Exercises in Algebraic Number Theory

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Sheet 13

To be handed in by: Friday, 29 January 2010, 2 pm.

1. (4 points) Let p be a prime number. Show that the multiplicative group $\mathbb{Q}_p^\times = \mathbb{Q}_p \setminus \{0\}$ is equal to the direct product $\mu_{p-1} \times (1 + p\mathbb{Z}_p) \times p^{\mathbb{Z}}$, where μ_{p-1} is the group of $(p-1)$th roots of unity.
2. (4 points) Let p be a prime number. Consider the quotient group $G := \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2}$, where $\mathbb{Q}_p^{\times 2}$ denotes the subgroup of squares.

   Show that G is an elementary abelian 2-group, i.e. that G is of the form $\underbrace{C_2 \times C_2 \times \cdots \times C_2}_{r-\text{times}}$. Moreover, show that $r = 2$ if $p > 2$ and that $r = 3$ if $p = 2$.
3. (4 points) Let p be a prime number and let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. For $f \in \mathbb{N}$, let ζ_f be a primitive $(p^f - 1)$th root of unity in $\overline{\mathbb{Q}}_p$.

   (a) Show that the series $\sum_{f=1}^{\infty} \zeta_f p^f$ does not converge in $\overline{\mathbb{Q}}_p$.

   (b) Show that $\overline{\mathbb{Q}}_p$ is not complete.
4. (4 points) In the lecture we proved the following corollary of Hensel's lemma (Corollary 14.4): Let K be a complete ultrametric field with valuation ring $\mathcal{O}$, valuation ideal $\mathfrak{P}$ and residue field $\mathbb{F} := \mathcal{O}/\mathfrak{P}$ and let $f \in \mathcal{O}[X]$ be a monic polynomial and $\overline{f} \in \mathbb{F}[X]$ its coefficient-wise reduction. If $\alpha \in \mathbb{F}$ is a simple zero of $\overline{f}$, then there is $\beta \in \mathcal{O}$ with $f(\beta) = 0$ and $\alpha = \beta + \mathfrak{P} \in \mathbb{F}$.

   (a) Show by giving a counterexample that the assumption 'simple' is necessary.

   (b) Show by giving a counterexample that the assumption 'complete' is necessary.