Math Prep Camp: Sets and Functions

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Exercise sheet 2

1. *Describing a map.*
   Let $A := \{0, 1, 2\}$ and $B := \{X, Y\}$ be sets.

   Which of the following lines describe a map $g : A \to B$? If not, why?

   (a) $g(2) = X$, $g(0) = Y$, $g(1) = X$

   (b) $g(0) = X$, $g(2) = Y$, $g(0) = X$

   (c) $g(0) = X$, $g(2) = Y$, $g(0) = Y$

   (d) $g(0) = Y$, $g(1) = Y$, $g(2) = Y$

2. *Image and preimage.*
   Consider the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$.

   (a) What is the image of 3 under f?

   (b) What is the image of $1 + x$ under f for $x \in \mathbb{R}$?

   (c) What is the image of the interval $[2, 3]$ under f?

   (d) What are the preimages of 4 under f?

   (e) What is the preimage of the interval $[1, 4]$ under f?

3. *Graphs.*

   (a) Sketch the graph of the function $f : [-2, 2] \to [0, 4], x \mapsto x^2$ (in the standard way: x-axis to the right, y-axis up).

   (b) Reflect your sketch at the line $x = y$ (i.e. swap x- and y-axis). Do you still have the graph of a function? Why?

4. *Injectivity, surjectivity, bijectivity.*
   For each of the following functions, state if they are injective, surjective or bijective (or none of these).

   (a) $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$

   (b) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}, x \mapsto x^2$

   (c) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$

   (d) $f : \mathbb{R} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$

   Here $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$.

5. *Composition.*

   (a) Let $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ and $g : \mathbb{R} \to \mathbb{R}, x \mapsto x + 1$. For $x \in \mathbb{R}$, compute $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$ and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}, x \mapsto \sqrt{x}$. For $x \in \mathbb{R}$, compute $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$.

6. *Inverse map.*
   Compute the inverse of each of the following bijective maps.

   (a) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$

   (b) Let $A := \{0, 1, 2\}$, $b := \{a, b, c\}$ and let $f : A \to B$ be given by $f(0) = b$, $f(1) = a$, $f(2) = c$.

7. *Injectivity, surjectivity, bijectivity (2).*
   Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d\}$.

   (a) Describe a surjective map from A to B.

   (b) Describe a map from A to B which is neither surjective nor injective.

   (c) Does there exist an injective map from A to B? Why?

   (d) Describe an injective map from B to A.

   (e) Describe a map from B to A which is neither surjective nor injective.

   (f) Does there exist a surjective map from B to A? Why?

8. *Domain, image, preimage*

   (a) Let f be a map from $\mathbb{N}$ to $\mathbb{Z}$ defined by $f(n) = n^3$ and g a map from $\mathbb{Z}$ to $\mathbb{N}$ defined by $g(n) = n^2$. Calculate the image of 2 under f and determine $f \circ g$.

   (b) Let f be a map from $E = \{1, 2, 3, 4\}$ to $F = \{0, 1, 3, 5, 7, 10\}$ such that $f(1) = 3$, $f(2) = 5$, $f(3) = 5$ and $f(4) = 0$.
   Determine $f(\{2, 3\})$, $\mathrm{im}(f)$.
   Determine $f^{-1}(\{5\})$, $f^{-1}(\{0, 1, 3\})$ and $f^{-1}(\{1, 10\})$.
   Is f injective, surjective, bijective?

9. *More on injectivity, surjectivity, bijectivity.*

   (a) Find an injective but not bijective map from $\mathbb{N}$ to $\mathbb{N}$.

   (b) Find a surjective but not bijective map from $\mathbb{N}$ to $\mathbb{N}$.

   (c) Find a bijection between $\mathbb{Z}$ and $\mathbb{N}$.

10. *Some proofs concerning maps.*
    Let A, B, C be sets and $f : A \to B$ and $g : B \to C$ maps. Prove:

    (a) If f and g are both injective (resp. surjective, resp. bijective), then $g \circ f$ is injective (resp. surjective, resp. bijective).

    (b) If $g \circ f$ is injective, then f is injective.

    (c) If $g \circ f$ is surjective, then g is surjective.

    (d) Suppose that both f and g are bijective with inverses f^{-1} and g^{-1}, respectively.
    Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

11. *Invertibility*
    Consider the function

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} |x + 1| \text{ if } x < 0 \\ |x - 1| \text{ if } x \geq 0. \end{cases}$$

    Make a sketch of f.

    (a) Is the function f a bijection?

    (b) Find the biggest closed interval $[a, 10] \subseteq \mathbb{R}$ such that f restricted to $[a, 10]$ is injective.

    (c) Write $g : [a, 10] \to f([a, 10])$ for the restriction of f to $[a, 10]$ with a from (b). Now g is bijective. Describe the inverse of g explicitly.

12. *Involution*
    Let E be a set and $f : E \to E$ a map such that: $f \circ f = \mathrm{id}_E$.
    Prove that f is bijective.
    What is its inverse?

13. *Sine function*
    Let $\sin : \mathbb{R} \to [-1, 1]$ be the sine function (known from school):

    (a) Is $\sin$ bijective?

    (b) Describe the preimage $\sin^{-1}(\{0\})$.

    (c) Describe the preimage $\sin^{-1}(\{1\})$.

14. *Maps and power sets.*
    Let E be a non-empty set, $\mathcal{P}(E)$ its power set, and $A, B \in \mathcal{P}(E)$. One defines

$$f : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto (A \cap X) \cup (B \cap \overline{X}^E),$$

    where $\overline{X}^E = E \setminus X$ is the complement of X in E.
    Analyse the equality $f(X) = \emptyset$.
    Deduce a necessary condition for f to be bijective.

15. *Increasing maps*.
    Let $I \subseteq \mathbb{R}$ and $J \subseteq \mathbb{R}$ be two intervals in $\mathbb{R}$. Let $f : I \longrightarrow J$ be a strictly increasing function.

    (a) Show that f is injective.

    (b) Determine the unique subset $K \subseteq J$ such that $f : I \longrightarrow K$ is bijective.

16. *Maps from* $\mathbb{N}$ *to* $\mathbb{N}$
    Consider a map $u : \mathbb{N} \to \mathbb{N}$ and assume that

$$\forall k \in \mathbb{N} : u(k + 1) > u(k).$$

    (a) Show rigorously (justifying each step of your argument) that for any $k, l \in \mathbb{N}$ with $k < l$, one has $u(k) < u(l)$.

    (b) Is u necessarily injective? Justify your answer by a precise argument or by a counter-example.

    (c) Is u necessarily surjective? Justify your answer by a precise argument or by a counter-example.