Exercises in Algebraic Number Theory

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Sheet 8

To be handed in by: Friday, 11 December 2009, 2 pm.

1. (4 points) Describe the quadratic subfields of $\mathbb{Q}(\zeta_n)$ for $n \geq 3$.
2. (4 points)
   (a) Let $n \in \mathbb{N}$. Conclude from Sheet 5, Exercise 4 (b), that there are infinitely many primes p that are congruent to 1 modulo n.
   (b) Let A be a finite abelian group. Show that there is a Galois extension $K/\mathbb{Q}$ with Galois group isomorphic to A.
3. (4 points) Prove that a subgroup $L \subset \mathbb{R}^n$ is of the form $L \cong \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \oplus \cdots \oplus \mathbb{Z}\omega_r$ for some $\omega_i \in \mathbb{R}^n$ if and only if it is discrete. For $n = 1$ show that L is either of the form $\mathbb{Z}\omega$ for some $\omega \in \mathbb{R}$ or a dense subgroup.
4. (4 points) Consider the lattice $L = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_r \subset \mathbb{R}^n$. Then the following conditions are equivalent:
   (i) L has maximal rank (i.e. $r = n$).
   (ii) The quotient group $\mathbb{R}^n/L$ is compact in the quotient topology.
   (iii) There exists a bounded subset $B \subset \mathbb{R}^n$ such that $L + B = \mathbb{R}^n$.