B_{st} and B_{cris}.

Proposition 4.1 *The rings B_{st} and B_{cris} are $(\mathbb{Q}_p, G_K)$-regular.*

The proof of this proposition is similar to the proof that we saw in Coung's talk for the case of B_{HT}. Let

$$\rho : G_K \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$$

be a p-adic Galois representation. We let

- $\mathbf{D}_{\mathrm{st}}(V) := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ and
- $\mathbf{D}_{\mathrm{cris}}(V) := (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$.

Purely formally, as proved in Ralf's talk, the regularity yields the following corollary.

Corollary 4.2 *Let* $\bullet$ *stand for* st *or* cris.

(a) There is an injective $B_\bullet[G_K]$*-linear homomorphism*

$$\alpha_\bullet(V) : B_\bullet \otimes_{K_0} \mathbf{D}_\bullet(V) \to B_\bullet \otimes_{\mathbb{Q}_p} V, \quad \lambda \otimes x \mapsto \lambda x$$

for the action of G_K *on* $B_\bullet \otimes_{K_0} \mathbf{D}_\bullet(V)$ *on the first component and the diagonal* G_K*-action on* $B_\bullet \otimes_{\mathbb{Q}_p} V$.

(b) $\dim_{K_0} \mathbf{D}_\bullet(V) \leq \dim_{\mathbb{Q}_p} V$.

(c) $\dim_{K_0} \mathbf{D}_\bullet(V) = \dim_{\mathbb{Q}_p} V \Leftrightarrow \alpha_\bullet(V)$ *is an isomorphism* $\Leftrightarrow V$ *is* $B_\bullet$*-adimissible.*

(d) The functors $\mathbf{D}_\bullet(V)$ *are compatible with* $\bigoplus$, $\bigotimes$ *and duals on* $B_\bullet$*-admissible* V.

Now we make the expected definition.

Definition 4.3
- *A* p*-adic Galois representation* V *of* G_K *is called* semi-stable *if it is* B_{st}*-admissible.*
- *A* p*-adic Galois representation* V *of* G_K *is called* crystalline *if it is* B_{cris}*-admissible.*

As B_{cris} is contained in B_{st}, we have that $B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V \leq B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$ (as $K_0[G_K]$-modules). As further taking G_K-invariants is left exact, we have $(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K} \leq (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ as K_0-vector spaces. From Corollary 4.2 we further obtain the inequality:

$$\dim_{K_0} \mathbf{D}_{\mathrm{cris}}(V) \leq \dim_{K_0} \mathbf{D}_{\mathrm{st}}(V) \leq \dim_{\mathbb{Q}_p} V.$$

This together with Corollary 4.2 immediately gives the following corollary.

Corollary 4.4 *Any crystalline representation is semi-stable.*

We also have:

Proposition 4.5 *(a) For any* p*-adic Galois representation we have* $K \otimes_{K_0} \mathbf{D}_{\mathrm{st}}(V) \leq \mathbf{D}_{\mathrm{dR}}(V)$ *as* K*-vector spaces.*

(b) If V *is semi-stable, then* $K \otimes_{K_0} \mathbf{D}_{\mathrm{st}}(V) = \mathbf{D}_{\mathrm{dR}}(V)$ *as* K*-vector spaces.*

(c) Any semi-stable representation is de Rham.

Proof. The basic (and only) ingredient is the following injection (from Stefan's talk):

$$K \otimes_{K_0} B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}.$$

As above, we tensor with V and take G_K-invariants and obtain the injection of K-vector spaces

$$\big((K \otimes_{K_0} B_{\mathrm{st}}) \otimes_{\mathbb{Q}_p} V\big)^{G_K} \hookrightarrow \big(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\big)^{G_K} = \mathbf{D}_{\mathrm{dR}}(V).$$

Noticing the trivial equality $K \otimes_{K_0} \mathbf{D}_{\mathrm{st}}(V) = K \otimes_{K_0} \big(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V\big)^{G_K} = \big((K \otimes_{K_0} B_{\mathrm{st}}) \otimes_{\mathbb{Q}_p} V\big)^{G_K}$ leads us to conclude $K \otimes_{K_0} \mathbf{D}_{\mathrm{st}}(V) \leq \mathbf{D}_{\mathrm{dR}}(V)$ as K-vector spaces, i.e. (a). Using the $(\mathbb{Q}_p, G_K)$-regularity of B_{dR}, we obtain the inequality

$$\dim_{K_0} \mathbf{D}_{\mathrm{st}}(V) \leq \dim_K \mathbf{D}_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p} V,$$

from which the other parts of the proposition follow. □

5 Towards filtered (φ, N)-modules

In this part we will approach the definition of (φ, N)-modules via its main example: $\mathbf{D}_{\mathrm{st}}(V)$.

Let us recall from Stefan's talk:

- The "Frobenius" φ uniquely extends to B_{st} by requiring $\varphi(\log[\varpi]) = p\log[\varpi]$.
- On B_{st} there is the "monodromy operator" $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$, which is defined by

$$N\Big(\sum_{n\in\mathbb{N}} b_n(\log[\varpi])^n\Big) = \sum_{n\in\mathbb{N}} nb_n(\log[\varpi])^{n-1}.$$

- $g\varphi = \varphi g$ and $gN = Ng$ for every $g \in G_{K_0}$, i.e. φ and N commute with the Galois action.
- $N\varphi = p\varphi N$.
- The sequence

$$0 \to B_{\mathrm{cris}} \to B_{\mathrm{st}} \xrightarrow{N} B_{\mathrm{st}} \to 0$$

  is exact.

This implies the following for a p-adic Galois representation $\rho : G_K \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$:

- On $\mathbf{D}_{\mathrm{st}}(V) = \big(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V\big)^{G_K}$ we define the "Frobenius" by $\varphi : \mathbf{D}_{\mathrm{st}}(V) \to \mathbf{D}_{\mathrm{st}}(V)$ by $\varphi(b \otimes v) = \varphi(b) \otimes v$.
- On $\mathbf{D}_{\mathrm{st}}(V) = \big(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V\big)^{G_K}$ we define the "monodromy operator" $N : \mathbf{D}_{\mathrm{st}}(V) \to \mathbf{D}_{\mathrm{st}}(V)$ by $N(b \otimes v) = N(b) \otimes v$.
- On $\mathbf{D}_{\mathrm{st}}(V)$ we still have the formulae $g\varphi = \varphi g$ and $gN = Ng$ for every $g \in G_{K_0}$, i.e. φ and N commute with the Galois action. This is clear, since the action is only on the first component.

- We also have $N\varphi = p\varphi N$ on $\mathbf{D}_{\mathrm{st}}(V)$ for the same reason.
- Because of $\dim_{K_0} \mathbf{D}_{\mathrm{st}}(V) \le \dim_{\mathbb{Q}_p} V$, this dimension is finite.
- The Frobenius φ is an isomorphism on $\mathbf{D}_{\mathrm{st}}(V)$, since it is injective on B_{st} (and, consequently, also on $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$, and thus on $\left(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}$).
- The sequence

$$0 \to \mathbf{D}_{\mathrm{cris}}(V) \to \mathbf{D}_{\mathrm{st}}(V) \xrightarrow{N} \mathbf{D}_{\mathrm{st}}(V)$$

is exact, as $\cdot \otimes_{\mathbb{Q}_p} V$ is exact and $(\cdot)^{G_K}$ is left exact.

Let V be semi-stable. Then we have

$$V \text{ crystalline } \Leftrightarrow \dim_{K_0} \mathbf{D}_{\mathrm{cris}}(V) = \dim_{K_0} \mathbf{D}_{\mathrm{st}}(V) = \dim_{\mathbb{Q}_p} V \Leftrightarrow N = 0.$$

Further, we recall from Christian's talk that there is a descending filtration of K-vector spaces on $\mathbf{D}_{\mathrm{dR}}(V)$:

$$\cdots \supseteq \mathrm{Fil}^{i-1} \mathbf{D}_{\mathrm{dR}}(V) \supseteq \mathrm{Fil}^{i} \mathbf{D}_{\mathrm{dR}}(V) \supseteq \mathrm{Fil}^{i+1} \mathbf{D}_{\mathrm{dR}}(V) \supseteq \ldots.$$

We use it for defining a descending filtration of K-vector spaces on $D_K := K \otimes_{K_0} \mathbf{D}_{\mathrm{st}}(V) \le \mathbf{D}_{\mathrm{dR}}(V)$ (see Proposition 4.5) by putting

$$\mathrm{Fil}^i D_K := D_K \cap \mathrm{Fil}^i \mathbf{D}_{\mathrm{dR}}(V).$$

This will make $\mathbf{D}_{\mathrm{st}}(V)$ into a filtered (φ, N)-module over K of finite dimension with bijective φ. The filtration is separated and exhaustive.

6 Filtered (φ, N)-modules - definitions and simple properties

We now use as definition the properties that we just saw for $\mathbf{D}_{\mathrm{st}}(V)$.

Definition 6.1 *A* (φ, N)-module *over K_0 (or k) is a K_0-vector space D together with two maps*

$$\varphi : D \to D \text{ “Frobenius” and } N : D \to D \text{ “monodromy”}$$

such that

(1) φ is semi-linear,

(2) N is K_0-linear and

(3) $N\varphi = p\varphi N$.

Definition 6.2 *Let D_1 and D_2 be two (φ, N)-modules with operators φ_i and N_i ($i = 1, 2$). A morphism $\eta : D_1 \to D_2$ of (φ, N)-modules is a K_0-linear map such that $\varphi_2 \circ \eta = \eta \circ \varphi_1$ and $N_2 \circ \eta = \eta \circ N_1$.*

Definition 6.3 *Let D_1 and D_2 be two (φ, N)-modules with operators φ_i and N_i ($i = 1, 2$). The* tensor product *$D_1 \otimes D_2$ is defined as the K_0-vector space $D_1 \otimes D_2 := D_1 \otimes_{K_0} D_2$ equipped with Frobenius*

$$\varphi(d_1 \otimes d_2) := \varphi(d_1) \otimes \varphi_2(d_2)$$

and the monodromy

$$N(d_1 \otimes d_2) := N_1(d_1) \otimes d_2 + d_1 \otimes N_2(d_2).$$

Here we note (the book does not do this) that the definition is symmetric in D_1 and D_2. Hence, we obtain (φ, N)-modules $\mathrm{Sym}^r D$ (r-fold symmetric product) as well $\bigwedge^r D$ (r-fold exterior product). (The book mentions at one point that one can see the exterior product as a sub-object of the tensor-product. This is correct, but only because we are over a field of characteristic zero. Otherwise, the correct way is to see the symmetric product as a quotient of the tensor product and the exterior product as a sub-object of the symmetric product.)

Definition 6.4 *Let D be a (φ, N)-module such that D_0 is finite dimensional as K_0-vector space and such that φ is bijective. The* dual *D^* of D is defined as the K_0-vector space $\mathrm{Hom}_{K_0-\text{linear}}(D, K_0)$ equipped with the Frobenius*

$$\varphi^*(\alpha) := \big(D \xrightarrow{\varphi^{-1}} D \xrightarrow{\alpha} K_0 \xrightarrow{\sigma} K_0\big)$$

and monodromy

$$N^*(\alpha) := -\alpha \circ N.$$

There is a "category-way" of seeing (φ, N)-modules. Namely, they are modules over the non-commutative ring generated by K_0, N and φ subject to the relations $\varphi a = \sigma(a)\varphi$, $Na = aN$ for all $a \in K_0$ and the relation $N\varphi = p\varphi N$.

Proposition 6.5 *Let D be a finite dimensional (φ, N)-module over K_0 with bijective φ.*

(a) N decreases slopes by 1, i.e. $N(D_\alpha) \subseteq D_{\alpha-1}$.

(b) N is nilpotent.

Proof. (a) We may test this after base change to $\mathbb{Q}_p^{\mathrm{unr}}$. Let $d \in \mathbb{Q}_p^{\mathrm{unr}} \otimes_{K_0} D_\alpha$ and $\lambda \in \overline{\mathbb{Q}}_p$ such that $v_p(\lambda) = \alpha$ and $\varphi(d) = \lambda d$. We compute

$$\varphi N d = \frac{1}{p} N \varphi(d) = \frac{1}{p} N \lambda d = \frac{\lambda}{p} N d$$

and conclude that Nd is an eigenvector for φ with valuation $\alpha - 1$, whence $Nd \in D_{\alpha-1}$.

(b) (First proof.) By (a) and the fact that the decomposition $D = \bigoplus_{j=1}^{s} D_{\alpha_j}$ is finite, $N^s = 0$.

(Second proof, not using (a).) Let $0 \neq \lambda \in \overline{\mathbb{Q}}_p$ be an eigenvalue of N such that the associated eigenspace $V \subseteq D \otimes \overline{\mathbb{Q}}_p$ is non-trivial. Let $v \in V$. Because of $N\varphi v = p\varphi N v = p\varphi\lambda v = p\lambda\varphi v$, it follows that N acts on $\varphi(V)$ by multiplication with $p\lambda$, whence $\varphi(V) \cap V = (0)$. It follows that

$\lambda = 0$, as iterating the application of φ would imply that V is infinite-dimensional. Hence, N has only 0 as eigenvalue and is hence nilpotent. □

Now we introduce another important structure on (φ, N)-modules, namely the filtration. In the example $\mathbf{D}_{\mathrm{st}}(V)$ we have a filtration on $K \otimes_{K_0} \mathbf{D}_{\mathrm{st}}(V)$: the de Rham-filtration.

Definition 6.6 *A* filtered (φ, N)-module over K *is a (φ, N)-module D over K_0 together with a filtration $\mathrm{Fil}^\bullet D_K$ on the K-vector space $D_K := K \otimes_{K_0} D$ which is decreasing, separated and exhaustive.*

The category of filtered (φ, N)-modules over K is denoted by $\mathrm{MF}_K(\varphi, N)$.

Definition 6.7 *A* morphism of filtered (φ, N)-modules over K *is a morphism $\eta : D_1 \to D_2$ of (φ, N)-modules over K_0 such that the induced map $\eta_K : D_{1,K} \to D_{2,K}$ is a homomorphism of filtered K-vector spaces as defined earlier in this talk.*

Definition 6.8 *Let $D_1, D_2, \ldots, D_r$ be filtered (φ, N)-modules over K. The* tensor product $D_1 \otimes D_2 \otimes \cdots \otimes D_r$ *in the category of filtered (φ, N)-modules over K is the tensor product $D := D_1 \otimes D_2 \otimes \cdots \otimes D_r$ in the category of (φ, N)-modules over K_0 equipped with the filtration on D_K as defined earlier in this talk. As this definition is symmetric in the D_i the filtration descends to give rise to Sym^r and $\bigwedge^r$ in the category of filtered (φ, N)-modules over K.*

Definition 6.9 *Let D be a filtered (φ, N)-module over K such that D is a finite-dimensional K_0-vector space and such that φ is bijective. The* dual filtered (φ, N)-module D^* over K *of D is the dual D^* in the category of (φ, N)-modules over K_0 equipped with the filtration*

$$\mathrm{Fil}^i(D^*)_K := (\mathrm{Fil}^{-i+1} D_K)^*.$$

For a filtered (φ, N)-module D over K, we define the *Hodge number* of D as

$$t_H(D) := t_H(D_K)$$

and the *Newton number* $t_N(D)$ as before. We have the following properties from earlier on.

Proposition 6.10 *(a) If $0 \to D' \to D \to D'' \to 0$ is a short exact sequence of filtered (φ, N)-modules over K, then we have*

$$t_N(D) = t_N(D') + d_N(D'') \text{ and } t_H(D) = t_H(D') + d_H(D'').$$

(b) Let D_1 and D_2 be two filtered (φ, N)-modules over K. Then we have

$$t_N(D_1 \otimes D_2) = t_N(D_1)\dim_{K_0}(D_2) + t_N(D_2)\dim_{K_0}(D_1)$$

and

$$t_H(D_1 \otimes D_2) = t_H(D_1)\dim_{K_0}(D_2) + t_H(D_2)\dim_{K_0}(D_1).$$

(c) *For a finite dimensional* (φ, N)*-module* D *with bijective* φ*, we have* $t_N(D^*) = -t_N(D)$ *and* $t_H(D^*) = -t_H(D)$*.*

Definition 6.11 *A filtered* (φ, N)*-module over* K *is called* admissible *if*

(i) $\dim_{K_0} D < \infty$,

(ii) φ *is bijective on* D,

(iii) $t_H(D) = t_N(D)$ *and*

(iv) *for any subobject* $D' \leq D$ *in the category of filtered* (φ, N)*-modules over* K *the inequality*

$$t_H(D') \leq t_N(D')$$

holds.

The category of admissible filtered (φ, N)*-modules over* K *is denoted by* $\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$.

Let D be an admissible (φ, N)-module over K and D' be a sub-object. A very useful statement is that the Hodge polygon of D' stays below the Newton polygon of D' (we allow that they "touch", of course).

The argument is best given in a picture. We sketch it. If D' only has a single Newton slope α, the statement is clear. Note that all Hodge slopes occuring in the polygon of D'_α also occur in the Hodge polygon of D', but possibly on longer line segments. If α was the smallest Newton slope, then we conclude that up to $\dim D'_\alpha$ the Hodge polygon remains below the Newton polygon. (Note that whereas the Newton polygon can be obtained by concatenating the Newton polygons of all D'_α, this is not true for Hodge polygons.) If now $D' = D'_\alpha \oplus D'_\beta$, we get the statement from our previous observation and the inequality $t_H(D'_\alpha \oplus D'_\beta) \leq t_N(D'_\alpha \oplus D'_\beta)$. We repeat the argument from above that all Hodge slopes here have to occur in the Hodge polygon for D', but possibly on longer line segments. This again implies that now up to $\dim D'_\alpha + \dim D'_\beta$ the Hodge polygon is below the Newton polygon. Like this we continue.

7 Examples of admissible filtered (φ, N)-modules

7.1 Trivial filtration

A filtration on a K-vector space V is called *trivial* if

$$\mathrm{Fil}^0(V) = V \text{ and } \mathrm{Fil}^1(V) = (0).$$

This means that the Hodge polygon is the straight line from $(0, 0)$ to $(h, 0)$ with $h = \dim_K V$.

Lemma 7.1 *Let* D *be a filtered* (φ, N)*-module over* K *with trivial filtration. Then* D *is admissible if and only if* D *is of slope* 0*. In that case,* $N = 0$*.*

Proof. If D is admissible, then the Newton polygon has to be above the Hodge polygon (i.e. above 0) with endpoint $t_H(D) = t_N(D) = 0$, so the Newton polygon also has to be the straight line from $(0, 0)$ to $(h, 0)$ with $h = \dim_K V$, whence all the slopes are zero.

Conversely, if the slope is zero, then the Newton polygon is the straight line from $(0, 0)$ to $(h, 0)$. The same holds for all sub-objects, whence D is admissible.

If all the slopes are zero, then $D = D_0$ and $ND \subseteq D_{-1} = (0)$. □

7.2 Tate twist

Let D be a filtered (φ, N)-module over K. For $i \in \mathbb{Z}$ define the *i-th Tate twist* $D\langle i\rangle$ as follows

- $D\langle i\rangle := D$ as K_0-vector space,
- $\mathrm{Fil}^r(D\langle i\rangle)_K := \mathrm{Fil}^{r+i} D_K$ for $r \in \mathbb{Z}$,
- N on $D\langle i\rangle$ is the same as N on D,
- φ on $D\langle i\rangle$ is defined as $p^{-i}\varphi$ on D.

Lemma 7.2 *(a)* $D\langle i\rangle$ *is a filtered* (φ, N)*-module over* K.

(b) $D\langle i\rangle$ *is admissible if and only if* D *is admissible.*

(c) $\mathbf{D}_{\mathrm{st}}(V(i)) \cong (\mathbf{D}_{\mathrm{st}}(V))\langle i\rangle$.

We skip the proof, which is by a computation. As a consequence of the lemma we have

$$\dim_{\mathbb{Q}_p} V(i) = \dim_{\mathbb{Q}_p} V \leq \dim_{K_0} \mathbf{D}_{\mathrm{st}}(V) = \dim_{K_0}(\mathbf{D}_{\mathrm{st}}(V(i))).$$

We immediately obtain the first of the equivalences:

- V is semi-stable $\Leftrightarrow$ $V(i)$ is semi-stable,
- V is de Rham $\Leftrightarrow$ $V(i)$ is de Rham,
- V is crystalline $\Leftrightarrow$ $V(i)$ is crystalline.

7.3 Dimension 1

We now suppose that we are given a 1-dimensional (φ, N)-module D over K. We choose a basis $d \in D$, so that $\varphi(d) = \lambda d$ for some $\lambda \in K_0$, whence $t_N(D) = v_p(\lambda)$. The monodromy operator N must be zero, as it is nilpotent. Due to 1-dimensionality, the filtration on D_K has a single jump, which by definition occurs at $t_H(D)$.

Here is the general construction of admissible (φ, N)-modules of dimension 1 over K. It only depends on $\lambda \in K_0^\times$. We define an associated (φ, N)-module D_λ as follows.

$$D_\lambda = K_0, \quad \varphi = \lambda\sigma, \quad N = 0$$

with the filtration

$$\mathrm{Fil}^r(D_K) = \begin{cases} D_K & \text{for } r \le v_p(\lambda), \\ 0 & \text{for } r > v_p(\lambda). \end{cases}$$

We have $D_\lambda \cong D_\mu$ as (φ, N)-modules if and only if there is u in the unit group of the integers of K_0 such that $\mu = \lambda \frac{\sigma(u)}{u}$. For, if such u is given, then the isomorphism $D_\lambda \to D_\mu$ of K_0-vector spaces is given by multiplication by u. Conversely, any isomorphism $D_\lambda \to D_\mu$ must be multiplication by some u and an easy calculation gives the relation $\mu = \lambda \frac{\sigma(u)}{u}$.

7.4 Dimension 2

The aim of this section is to classify all admissible (φ, N)-modules over $K = \mathbb{Q}_p$ of dimension 2.

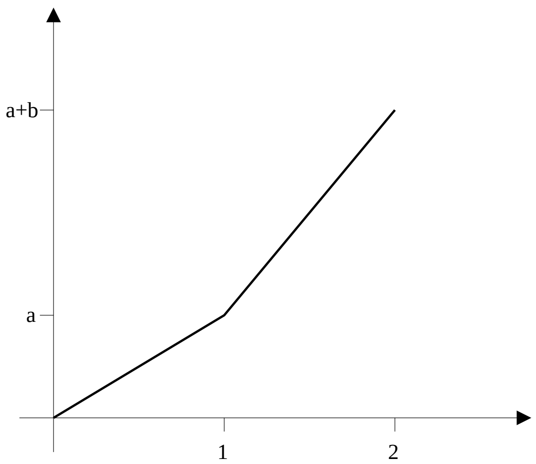
The case of trivial filtration was treated above. By the Tate twist we can and will from now on assume that there are two jumps occuring at 0 and j. Hence, we have

$$\mathrm{Fil}^r D_K = \begin{cases} D_K & \text{if } r \le 0, \\ L & \text{if } 1 \le r \le j, \\ (0) & \text{if } r > j, \end{cases}$$

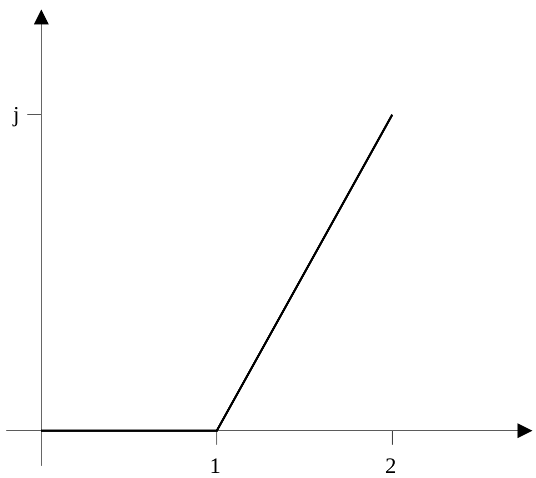
with some 1-dimensional $\mathbb{Q}_p$-vector space L.

We now compute and plot the Newton and the Hodge polygon. The Hodge polygon is by definition the polygon with vertices $(0,0)$, $(1,0)$, $(2,j)$.

Let $f(X) = X^2 + uX + v \in \mathbb{Q}_p[X]$ be the characteristic polynomial of φ. The Newton polygon of D is just the usual Newton polygon of f, i.e. the convex hull of $(0,0)$, $(1, v_p(u))$, $(2, v_p(v))$. A different description is as follows. Factor $f(X) = (X - \lambda_1)(X - \lambda_2)$ in $\overline{\mathbb{Q}}_p[X]$, where we order λ_1 and λ_2 such that $a := v_p(\lambda_1) \le b := v_p(\lambda_2)$: the first line segment has slope a, the second one has slope b.



Newton polygon



Hodge polygon

We see that if D is admissible, then

$$t_N(D) = a + b = j = t_H(D) \text{ and } a \ge 0.$$

The latter condition comes from the fact that the Newton polygon has to be above the Hodge polygon.

7.4.1 The non-crystalline case $N \neq 0$

Let v be an eigenvector with eigenvalue $\lambda \in \overline{\mathbb{Q}}_p$ (we will shortly see that $\lambda \in \mathbb{Q}_p$). We have

$$\varphi N v = \frac{1}{p} N \varphi v = \frac{1}{p} N \lambda v = \frac{\lambda}{p} N v.$$

Hence, Nv is either 0 or an eigenvector of φ with eigenvalue $\frac{\lambda}{p}$. Applying this with an eigenvector v_1 with eigenvalue λ_1, we find $Nv_1 = 0$, as the eigenvalue of Nv_1 would have valuation smaller than $v_p(\lambda_2) \geq v_p(\lambda_1)$, which is a contradiction. This also shows that $v_p(\lambda_1) \neq v_p(\lambda_2)$ (otherwise $N = 0$).

Let v_2 an eigenvector with eigenvalue λ_2. It follows that $Nv_2 \neq 0$, as otherwise $N = 0$, since v_1 and v_2 form a basis of D. This gives $p\lambda_1 = \lambda_2$. It follows that $\lambda_1, \lambda_2 \in \mathbb{Q}_p$ and

$$j = t_H(D) = t_N(D) = 1 + 2v_p(\lambda_1).$$

Now choose the basis $\{e_1, e_2\}$ of D with $e_2 := v_2$ and $e_1 = Ne_2$. Then we have

$$\varphi = \begin{pmatrix} \lambda_1 & 0 \\ 0 & p\lambda_1 \end{pmatrix} \text{ and } N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We now determine L explicitly. There is a unique 1-dimensional subobject $D' \leq D$ because it has to be fixed by φ and N, namely $D' = \langle e_1 \rangle$. Obviously, $t_N(D') = v_p(\lambda_1) = a < j = 2a + 1$. The filtration on D' is the one induced from D, i.e.

$$\operatorname{Fil}^r D' = D' \cap \operatorname{Fil}^r D = \begin{cases} D' & \text{if } r \leq 0, \\ D' \cap L & \text{if } 1 \leq r \leq j, \\ 0 & \text{if } r > j. \end{cases}$$

Hence, we have

$$t_H(D') = 0 \text{ if } L \neq D' \text{ and } t_H(D') = j \text{ if } L = D'. \tag{7.1}$$

It follows that

$$t_H(D') \leq t_N(D') = a \Leftrightarrow D' \neq L.$$

The admissibility thus implies $D' \neq L$, whence $L = \langle e_2 + \alpha e_1 \rangle$ for a unique $\alpha \in \mathbb{Q}_p$.

Conversely, choosing $\alpha \in \mathbb{Q}_p$ and $0 \neq \lambda \in \mathbb{Z}_p$ and putting (for the standard basis on the 2-dimensional $\mathbb{Q}_p$-vector space D)

$$\varphi = \begin{pmatrix} \lambda & 0 \\ 0 & p\lambda \end{pmatrix} \text{ and } N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

as well as

$$\operatorname{Fil}^r D = \begin{cases} D & \text{if } r \leq 0, \\ \langle \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \rangle & \text{if } 1 \leq r \leq j, \\ (0) & \text{if } r > j, \end{cases}$$

we obtain an admissible (φ, N)-module over $\mathbb{Q}_p$. By Tate twisting we obtain all admissible (φ, N)-modules over $\mathbb{Q}_p$.

7.4.2 The crystalline case: $N = 0$

First case: $f(X) = X^2 + uX + v$ is irreducible over $\mathbb{Q}_p$.

As there is no non-trivial subobject (it would be a line with eigenvalue in $\mathbb{Q}_p$), admissibility of D is equivalent to $a + b = t_N(D) = t_H(D) = j$.

Suppose that D is admissible and pick any vector $0 \neq e_1 \in L$. Then $\{e_1, e_2\}$ with $e_2 = \varphi(e_1)$ form a basis of D. The characteristic polynomial forces the following shape:

$$\varphi = \begin{pmatrix} 0 & -v \\ 1 & -u \end{pmatrix} \text{ and } N = 0$$

and

$$\mathrm{Fil}^r D = \begin{cases} D & \text{if } r \leq 0, \\ \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle & \text{if } 1 \leq r \leq j, \\ (0) & \text{if } r > j. \end{cases}$$

Conversely, given $u, v \in \mathbb{Q}_p$ with $j = v_p(v) > 0$ such that $X^2 + uX + v$ is irreducible in $\mathbb{Q}_p[X]$, by the above formulae we can associate to it an irreducible admissible (φ, N)-module over $\mathbb{Q}_p$. By Tate twisting we obtain all of this type.

Second case: $f(X) = (X - \lambda_1)(X - \lambda_2)$ with $\lambda_1, \lambda_2 \in \mathbb{Q}_p$.

We first treat the case $\lambda := \lambda_1 = \lambda_2$ such that (for some basis $\{e_1, e_2\}$) $\varphi = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. I do not find this case treated in the text, but it does not seem to be excluded. In this case, there is a unique subobject, namely, $D' = \langle e_1 \rangle$. We have $t_N(D') = v_p(\lambda) = a < 2a = j$. By Equation 7.1, admissibility hence implies $t_H(D') = 0$, whence $L \neq \langle e_1 \rangle$.

Suppose now that there is a basis of eigenvectors $\{e_1, e_2\}$ with eigenvalues λ_1 and λ_2, respectively (we allow $\lambda_1 = \lambda_2$). There are two stable subobjects, namely $\langle e_1 \rangle$ and $\langle e_2 \rangle$. Admissibility implies as above that L is neither of them. By rescaling e_1 and e_2 we can assume that $L = \langle e_1 + e_2 \rangle$.

Hence, we obtain in this case

$$\varphi = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ and } N = 0$$

and

$$\mathrm{Fil}^r D = \begin{cases} D & \text{if } r \leq 0, \\ \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle & \text{if } 1 \leq r \leq j, \\ (0) & \text{if } r > j. \end{cases}$$

Conversely, given $\lambda_1, \lambda_2 \in \mathbb{Z}_p$ with $v_p(\lambda_1) \leq v_p(\lambda_2)$ and $j = v_p(\lambda_1) + v_p(\lambda_2)$, the above formulae give rise to an admissible (φ, N)-module over $\mathbb{Q}_p$. We may again apply the Tate twist.

I do not state Proposition 7.11 of the book because the case $\varphi = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ seems to be missing. By Dieudonné's theorem we know that φ can be diagonalised after base change to $\mathbb{Q}_p^{\mathrm{unr}}$ as a semi-linear map. But, I do not see where an isomorphism as (φ, N)-module with any of the diagonal ones should come from. In fact, it cannot exist, since the minimal polynom of φ on the non-diagonal module is different from the minimal polynomial on the diagonal one. Could it be that the modules become isomorphic over $\mathbb{Q}_p^{\mathrm{unr}}$?

8 Theorem of Fontaine-Colmez (Theorem B)

So far we have described the functor

$$\mathbf{D}_{\mathrm{st}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K(\varphi, N), \ \ V \mapsto \mathbf{D}_{\mathrm{st}}(V).$$

Proposition 8.1 (Theorem B(1)) *If V is a semi-stable p-adic Galois representation of G_K, then $\mathbf{D}_{\mathrm{st}}(V)$ is an admissible filtered (φ, N)-module over K with φ and N as defined before. More precisely, we have the functor*

$$\mathbf{D}_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}^{\mathrm{ad}}_K(\varphi, N), \ \ V \mapsto \mathbf{D}_{\mathrm{st}}(V).$$

It is compatible with tensor products and duals.

Definition 8.2 *For D a filtered (φ, N)-module over K, let*

$$\mathbf{V}_{\mathrm{st}}(D) := \{v \in B_{\mathrm{st}} \otimes D | \varphi(v) = v, \ N(v) = 0, \ 1 \otimes v \in \mathrm{Fil}^0(K \otimes_{K_0} (B_{\mathrm{st}} \otimes D))\},$$

where the tensor product $B_{\mathrm{st}} \otimes D$ is the tensor product in the category of filtered (φ, N)-modules over K.

We have that $\mathbf{V}_{\mathrm{st}}(D)$ is a sub-$\mathbb{Q}_p$-vector space of $B_{\mathrm{st}} \otimes D$ (that is clear), which is stable under G_K. For the latter we need that G_K respects the filtration on B_{dR} (I think).

Proposition 8.3 (Theorem B(2)) *If D is an admissible filtered (φ, N)-module over K, then $\mathbf{V}_{\mathrm{st}}(D)$ is a semi-stable p-adic representation of G_K. More precisely, we have the functor*

$$\mathbf{V}_{\mathrm{st}} : \mathrm{MF}^{\mathrm{ad}}_K(\varphi, N) \to \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K), \ \ D \mapsto \mathbf{V}_{\mathrm{st}}(D).$$

It is compatible with tensor products and duals.

Finally we can state the main part of Theorem B.

Theorem 8.4 (Theorem B(3)) *The functor*

$$\mathbf{D}_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}^{\mathrm{ad}}_K(\varphi, N), \ \ V \mapsto \mathbf{D}_{\mathrm{st}}(V).$$

is an equivalence of categories with quasi-inverse

$$\mathbf{V}_{\mathrm{st}} : \mathrm{MF}^{\mathrm{ad}}_K(\varphi, N) \to \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K), \ \ D \mapsto \mathbf{V}_{\mathrm{st}}(D).$$