Exercises in Algebraic Number Theory

Winter term 2009/2010

Universität Duisburg-Essen **Sheet 0**
Institut für Experimentelle Mathematik
Prof. Dr. Gabor Wiese, Dr. Tommaso Centeleghe

These exercises are suggestions for the first exercise class and need not be handed in.

The aim of this exercise sheet is to recall some terminology and results from previous lectures. **If you do not know this terminology, then ask!** *We will then include it in the lecture.*

1. (a) When is a ring R called *Noetherian*, when *Artinian*?
   (b) Is $\mathbb{Z}$ Artinian as a $\mathbb{Z}$-module? Give a proof or a counterexample.
   (c) Is $\mathbb{Z}$ Noetherian as a $\mathbb{Z}$-module? Give a proof or a counterexample.
   (d) Is $\mathbb{Q}/\mathbb{Z}$ Artinian as a $\mathbb{Z}$-module? Give a proof or a counterexample.
   (e) Is $\mathbb{Q}/\mathbb{Z}$ Noetherian as a $\mathbb{Z}$-module? Give a proof or a counterexample.

2. Let R be a commutative ring and $S \subseteq R$ a multiplicatively closed subset not containing 0.

   (a) Describe the ring $S^{-1}R$, called the *ring of fractions of R with respect to S*.
   (b) Show that for a prime ideal $\mathfrak{P} \lhd R$, the set $S = R \setminus \mathfrak{P}$ is such a multiplicatively closed subset. We write $R_{\mathfrak{P}} = S^{-1}R$ and call it the *localisation of R at $\mathfrak{P}$*.
   (c) Describe how the set of ideals of $S^{-1}R$ corresponds to a subset of the ideals of R.
   (d) Prove: Is R Noetherian, then $S^{-1}R$ is Noetherian, too.

3. (a) Give the definition of the *Krull dimension* of a ring.
   (b) Compute the Krull dimension of any field.
   (c) Compute the Krull dimension of $\mathbb{Z}$.

4. Let K be a field. We consider the polynomial ring R over K in countably (infinitely) many variables, i.e. $R := K[X_1, X_2, X_3, \dots]$.

   (a) Is R a Noetherian ring? Give a proof or a counterexample.
   (b) Compute the Krull dimension of R.
   (c) Is R an integral domain? Give a proof or a counterexample.
   (d) Is R a factorial ring? Give a proof or a counterexample.

   Hint: Use well-known statements on polynomial rings in finitely many variables.

5. Let R be a commutative ring. If A_i for $i \in \mathbb{N}$ are R-modules, then we say that the sequence

   $$\cdots \to A_{i-1} \xrightarrow{\phi_{i-1}} A_i \xrightarrow{\phi_i} A_{i+1} \to \cdots$$

   is a *complex* if $\mathrm{im}(\phi_{i-1}) \subseteq \ker(\phi_i)$ for all i. It is called *exact* if $\mathrm{im}(\phi_{i-1}) = \ker(\phi_i)$ for all i. Furthermore, an exact sequence of R-modules of the form

   $$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{0.1}$$

   is called a *short exact sequence*. One says that it *splits* if there is an R-homomorphism $s : C \to B$ such that $\beta \circ s = \mathrm{id}_C$.

(a) Show: If the short exact sequence (0.1) splits, then there is an R-isomorphism $B \cong A \oplus C$.

(b) Let M be an R-module. An endomorphism $f \in \mathrm{End}_R(M)$ is called a *projection* if $f \circ f = f$ holds.

Show that the canonical exact sequence $0 \to \ker(f) \to M \to \mathrm{im}(f) \to 0$ splits. Thus, there is an R-isomorphism $M \cong \ker(f) \oplus \mathrm{im}(f)$.

(c) Let C in (0.1) be a projective R-module. Show that the sequence (0.1) splits.

6. Let R be a commutative ring.

(a) Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ and $0 \to C \xrightarrow{\gamma} D \xrightarrow{\delta} E \to 0$ be short exact sequences.

Prove: The sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\gamma\circ\beta} D \xrightarrow{\delta} E \to 0$ is exact.

(b) Conclude from (a) that for $k \geq 3$ every long exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to \cdots \to A_{k-1} \to A_k \to 0$$

of R-modules can be formed from $k-2$ short exact sequences.

Hint: Induction.

(c) Let $R = K$ be a field. Let V_i be finite dimensional K-vector spaces for $i = 1, \dots, k$. Let $0 \to V_1 \to V_2 \to \cdots \to V_{k-1} \to V_k \to 0$ be an exact sequence.

Prove: $0 = \sum_{i=1}^{k} (-1)^i \dim_K V_i$.

Hint: Induction.