Exercises in Algebraic Number Theory

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Sheet 11

To be handed in by: Friday, 15 January 2010, 2 pm.

1. (4 points) Let K be a field and $|\cdot|$ an absolute value on K.
   (a) Let $0 < s < 1$. Show that $|\cdot|^s$ is also an absolue value. Find a counterexample for $s > 1$. Show that if $|\cdot|$ is ultrametric, then $|\cdot|^s$ is an absolue value for all $s > 0$.
   (b) Let $M \subseteq K$ be a subfield and assume that $|\cdot|$ is trivial on M. Show that $|\cdot|$ is trivial on K if K/M is an algebraic field extension.

2. (4 points) Let K be a field and $K(T) = \mathrm{Frac}(K[T])$ be the function field over K in one variable. Let ρ be any element of $\mathbb{R}$ satisfying $0 < \rho < 1$.
   (1) Let $p(T) \in K[T]$ be an irreducible nonconstant polynomial. Let $v_{p(T)}$ be the $p(T)$-valuation on $K(T)$. Recall that $v_{p(T)}(\frac{g(T)}{h(T)})$ with $g, h \in K[T]$ is defined as $v_{p(T)}(g(T)) - v_{p(T)}(h(T))$, where $v_{p(T)}(r(T))$ for $r \in K[T]$ is the multiplicity with which $p(T)$ divides $r(T)$.
   For $f \in K(T)$ define
   $$|f|_{p(T)} := (\rho^{\deg(p(T))})^{v_{p(T)}(f(T))}.$$
   (2) For $f(T) = \frac{g(T)}{h(T)} \in K(T)$ define
   $$|f|_\infty := \rho^{(\deg(h(T)) - \deg(g(T)))}.$$

   Show that $|\cdot|_{p(T)}$ and $|\cdot|_\infty$ define ultrametric absolute values on $K(T)$.

3. (4 points) Compute with a precision of 4 places:
   (a) $0.2643 \cdot 0.1234 \in \mathbb{Q}_7$,
   (b) $1/0.1234 \in \mathbb{Q}_5$,
   (c) $0.05 - 2.345 \in \mathbb{Q}_{11}$.

4. (4 points) Show that the p-adic expansion of any $x \in \mathbb{Q}$ becomes periodic.