Exercises in Algebraic Number Theory

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Sheet 12

To be handed in by: Friday, 22 January 2010, 2 pm.

1. (4 points) Show that $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x| \leq 1\}$ is a discrete valuation ring. Consequently, it is integrally closed in its field of fractions $\mathbb{Q}_p$.
2. (4 points) Compute in $\mathbb{Q}_7$ with a precision of at least 4 places the following numbers: $-\frac{1}{6}$, $\sqrt{-3}$, $\sqrt[3]{-1}$.
3. (4 points)

   (a) The p-adic Newton method. Let $f \in \mathbb{Z}_p[X]$ be a polynomial and $a \in \mathbb{Z}_p$ such that $|f(a)|_p < 1$ and $|\frac{f(a)}{(f'(a))^2}|_p = \epsilon < 1$. Define a sequence $(a_n)_n$ by

   $$a_0 := a, \quad a_{n+1} := a_n - \frac{f(a)}{f'(a)}.$$

   Show that the sequence $(a_n)_n$ converges to some $x \in \mathbb{Z}_p$ with $f(x) = 0$ and

   $$|x - a_n|_p \leq \epsilon^{2^n}.$$

   Hint: Show by induction the following assertions:

   - $|f(a_n)/f'(a_n)|_p \leq \epsilon^{2^n} |f'(a)|_p$,
   - $|f(a_n)|_p \leq \epsilon^{2^n} (|f'(a)|_p)^2$,
   - $|f(a_n)|_p = |f'(a)|_p$.

   (b) Show that $\mathbb{Z}_p$ contains the group of $(p-1)$th roots of unity.

4. (4 points) Let K be a field and $p(T)$ a nonconstant irreducible polynomial in $K[T]$. In Exercise 2, Sheet 11, the $p(T)$-absolute value $|\cdot|_{p(T)}$ on $K(T)$ was defined. Put $L := K[X]/(p(T))$. Show that the completion of $K(T)$ with respect to $|\cdot|_{p(T)}$ is equal to $L((T))$, the field of formal power series over L.