Exercises in Algebraic Number Theory

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Sheet 2

To be handed in by: Friday, 30 October 2009, 2 pm.

1. (4 points) Let R be a commutative ring and let $S_1, \dots, S_n$ be integral ring extensions of R.
   (a) Prove that $\prod_{i=1}^n S_i$ is an integral ring extension of R.
   (b) Let $T \subseteq R$ be a subring such that $R \setminus T$ is multiplicatively closed. Prove that T is integrally closed in R.
2. (4 points) Let K be a field. A subring $R \subseteq K$ is called a *valuation ring* of K if for each $x \in K^\times$ we have $x \in R$ or $x^{-1} \in R$.
   (a) Show that every valuation ring of K is a local ring.
   (b) Show that any valuation ring of K is integrally closed.
3. (4 points) Show that the discriminant d_K of any number field $K/\mathbb{Q}$ is always congruent to 0 or 1 modulo 4.

   Hint: Use the Leibniz formula for computing the determinant of the matrix $(\sigma_i(\alpha_j))_{i,j}$, i.e. (in the notation of the lecture)
   $$\sum_{\tau \in S_n} \operatorname{sgn}(\tau) \prod_{i=1}^n \sigma_i(\alpha_{\tau(i)}),$$
   and divide it up into the sum A over the even permutations minus the sum B over the odd permutations. Show next that $A+B$ and AB are rational integers. Conclude from this.
4. (4 points) Let R be an integral domain whose field of fractions $K := \operatorname{Frac}(R)$ is a number field. Prove that the ideal quotient of fractional ideals satisfies the following properties:
   $$H : (IJ) = (H : I) : J, \quad (\bigcap_k I_k) : J = \bigcap_k (I_k : J), \quad I : (\sum_k J_k) = \bigcap_k (I : J_k).$$