Exercises in Algebraic Number Theory

Winter term 2009/2010

Universität Duisburg-Essen
Institut für Experimentelle Mathematik
Prof. Dr. Gabor Wiese, Dr. Tommaso Centeleghe

Sheet 9

To be handed in by: Friday, 18 December 2009, 2 pm.

1. (4 points) [Minkowski's theorem on linear forms] Suppose that n linear forms on $\mathbb{R}^n$ with real coefficients are explicitly given by $L_i(x) = \sum_{j=1}^n a_{i,j}x_j$ for $x = (x_1, \dots, x_n)$ such that the matrix $A = (a_{i,j})_{i,j}$ has a nonzero determinant. Let $c_1, \dots, c_n$ be positive real numbers with the property $\prod_{i=1}^n c_i > |\det(A)|$.

   Show that there exists a nonzero $x \in \mathbb{Z}^n$ such that

   $$|L_i(x)| < c_i \quad \text{for all } i = 1, \dots, n.$$

2. (4 points) Determine the class number of $\mathbb{Q}(\sqrt{d})$ for $d = -41, -47, -163$.
3. (4 points) Show that all real quadratic fields of discriminant $d < 40$ have class number 1. What is the class number of $\mathbb{Q}(\sqrt{10})$?
4. (4 points) Let K be a number field and $R \subset K$ a subring of K that is a Dedekind ring. Show that there exists a set of primes S of $\mathcal{O} := \mathcal{O}_K$ (the ring of integers of K) such that

   $$R = \bigcap_{\mathfrak{p} \notin S} \mathcal{O}_{\mathfrak{p}} = \{x \in K \mid \operatorname{ord}_{\mathfrak{p}}(x) \geq 0 \ \forall\, \mathfrak{p} \notin S\}.$$

   [This exercise only needs the tools from Sections 3 and 4 of the lecture.]