The Galois Representation Attached to a Hilbert Modular Form

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Abstract

This talk is the last one in the Essen seminar on quaternion algebras. It is based on the paper by Takeshi Saito on Hilbert modular forms and p-adic Hodge theory, but I also used talk notes by Gerard van der Geer and Theo van den Bogaart. However, I made some changes in the presentation, which may have led to the introduction of errors. The reader be warned.

1 Hilbert modular forms and automorphic representations

Notation 1.1 (First part) *We fix the following data.*

- $F/\mathbb{Q}$*, a totally real number field with* $[F:\mathbb{Q}] = n > 1$.
- $I = \{\tau_1, \dots, \tau_n\} = \mathrm{Hom}(F, \mathbb{R})$*, the embeddings of* F *into* $\mathbb{R}$.
- $\mathcal{O}_F$*, the ring of integers of* F.
- $D^{-1} = \{b \in F | \, \mathrm{Tr}_{F/\mathbb{Q}}(\mathcal{O}_F b) \subset \mathbb{Z}\}$*, the codifferent ideal.*
- $v \lhd \mathcal{O}_F$*, a fixed place which we only need and define if* n *is even.*
- $(k) = (k_1, \dots k_n, w)$ *an* $n+1$*-tuple of integers such that* $w \geq k_i \geq 2$ *and* $k_i \equiv w \mod 2$.
- $X = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$*, the union of the upper and the lower half planes,* X^+ *the upper half plane.*

Adelic Hilbert modular forms

We quickly recall the definition of adelic modular forms, following Saito.

Let X^I be the n-fold product of X with the left $\mathrm{GL}_2(\mathbb{R})^I$-action

$$\gamma . z = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d} = \Big(\frac{a_i z_i + b_i}{c_i z_i + d_i}\Big)_i \in X^I$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \Big(\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \Big)_i$ and $z = (z_i)_i$. Note that

$$\mathrm{GL}_2(\mathbb{A}_F) = \mathrm{GL}_2(\mathbb{R})^I \times \mathrm{GL}_2(\mathbb{A}_{F,f}).$$

Via the embeddings (τ_i) and the diagonal $F \hookrightarrow \mathbb{A}_{F,f}$, we obtain a natural left action of $\mathrm{GL}_2(F)$ on $X^I \times \mathrm{GL}_2(\mathbb{A}_{F,f})$ by $\gamma.(z,g) = (\gamma z, \gamma g)$. There is also a right action of $\mathrm{GL}_2(\mathbb{A}_{F,f})$ on $X^I \times \mathrm{GL}_2(\mathbb{A}_{F,f})$ by right multiplication on the second factor.

A function $X^I \times \mathrm{GL}_2(\mathbb{A}_{F,f}) \to \mathbb{C}$ is called *holomorphic* if it induces a locally constant map

$$\mathrm{GL}_2(\mathbb{A}_{F,f}) \xrightarrow{g \mapsto (z \mapsto f(z,g))} \mathrm{Hol}(X^I, \mathbb{C}).$$

There is a right $\mathrm{GL}_2(F)$-action and a left $\mathrm{GL}_2(\mathbb{A}_{F,f})$-action on $\mathrm{Hol}(X^I \times \mathrm{GL}_2(\mathbb{A}_{F,f}), \mathbb{C})$, which are defined as follows. Let $f \in \mathrm{Hol}(X^I \times \mathrm{GL}_2(\mathbb{A}_{F,f}), \mathbb{C})$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(F)$, $z \in X^I$ and $g \in \mathrm{GL}_2(\mathbb{A}_{F,f})$:

$$(\gamma^* f)(z,g) = \frac{\det(\gamma)^{\frac{w+k-2}{2}}}{(cz+d)^k} f(\gamma z, \gamma g) = \big(\prod_i \frac{\det(\gamma_i)^{\frac{w+k_i-2}{2}}}{(c_i z_i + d_i)^{k_i}}\big) f(\gamma z, \gamma g).$$

Let $g' \in \mathrm{GL}_2(\mathbb{A}_{F,f})$,

$$(g'_* f)(z,g) = f(z, gg').$$

For $K \subset \mathrm{GL}_2(\mathbb{A}_{F,f})$ open compact subgroup, the space of *adelic Hilbert modular form of multi-weight* (k) *on* K is

$$\mathcal{M}^{(k)}_{\mathbb{C},K} = \{f \in \mathrm{Hol}(X^I \times \mathrm{GL}_2(\mathbb{A}_{F,f}), \mathbb{C}) \mid \gamma^* f = f, g_* f = f \ \forall \gamma \in \mathrm{GL}_2(F) \ \forall g \in \mathrm{GL}_2(\mathbb{A}_{F,f})\}.$$

The union (direct limit) over all open compact K is denoted by $\mathcal{M}^{(k)}_{\mathbb{C}}$.

As explained in Hai's talk, such adelic Hilbert modular forms have a Fourier expansion, which, however, I do not intend to recall. Let us just say that for a Hecke eigenform (to be defined in a moment) the Fourier coefficients are (up to some normalisation factor) equal to Hecke eigenvalues. We let $S^{(k)}_{\mathbb{C},K}$ and $S^{(k)}_{\mathbb{C}}$ be the *cuspidal subspaces*, i.e. the subspaces where all 0-th Fourier coefficients vanish.

For the applications to Galois representations we introduce one special open compact subgroup for each integral ideal $\mathfrak{n} \subset \mathcal{O}_F$. Let $\hat{T} = \widehat{\mathcal{O}}_F \oplus D^{-1}\widehat{\mathcal{O}}_F$ be a lattice of $\mathbb{A}^2_{F,f}$. Let

$$K_1(\mathfrak{n}) = \{g \in \mathrm{GL}_2(\mathbb{A}_{F,f}) | g\hat{T} = \hat{T}, g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \mod \mathfrak{n}\hat{T}\}.$$

Hecke operators

Here we present two points of view on Hecke operators. Let $g \in \mathrm{GL}_2(\mathbb{A}_{F,f})$. To g we attach the operator T_g defined as follows:

$$S^{(k)}_{\mathbb{C},K} \xrightarrow{\text{res}} S^{(k)}_{\mathbb{C},g^{-1}Kg \cap K} \xrightarrow{g_*} S^{(k)}_{\mathbb{C},K \cap gKg^{-1}} \xrightarrow{\text{Tr}} S^{(k)}_{\mathbb{C},K},$$

where the *trace map* is given by $f \mapsto \sum_{h \in K/K \cap gKg^{-1}} h_* f$, supposing, of course, that K is such that this sum is finite. This description of Hecke operators is nice because it will be very similar to the description on Shimura curves to be given later on. But, there is also the equivalent double coset point of view: Let $T = T_g = KgK$ or let T be any other subset K-invariant from the left and the right. Then we have/put

$$T : f \mapsto \sum_{h \in T/K} h_* f.$$

We define two important types of Hecke operators:

- Let $\mathfrak{p}$ be a prime of F and $\pi_\mathfrak{p}$ a uniformiser of $(\mathcal{O}_F)_\mathfrak{p}$. Let

$$T_\mathfrak{p} := T_g \text{ with } g = \left(\begin{smallmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

- Let $\mathfrak{p}$ be a prime of F and $\pi_\mathfrak{p}$ a uniformiser of $(\mathcal{O}_F)_\mathfrak{p}$. Let

$$R_\mathfrak{p} := T_g \text{ with } g = \left(\begin{smallmatrix} \pi_\mathfrak{p} & 0 \\ 0 & \pi_\mathfrak{p} \end{smallmatrix}\right).$$

(It may be that one has to impose some conditions on K. But for sure, the definition is correct with $K_1(\mathfrak{n})$ and $(\mathfrak{p}, \mathfrak{n}) = 1$.)

Let $L \subset \mathbb{C}$ be a field containing the Galois closure of F over $\mathbb{Q}$.

Fact 1.2 *There are L-structures $S_{L,K}^{(k)}$ and $S_L^{(k)}$ in $S_{\mathbb{C},K}^{(k)}$ and $S_\mathbb{C}^{(k)}$, respectively.*

Moreover, each $S_{L,K}^{(k)}$ is a finite dimensional L-vector space.

Definition 1.3 *The* Hecke algebra *of $S_{L,K}^{(k)}$ is defined as*

$$\mathbb{T}_{L,K}^{(k)} = \langle T_\mathfrak{p}, R_\mathfrak{p} \in \mathrm{End}_L\big(S_{L,K}^{(k)}\big) | \mathfrak{p} \subset \mathcal{O}_F \rangle_{L\text{-algebra}} = \langle T_\mathfrak{p}, R_\mathfrak{p} \in \mathrm{End}_\mathbb{C}\big(S_{\mathbb{C},K}^{(k)}\big) | \mathfrak{p} \subset \mathcal{O}_F \rangle_{L\text{-algebra}}.$$

Fact 1.4 *The Hecke algebra $\mathbb{T}_{L,K}^{(k)}$ is a finite dimensional <u>commutative</u> L-algebra.*

Hence, there exist *Hecke eigenforms*, i.e. elements of $S_{\mathbb{C},K}^{(k)}$ that are eigenvectors for all elements of the Hecke algebra. Let f be a Hecke eigenform. The *system of eigenvalues attached to f* is described by the L-algebra homomorphism

$$\Theta_f : \mathbb{T}_{L,K}^{(k)} \to \mathbb{C}, \;\; T \mapsto \lambda_T,$$

where λ_T is the eigenvalue of T, i.e. $Tf = Tf = \lambda_T f$.

As already said above, if f is suitably normalised, the eigenvalue of $T_\mathfrak{p}$ is equal to the Fourier coefficient at $\mathfrak{p}$ (times the norm of $\mathfrak{p}$, according to Saito). But, we will not need Fourier coefficients here (not explicitly, at least).

We let $L(f) = \mathrm{im}(\Theta_f)$, the *coefficient field of f* (with respect to L). It is a finite extension of L due to the finite dimensionality of the Hecke algebra. In particular, if L is a number field (e.g. the Galois closure of F), then so is $L(f)$.

The automorphic representation attached to a Hilbert newform

We let π_f be the $\mathrm{GL}_2(\mathbb{A}_{F,f})$-orbit of f in $S_{\mathbb{C}}^{(k)}$ and call it the *automorphic* $\mathrm{GL}_2(\mathbb{A}_{F,f})$*-representation attached to* f.

Fact 1.5 *As* $\mathrm{GL}_2(\mathbb{A}_{F,f})$*-representations (over* $\mathbb{C}$*) we have an isomorphism*

$$S_{\mathbb{C}}^{(k)} \cong \bigoplus_{f \text{ newform}} \pi_f.$$

The term newform *here only refers to the fact that we do not distinguish between systems of eigenvalues such that the* Θ_f *differ only at finitely many* $\mathfrak{p}$.

Each π_f *can be defined over* $L(f)$*. To be precise, we sometimes write* $\pi_{f,L(f)}$*. Then we have by definition* $\pi_{f,L(f)} \otimes_{L(f)} \mathbb{C} = \pi_f$.

We do not need the following for the sequel but list it nevertheless: We call two Hecke eigenforms f_1 *and* f_2 Galois conjugate *if there exist embeddings* $\iota_i : L(f_i) \hookrightarrow \mathbb{C}$ *such that* $\iota_1(f_1) = \iota_2(f_2)$*, i.e.* $\iota_1 \circ \Theta_{f_1} = \iota_2 \circ \Theta_{f_2}$*. The* $\mathrm{GL}_2(\mathbb{A}_{F,f})$*-orbit of* $\langle \iota(f) | \iota : L(f) \hookrightarrow \mathbb{C}, \iota|_L = \mathrm{id} \rangle$ *defines an automorphic representation* $\pi_{L(f)}$ *over* L *such that*

$$\pi_{L(f)} \otimes_L \mathbb{C} = \prod_{\iota} \pi_{\iota(f)}$$

with the product running through the ι *as above. This yields an isomorphism of* $\mathrm{GL}_2(\mathbb{A}_{F,f})$*-representations (over* L*)*

$$S_L^{(k)} \cong \bigoplus_{f \text{ newform up to Galois conjugacy}} \pi_{L(f)}.$$

2 Main Theorem

We now come to Galois representations.

We fix algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ for all p and we consider F as a subfield of $\overline{\mathbb{Q}}$ (i.e. we fix an embedding) and $F_{\mathfrak{p}}$ as a subfield of $\overline{\mathbb{Q}}_p$ (also by fixing an embedding) for every prime $\mathfrak{p}$ of F. We choose embeddings

$$\iota_{\mathfrak{p}} : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$$

whose restriction to F is equal to

$$F \hookrightarrow F_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{Q}}_p$$

via the natural (resp. fixed) embeddings. Note that the choice of $\iota_{\mathfrak{p}}$ corresponds to the choice of a prime ideal for each finite extension $F \subseteq M \subset \overline{\mathbb{Q}}$ which are compatible with intersection. We also obtain an embedding of absolute Galois groups

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}}) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/F), \quad \sigma \mapsto \iota_{\mathfrak{p}}^{-1} \circ \sigma \circ \iota_{\mathfrak{p}}.$$

Note that this definition makes sense, since $\overline{\mathbb{Q}}/F$ is a normal extension. If we have two such embeddings $\iota_{\mathfrak{p}_1}$ and $\iota_{\mathfrak{p}_2}$, then the two embeddings of Galois groups are conjugate by $\iota_{\mathfrak{p}_1} \circ \iota_{\mathfrak{p}_2}^{-1}$.

Call $\mathbb{F}_\mathfrak{p}$ the residue field of $\mathfrak{p}$. We have the natural exact sequence

$$0 \to I_\mathfrak{p} \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p}) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_\mathfrak{p}) \to 0,$$

where $I_\mathfrak{p}$ is the inertia group at $\mathfrak{p}$. Of course, we suppose that $\mathfrak{p}$ divides the rational prime p. The right hand side map is the natural one. By $\mathrm{Frob}_\mathfrak{p}$ we denote the *arithmetic Frobenius element* in $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_\mathfrak{p})$, i.e. the one given by $x \mapsto x^q$ with $q = \#\mathbb{F}_\mathfrak{p}$. (There is always some confusion about geometric and arithmetic Frobenius elements. I prefer the latter.) We also denote by $\mathrm{Frob}_\mathfrak{p}$ any preimage in $\mathrm{Gal}_{\overline{\mathbb{Q}}_p/F_\mathfrak{p}}$, which is, of course, not well defined. So we have to handle it with care, but we will...

Theorem 2.1 *Let $f \in S^{(k)}_{\mathbb{C},K_1(\mathfrak{n})}$ be a newform corresponding to the L-algebra homomorphism $\Theta_f : \mathbb{T}^{(k)}_{L,K_1(\mathfrak{n})} \to \mathbb{C}$. Let Λ be a prime of $L(f)$. Then there is a* Galois representation, *i.e. a continuous group homomorphism*

$$\rho_{f,\Lambda} = \rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(L(f)_\Lambda)$$

which satisfies:

- *It is unramified outside $\mathfrak{n}\ell$ with $(\ell) = \Lambda \cap \mathbb{Z}$, i.e. $\rho(I_\mathfrak{p}) = 0$ for all $\mathfrak{p}$ with $(\mathfrak{p}, \mathfrak{n}\ell) = 1$; hence, $\rho(\mathrm{Frob}_\mathfrak{p})$ is well-defined for these $\mathfrak{p}$.*
- $\mathrm{Tr}(\rho(\mathrm{Frob}_\mathfrak{p})) = \Theta_f(T_\mathfrak{p})$ *for all* $(\mathfrak{p}, \mathfrak{n}\ell)$.
- $\det(\rho(\mathrm{Frob}_\mathfrak{p})) = \Theta_f(R_\mathfrak{p})\,\mathrm{Nm}(\mathfrak{p})$ *for all* $(\mathfrak{p}, \mathfrak{n}\ell)$.

This theorem is due to many people, in particular Carayol, Blasius, Rogawski and Taylor. I think it is proved in the above generality, but I have not checked it. In this lecture we will need the additional assumption (if $[F : \mathbb{Q}]$ is even) that π_f is *discrete series at v*. I won't explain what this means.

The theorem is, in fact, more precise. The restriction of ρ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})$ can be described at all places $\mathfrak{p}$, not only the unramified ones. This can be formulated in terms of Weil-Deligne representations (see the seminar a year ago). For the places above ℓ, this is the result proved in Saito's article.

3 Quaternionic automorphic forms and epresentations

Notation 3.1 (Second part)

- *B, the quaternion algebra (unique up to isomorphism) over F which is split at τ_1 and ramified at $\tau_2, \ldots, \tau_n$ (and v, if n is even). I.e. we have*

$$B \otimes_\mathbb{Q} \mathbb{R} = \mathrm{Mat}_2(\mathbb{R}) \times \underbrace{\mathbb{H} \times \cdots \times \mathbb{H}}_{n-1 \text{ copies}}$$

with $\mathbb{H}$ the Hamiltonian quaternion algebra.

- *$G = \mathrm{Res}_{F/\mathbb{Q}} B^\times$, the Weil restriction.*

- *$L/\mathbb{Q}$, a Galois number field containing F and splitting B, i.e.*

$$B \otimes_F L \cong \mathrm{Mat}_2(L).$$

Note that

$$G(\mathbb{A}_f) = \prod_p \prod_{\mathfrak{p}|p, \mathfrak{p} \neq v} \mathrm{GL}_2(F_\mathfrak{p}) \times (B \otimes_F F_v)^\times.$$

Recall the Shimura curve

$$M_K(\mathbb{C}) = G(\mathbb{Q}) \backslash \big(X \times G(\mathbb{A}_f)/K \big)$$

of level K. It has a model M_F over F. We let

$$M(\mathbb{C}) = G(\mathbb{Q}) \backslash \big(X \times G(\mathbb{A}_f) \big).$$

Kay defined several objects. We shall only list them here (maybe, we even slightly change them), but will not recall the precise definitions.

- $\mathcal{P}^{(k)}_{K,L}$, a constructible sheaf of L-vector spaces on $M_K(\mathbb{C})$.
- $\mathcal{P}^{(k)}_L$, a sheaf of L-vector spaces on $M(\mathbb{C})$.
- $\mathcal{P}^{(k)}_\lambda$, an étale sheaf of L_λ-vector spaces on M_F for some maximal ideal $\lambda \lhd \mathcal{O}_F$ such that

$$\mathrm{H}^i(M(\mathbb{C}), \mathcal{P}^{(k)}_L) \otimes_L L_\lambda \cong \mathrm{H}^i_{\text{et}}(M_F \times \overline{\mathbb{Q}}, \mathcal{P}^{(k)}_\lambda).$$

- $\mathcal{V}^{(k)}_K$, a locally free $\mathcal{O}_{M_K(\mathbb{C})}$-module of rank 1.
- $\mathcal{V}^{(k)}$, a locally free $\mathcal{O}_{M(\mathbb{C})}$-module of rank 1.
- Let $\mathcal{W}^{(k)}_K := \mathcal{V}^{(k)}_K \otimes_{\mathcal{O}_{M_K(\mathbb{C})}} \Omega^1_{M_K(\mathbb{C})}$.
- Let $\mathcal{W}^{(k)} := \mathcal{V}^{(k)} \otimes_{\mathcal{O}_{M(\mathbb{C})}} \Omega^1_{M(\mathbb{C})}$.

The principal result from Kay's talk is the following theorem.

Theorem 3.2 (Analog of Eichler-Shimura) *There is an isomorphism:*

$$\mathrm{H}^1(M(\mathbb{C}), \mathcal{P}^{(k)}_L) \otimes_L \mathbb{C} \cong \mathrm{H}^0(M(\mathbb{C}), \mathcal{W}^{(k)}) \oplus \overline{\mathrm{H}^0(M(\mathbb{C}), \mathcal{W}^{(k)})}.$$

A similar result holds at finite level K.

Definition 3.3 *We call*

$$S'^{(k)}_{\mathbb{C},K} = \mathrm{H}^0(M_K(\mathbb{C}), \mathcal{W}^{(k)}_K)$$

the space of quaternionic automorphic forms of level K and multi-weight (k). *Analogously, we let (taking $\varinjlim_K$)*

$$S'^{(k)}_{\mathbb{C}} = \mathrm{H}^0(M(\mathbb{C}), \mathcal{W}^{(k)}).$$

This definition is unsatisfactory, it should be made explict. But that is impossible during this talk, for time reasons (also time reasons during preparation...). I think/hope that one will find a description similar to that of adelic Hilbert modular forms.

Hecke operators

The definition is analogous to the one given for adelic Hilbert modular forms. Let $g \in G(\mathbb{A}_f)$. We start on the Shimura curve (on $\mathbb{C}$-points, but also on the model over F; the map g is right multiplication by g on the second factor):

$$M_K \leftarrow M_{K \cap g^{-1}Kg} \overset{g}{\leftarrow} M_{gKg^{-1} \cap K} \rightarrow M_K,$$

where the outer maps are the natural projections.

On the quaternionic automorphic forms and, more generally, on cohomology these maps induce an operator T_g, as follows:

$$\mathrm{H}^i(M_K, \cdot) \xrightarrow{\pi_*} \mathrm{H}^i(M_{K \cap g^{-1}Kg}, \cdot) \xrightarrow{g_*} \mathrm{H}^i(M_{gKg^{-1} \cap K}, \cdot) \xrightarrow{\pi^*} \mathrm{H}^i(M_K, \cdot).$$

Of course, the T_g also give maps on $\varinjlim_K$, i.e. a $G(\mathbb{A}_f)$-action, in particular on ${S'}_K^{(k)}$ and on the $\mathrm{H}^1(\cdot)$. The naturality of all maps in the above theorem makes the following theorem believable.

Theorem 3.4 *The map from Theorem 3.2 is compatible with the $G(\mathbb{A}_f)$-action.*

Let $\mathfrak{p} \neq v$. Since $B \otimes_F F_\mathfrak{p} = \mathrm{GL}_2(\mathbb{F}_p)$, it makes sense to define $T_\mathfrak{p}$ and $R_\mathfrak{p}$ as for Hilbert modular forms, i.e. as T_g for $g = \left(\begin{smallmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $g = \begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & \pi_\mathfrak{p} \end{pmatrix}$, respectively. However, we won't need them for the sequel.

Automorphic representation of $G(\mathbb{A}_f)$

An *automorphic representation of* $G(\mathbb{A}_f)$ is an irreducible constituent of ${S'}_\mathbb{C}^{(k)} = \mathrm{H}^0(M(\mathbb{C}), \mathcal{W}^{(k)})$.

4 The Jacquet-Langlands correspondence and hint on the proof

Theorem 4.1 (Jacquet-Langlands) *Let f be an adelic Hilbert newform and π_f the associated automorphic $\mathrm{GL}_2(\mathbb{A}_{F,f})$-representation such that $\pi_{f,v}$ is discrete series (if $[K : \mathbb{Q}]$ is even). Then there exists a unique automorphic representation π'_f of $G(\mathbb{A}_f)$ such that $\pi_{f,\mathfrak{p}} \cong \pi'_{f,\mathfrak{p}}$ as $\mathrm{GL}_2(F_\mathfrak{p})$-representations for all $\mathfrak{p} \neq v$. Moreover, π'_f has a model over $L(f)$, denoted by $\pi'_{f,L(f)}$ (as has π_f, see above).*

Theorem 4.2 (Multiplicity one) *There is an isomorphism of $G(\mathbb{A}_f)$-representations (over $\mathbb{C}$)*

$${S'}_\mathbb{C}^{(k)} = \mathrm{H}^0(M(\mathbb{C}), \mathcal{W}^{(k)}) \cong \bigoplus_{f \text{ newform, not discrete series at } v} \pi'_f.$$

Corollary 4.3 *There is an isomorphism of $G(\mathbb{A}_f)$-representations (over $\mathbb{C}$)*

$$\mathrm{H}^1(M(\mathbb{C}), \mathcal{P}_L^{(k)}) \otimes_L \mathbb{C} \cong \bigoplus_{f \text{ newform, not discrete series at } v} (\pi'_f \oplus \overline{\pi'_f}).$$

Hence, $\mathrm{H}^1(M(\mathbb{C}), \mathcal{P}_L^{(k)}) \otimes_L L(f)$ contains $\pi'_{f,L(f)}$ precisely twice, since $\pi'_{f,L(f)} = \overline{\pi'_{\overline{f},L(\overline{f})}}$ and all other constituents are non-isomorphic to $\pi'_{f,L(f)}$.

Corollary 4.4 *(a)* $\mathrm{Hom}_{G(\mathbb{A}_f)}\big(\pi'_{f,L(f)}, \mathrm{H}^1(M(\mathbb{C}), \mathcal{P}_L^{(k)}) \otimes_L L(f)\big) = L(f) \oplus L(f)$ *(since $\pi'_{f,L(f)}$ is absolutely irreducible).*

(b) Let Λ be a prime ideal of $L(f)$ dividing λ. Then

$$\mathrm{Hom}_{G(\mathbb{A}_f)}\big(\pi'_{f,L(f)} \otimes_{L(f)} L(f)_\Lambda, \mathrm{H}^1(M(\mathbb{C}), \mathcal{P}_L^{(k)}) \otimes_L L(f)_\Lambda\big) = L(f)_\Lambda \oplus L(f)_\Lambda.$$

Now we use the comparison from above (after tensoring with $L(f)_\Lambda$ over L_λ):

$$\mathrm{H}^i(M(\mathbb{C}), \mathcal{P}_L^{(k)}) \otimes_L L(f)_\Lambda \cong \mathrm{H}^i_{\text{et}}(M_F \times \overline{\mathbb{Q}}, \mathcal{P}_\lambda^{(k)}) \otimes_{L_\lambda} L(f)_\Lambda.$$

Corollary 4.5 *The 2-dimensional $L(f)_\Lambda$-vector space*

$$\mathrm{Hom}_{G(\mathbb{A}_f)}\big(\pi'_{f,L(f)} \otimes_{L(f)} L(f)_\Lambda, \mathrm{H}^1_{\text{et}}(M_F \times \overline{\mathbb{Q}}, \mathcal{P}_\lambda^{(k)}) \otimes_{L_\lambda} L(f)_\Lambda\big) = L(f)_\Lambda \oplus L(f)_\Lambda$$

carries a continuous linear $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-action.

This is the $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-representation that we are looking for. Unfortunately, we cannot check the claimed properties in this talk.