A LOCAL LORENTZIAN FERRAND-OBATA THEOREM FOR CONFORMAL VECTOR FIELDS

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ABSTRACT. For a conformal vector field on a closed, real-analytic, Lorentzian manifold we prove that the flow is locally isometric—that it preserves a metric in the conformal class on a neighborhood of any point—or the metric is everywhere conformally flat. The main theorem can be viewed as a local version of the Lorentzian Lichnerowicz conjecture in the real-analytic setting. The key result is an optimal improvement of the local normal forms for conformal vector fields of [FM13], which focused on non-linearizable singularities. This article is primarily concerned with essential linearizable singularities, and the proofs include global arguments which rely on the compactness assumption.

1. INTRODUCTION

The Ferrand-Obata Theorem on conformal groups of Riemannian manifolds is paradigmatic in the Zimmer-Gromov program, which aims to classify compact or finite-volume manifolds with rigid geometric structures admitting large group of automorphisms. This theorem was conjectured by Lichnerowicz and proved independently by J. Ferrand and M. Obata; it characterizes closed Riemannian manifolds with non-compact conformal transformation group (see Theorem 1.1 below). In contrast, the Lorentzian version of the Lichnerowicz conjecture is still open, although significant progress has been made in certain cases.

In this article, we establish a local version of the Lorentzian Lichnerowicz conjecture in the real analytic category. Our results shed light on the behavior around singularities of conformal vector fields and also constitute a maximality result with respect to conformal embeddings, having as a consequence that certain noncompact Lorentzian manifolds cannot be conformally compactified; thus they link local analysis with global geometric rigidity.

1.1. The Lorentzian Lichnerowicz conjecture.

For a pseudo-Riemannian metric g on a manifold, a subgroup G of its conformal group $\mathrm{Conf}(M, g)$ is called *essential* if it does not preserve any pseudo-Riemannian metric $g' = e^{2\lambda} g$ in the conformal class of g. In the case of a Riemannian metric on a compact manifold, such a group G is essential if and only if it is non-compact. In this case it was proved independently and simultaneously by J. Ferrand and M. Obata :

Theorem 1.1 (Ferrand [LF71, Fer76], Obata [Oba71])**.** *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$. If $\mathrm{Conf}(M, g)$ is essential then (M, g) is conformally diffeomorphic to the round sphere $\mathbf{S}^n$.*

Date: November 6, 2025.
2020 *Mathematics Subject Classification.* 53B30, 53C24, 53A30.
Key words and phrases. Lorentz metric, conformal geometry, Lorentz Lichenrowicz conjecture, Killing field.
We are grateful to the organizers of the IHP trimester Group Actions and Rigidity: Around the Zimmer Program, to the IHP itself (UAR 839 CNRS-Sorbonne Université), and to LabEx CARMIN (ANR-10-LABX-59-01), for providing the stimulating environment in which this work was initiated. This work has also been supported by the French government, through the UCA$^{\mathrm{JEDI}}$ Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR-15-IDEX-01, by the ANR grant IsoMoDyn ANR-25-CE40-1360-03 and by the Luxembourg National Research Fund (FNR), grant reference O24/18936913/RiGA. We wish to expressly thank Thomas Leistner for helpful conversations and comments.

This theorem positively answers a question asked by A. Lichnerowicz.

A similar question in the pseudo-Riemannian framework was asked subsequently in [DG91, Section 6.2]. In higher signature, essentiality of the conformal group does not determine the topology of the manifold. In fact, in the Lorentzian case, the second author proved that, in any dimension $n \geq 3$, there are infinitely many topological types of compact manifolds bearing infinitely many non-conformally equivalent Lorentzian metrics admitting an essential conformal flow [Fra05]. Nevertheless, all these Lorentzian metrics are *conformally flat*, meaning they are locally conformal equivalent with flat Minkowski space. The analogue of the original Lichnerowicz Conjecture is thus:

Lorentzian Lichnerowicz Conjecture: *Let (M, g) be a compact Lorentzian manifold of dimension $n \geq 3$. If* $\mathrm{Conf}(M, g)$ *is essential then (M, g) is conformally flat.*

Important results which support the above Lorentzian Lichnerowicz Conjecture were obtained by the second and the third authors for real-analytic three-dimensional manifolds [FM23] and by the third and the fourth authors for simply connected real-analytic manifolds [MP22a]. The fourth author also proved the Lorentzian Lichnerowicz Conjecture in the case where $\mathrm{Conf}(M, g)$ contains an essential connected simple Lie subgroup [Pec17, Pec18]. This last result is a step towards the classification of conformal groups of compact Lorentz manifolds, in the vein of the classification of their isometry groups settled in [AS97a, AS97b, Zeg98]. The recent preprint [Meh25] relates to the Lorentzian Lichnerowicz conjecture for locally homogeneous Lorentzian manifolds, and proves that compact quotients of conformally homogeneous Lorentzian spaces have essential conformal group if and only if they are flat.

In higher signature the analogous pseudo-Riemannian Lichnerowicz Conjecture fails: non conformally flat pseudo-Riemannian metrics of signature (p, q), with $p, q \geq 2$, admitting essential conformal flows, were constructed by the second author on Hopf manifolds [Fra15]. The pseudo-Riemannian manifolds constructed in [Fra15] come from non conformally flat polynomial deformations g of the (p, q)-Minkowski metric, with Hopf-type compact quotients of $(\mathbf{R}^{p+q} \setminus \{0\}, g)$, diffeomorphic to $\mathbf{S}^1 \times \mathbf{S}^{p+q-1}$, which admit essential conformal flows.

1.2. **Lichnerowicz conjecture for conformal vector fields: Statement of the main theorem.** Several notions of essentiality can be investigated. It is for instance common to assume that the identity component $\mathrm{Conf}(M, g)^0$ is essential. Obata's version of the Riemannian Lichnerowicz conjecture is proved under this assumption. See [Fer99] for an account of the history. Another relevant notion is that of *strong essentiality*, which is the absence of an invariant measure of full support. It is also natural to assume that some individual conformal transformation or conformal one-parameter group of transformations is essential, a notion *a priori* stronger than the essentiality of $\mathrm{Conf}(M, g)^0$.

A conformal vector field Y on a semi-Riemannian manifold (M, g) is *inessential* if there exists a metric in the conformal class $[g]$ for which Y is a Killing field. A specific, but still challenging, case of the conjecture is the following.

Lorentzian Lichnerowicz Conjecture for conformal vector fields *Let (M, g) be a compact Lorentz manifold of dimension $n \geq 3$. If (M, g) is not conformally flat, then any conformal vector field on M is inessential.*

In this statement, it is crucial to make the global assumption that M is a compact manifold. For instance, [Ale85] builds a family of non-conformally flat, real-analytic, Lorentzian metrics on $\mathbf{R}^n$ conformally invariant by a linear flow which is locally essential at every singularity. Thus, the conjecture asserts in particular that such local essential phenomena cannot be compactified, that is, conformally and equivariantly embedded into a closed Lorentzian manifold.

We will not prove this conjecture in its full generality, but rather establish a version under a stronger hypothesis. A conformal vector field Y is *locally inessential* if each point $p \in M$ admits a neighborhood U on which Y is a Killing field for some metric in the conformal class $[g|_U]$. A singularity of Y is *locally essential* if it admits a fundamental system of neighborhoods on which Y is always essential. Our main result proved is the following local version of the Lorenztian Lichnerowicz Conjecture, in the real-analytic category:

Theorem 1.2. *Let (M,g) be a closed, real-analytic Lorentzian manifold of dimension ≥ 3. If (M,g) is not conformally flat, then any conformal vector field on M is locally inessential.*

Although the conclusion of the theorem as stated is local, the assumption that M is a compact manifold is needed because of the examples in [Ale85] mentioned above. Theorem 1.2 proves, in particular, that non conformally flat examples in [Ale85] do not admit real-analytic conformal compactifications. In [MP22b] the third and fourth authors proved this under the additional assumption that M is simply connected.

Given a conformal vector field Y on a Lorentzian manifold (M,g), and given a point $p \in M$, the *conformal distortion* of Y at p is the real number λ such that $(L_Y g)_p = 2\lambda g_p$. When p is a singularity of Y, the distortion at p is the same for every metric in the conformal class. This is a consequence of the formula $L_Y(fg) = (Y.f)g + fL_Y g$ for every smooth function f. Then Y has *nontrivial conformal distortion* at a singularity p if $\lambda \neq 0$. Observe that a conformal vector field with nontrivial conformal distortion at a singular point p cannot preserve any metric in the conformal class, even in a neighborhood of p. Theorem 1.2 has as a corollary:

Corollary 1.3. *Let (M,g) be a closed, real-analytic Lorentzian manifold of dimension ≥ 3, and let Y be a conformal vector field on M. If a singular point of Y has nontrivial conformal distortion, then (M,g) is conformally flat.*

1.3. **Global versus local essentiality.** By Theorem 1.1, Riemannian essential conformal vector fields always have singularities. In fact, as can be seen by examining the singularities of non-compact flows by Möbius transformations on $\mathbf{S}^n$, the notions of essentiality and local essentiality coincide for Riemannian conformal vector fields.

This property no longer holds in Lorentzian signature. Consider the following example, the Lorentzian Hopf manifold. Take $\mathbf{R}^3 \setminus \{0\}$ endowed with the metric $\tilde{g}_x := g_0/|x|^2$, where g_0 is the Minkowski metric $2dx_1dx_3 + dx_2^2$, and $|\cdot|$ denotes the Euclidean norm. Let h be the homothetic transformation $x \mapsto 2x$, and let Γ be the discrete group generated by h. The quotient of $\mathbf{R}^3 \setminus \{0\}$ by Γ is a smooth manifold M, diffeomorphic to $\mathbf{S}^1 \times \mathbf{S}^2$, endowed with the Lorentzian quotient metric, denoted g. The one-parameter group of homotheties of $\mathbf{R}^3$ descends to an isometric $\mathbf{S}^1$-action on (M,g); denote it by $\{k^t\}$. The linear $O(1,2)$ action descends to a conformal action on (M,g).

Consider a unipotent one-parameter subgroup $\{\tilde{u}^t\} < O(1,2)$. It induces a one-parameter subgroup $\{u^t\} <$ $\mathrm{Conf}(M,g)$, generated by a conformal vector field U. The isotropic line of fixed points of u^t projects onto two closed null geodesics Δ_i, $i = 1,2$, of (M,g), which are fixed pointwise by $\{u^t\}$. Given $p \in \Delta_i$ and an evenly covered neighborhood of p, it is clear that the conformal distortion of U at p is null. Alternatively, the metric g_0 descends to a Lorentzian metric in the conformal class, well-defined in a neighborhood of p, which is $\{u^t\}$-invariant on this neighborhood. In particular—see also Lemma 2.2 below—the flow $\{u^t\}$ is locally inessential at p.

However, $\{u^t\}$ is an essential conformal one-parameter subgroup of $\mathrm{Conf}(M,g)$. Indeed, all orbits of the flow spiral around either Δ_1 or Δ_2. If g' were a conformal u^t-invariant metric, then for any $p \in M$, the values

$$g'_p(U,U) = g'_{u^t(p)}(U,U) \to 0,$$

contradicting the fact that U is spacelike on an open-dense subset. Thus $\{u^t\}$ is an essential conformal flow of (M,g) all of whose singularities are locally inessential. Therefore local essentiality is stronger than essentiality.

The commutative product $\varphi^t = k^t u^t$ admits no singularity on M. It is another essential one-parameter group: indeed, if the convex k^t-invariant subset $\mathcal{I} = \{g' \in [g] \mid (\varphi^t)^* g' = g'\}$ were non-empty, averaging over $\{k^t\}$ would yield a metric in the conformal class which is both k^t and φ^t invariant, therefore u^t-invariant:

a contradiction. Hence, even in dimension 3, a Lorentzian conformal vector field can be everywhere non-singular and globally essential.

2. Outline of the proof of Theorem 1.2

2.1. **Linearizability of conformal vector fields.** It is well known that, in the neighborhood of a singular point, any Killing field is linear in exponential coordinates. This linearizability property, however, generally fails for conformal vector fields. Although some techniques are available to study such fields near a singularity, there is currently no classification of their possible local normal forms. The situation becomes much simpler when one deals with real-analytic structures. In this case, non-linearizability occurs only in the conformally flat case (and then Lie theory provides the list of possible normal forms). More precisely:

Theorem 2.1 ([FM13], Thm 1.2)**.** *Let (M, g) be a real-analytic Lorentzian manifold of dimension at least 3. Let X be a local conformal vector field admitting a singularity x. If (M, g) is not conformally flat, then X is linearizable in a neighborhood of x.*

This result will be key in proving Theorem 1.2, since under the assumptions of that theorem we may always assume that conformal vector fields are linearizable around their singularities.

2.2. **Taxonomy of singularities for a linearizable conformal vector field.** Our setting here is that of a local conformal vector field Y on a Lorentzian manifold (M, g). We assume that $p \in M$ is a singularity of Y, and that Y is linearizable in a neighborhood of p. In what follows, we denote by $\{\varphi_Y^t\}$ the local flow generated by Y (we do not assume completeness of Y, so $\{\varphi_Y^t\}$ may be only a local flow). The linearized flow $\{D_p\varphi_Y^t\}$ can be written as a product $\{e^{at}A^t\}$, where $a \in \mathbf{R}$ is the conformal distorsion of Y, and $\{A^t\}$ is a one-parameter group of $\mathrm{O}(1, n-1)$. Such one-parameter groups fall into three categories (see e.g. [Rat19], [Mat92]):

1. *Elliptic flows.* Up to conjugacy, these are one-parameter groups in the compact subgroup $\mathrm{O}(n-1) \subset \mathrm{O}(1, n-1)$.
2. *Parabolic flows.* They have exactly one fixed point on the boundary at infinity of real hyperbolic space $\mathbf{H}^{n-2}$, and can be written as a commuting product $\{K^tU^t\}$, where $\{K^t\}$ and $\{U^t\}$ are two one-parameter groups, respectively elliptic and unipotent.
3. *Loxodromic flows.* They have exactly two fixed points on the boundary at infinity of real hyperbolic space $\mathbf{H}^{n-2}$, and can be written as a commuting product $\{K^tD^t\}$, where $\{K^t\}$ and $\{D^t\}$ are two one-parameter groups, respectively elliptic and hyperbolic, namely nontrivial and $\mathbf{R}$-split in $\mathrm{O}(1, n-1)$.

We may then classify the different singularities of Y into the following categories:

Isometry-like singularities. These are singularities p where the conformal distortion of Y vanishes. The flow $\{D_p\varphi_Y^t\}$ belongs to $\mathrm{O}(1, n-1)$.

Contracting/expanding singularities. Singularities p for which $\lim_{t\to+\infty} |D_p\varphi^t| = 0$ (resp. $\lim_{t\to-\infty} |D_p\varphi^t| = 0$) are called *contracting* (resp. *expanding*).

For instance, this category includes all flows of the form $\{D_p\varphi^t\} = \{e^{-at}K^tU^t\}$ with $a \neq 0$, where K^t and U^t are commuting elliptic and parabolic flows, respectively (possibly trivial).

For flows of the form $\{D_p\varphi^t\} = \{e^{-at}L^t\}$ with $a \neq 0$ and L^t loxodromic, we may write in a suitable frame of T_pM:

$$D_p\varphi^t = \begin{pmatrix} e^{(a+b)t} & & \\ & e^{at}R^t & \\ & & e^{(a-b)t} \end{pmatrix}, \tag{1}$$

with $b \geq 0$, and $\{R^t\}$ a one-parameter group in $\mathrm{O}(n-2)$. Being contracting (expanding) is then equivalent to the condition $a < 0$ and $|b| < a$ (resp. $a > 0$ and $|b| < a$).

Observe that when p is a contracting singularity, the local flow $\{\varphi_Y^t\}$ is defined for all $t \geq 0$ on a neighborhood U of p. Moreover, for any compact subset $K \subset U$, we have $\lim_{t \to +\infty} \varphi_Y^t(K) = p$ in the Hausdorff topology.

Mixed singularities. This is the case where $\{D_p \varphi_Y^t\}$ has the form (1), and $|b| > |a| > 0$ in (1). Replacing Y by $-Y$ if necessary, the local dynamics of φ_Y^t in our linearizable setting exhibit a one-dimensional unstable manifold and a codimension-one strongly stable manifold.

Balanced singularities. This is the case $|b| = |a| \neq 0$ in (1). Looking at the linearized flow, we see that $\{\varphi_Y^t\}$, admits locally a null geodesic segment of fixed points together with a codimension-one strongly stable manifold.

Nonsingular points. These are points $p \in M$ for which $Y_p \neq 0$.

2.3. **Local inessentiality at regular points.** A conformal vector field is always inessential in a neighborhood of a nonsingular point $p \in M$. By the flow-box theorem, there exist coordinates $(t, x^1, \ldots, x^{n-1})$ on a neighborhood U of p in which Y is simply $\frac{\partial}{\partial t}$. We then define a Lorentzian metric h on U as follows:

$$h(t,x)(u,v) := g(0,x)\big(D_{(t,x)}\varphi^{-t}(u), D_{(t,x)}\varphi^{-t}(v)\big).$$

The metric h belongs to the conformal class $[g|_U]$, and by construction it is invariant under the local flow of Y.

2.4. **Local inessentiality at isometry-like singular points.** We state below a general result showing that linearizable conformal vector fields are locally homothetic. It is likely that this observation already appears in the literature, but we are not aware of any reference.

Lemma 2.2. *Let (M, g) be a pseudo-Riemannian manifold, and Y a conformal vector field with conformal distortion a at a singular point p. Assume that Y is linearizable in a neighborhood U of p. Then there exists a pseudo-Riemannian metric h in the conformal class $[g|_U]$ for which Y is homothetic with distortion a, namely $L_Y h = 2ah$.*

Under the hypotheses of Theorem 1.2, and because of the linearization result proved in Theorem 2.1, Lemma 2.2 implies, in particular, that in the neighborhood of any isometry-like singular point (corresponding to the case of zero conformal distortion in the lemma), a conformal vector field is inessential.

Proof. Since Y is a conformal vector field, we have $L_Y(e^{-2\sigma} g) = 0$ and therefore $L_Y g = 2(Y.\sigma)g$, for some smooth function $\sigma : M \to \mathbf{R}$. The metric g defines a volume form ω_g, for which $L_Y \omega_g = n(Y.\sigma)\omega_g$. On the neighborhood U where Y is linearizable, there exists another volume form ω_0, defined by the "Minkowski metric in linearized coordinates," for which $L_Y \omega_0 = na\omega_0$, with a being the distortion at the singular point p. By replacing ω_0 with $-\omega_0$ if necessary, we may write $\omega_g = e^{n\lambda}\omega_0$ on U, for some smooth $\lambda : U \to \mathbf{R}$. It follows that

$$L_Y \omega_g = (n(Y.\lambda) + na)e^{n\lambda}\omega_0 = (n(Y.\lambda) + na)\omega_g.$$

Together with $L_Y \omega_g = n(Y.\sigma)\omega_g$, this yields

$$Y.\sigma - Y.\lambda = a.$$

This means precisely that on the open set U

$$L_Y(e^{-2\lambda} g) = 2ae^{-2\lambda} g.$$

In other words, Y is a homothetic vector field for the metric $h := e^{-2\lambda} g$, with conformal distortion a. $\square$

2.5. **Existence of contracting/expanding singularities implies conformal flatness.** Here, the limit is understood with respect to the Hausdorff topology. The theorem below shows that such dynamical behavior forces conformal flatness (hence it does not occur under the hypotheses of Theorem 1.2).

Theorem 2.3 ([Fra12a], Thm 1.3 (2))**.** *Let (M, g) be a Lorentzian manifold. Assume that (f_k) in $\mathrm{Conf}^{loc}(M, g)$, is a sequence of local conformal transformations, all defined a same open set. If there exists an open set U, and a point x_0 such that $f_k(U) \to x_0$ in the Hausdorff topology, then U is conformally flat. In particular, if (M, g) is real-analytic, it is conformally flat.*

When Y is a local conformal vector field admitting a contracting/expanding singularity p, we may first turn Y into $-Y$ to make p contracting. Then, we already observed that there exists an open set U containing p such that $\lim_{t\to+\infty} \varphi_Y^t(U) = p$. We then get the

Corollary 2.4. *Let (M, g) be a real-analytic Lorentzian manifold. Assume that there exists on M a local conformal vector field with a contracting/expanding singularity. Then (M, g) is conformally flat.*

2.6. **Mixed and balanced singularities.** The discussion above shows that, given a real-analytic Lorentzian manifold (M, g) which is not conformally flat, and a conformal vector field Y on M, the field Y is locally inessential in a neighborhood of any point p which is either nonsingular for Y or an isometry-like singularity. Theorem 2.1 and Corollary 2.4 show that contracting/expanding singularities do not occur under those hypotheses.

To prove Theorem 1.2, it remains to consider singularities of mixed or balanced type. We shall show that under the additional assumption that M is compact, such singularities cannot occur under the hypotheses of the theorem. More precisely, a compact manifold M admitting a conformal vector field Y with such singularities must necessarily be conformally flat. This will be established in the forthcoming theorems. The proofs of these results are more subtle, as they rely on global dynamical arguments related to the compactness of M.

The case of a mixed-type singularity will be treated in Section 4. It will essentially be shown that the presence of a mixed-type singularity on a compact real-analytic Lorentzian manifold necessarily implies the existence of a contracting/expanding singularity. By Corollary 2.4, this in turn forces conformal flatness.

The case of a balanced-type singularity is more delicate and will occupy a large part of this article. It was proved in [MP22b, Prop 4.1] that existence of a balanced-type singularity implies conformal flatness or existence of a real-analytic codimension-one foliation. As in that proof, we show in Section 5 that in a neighborhood of a balanced-type singularity, there exists a gravitational pp-wave metric in the conformal class. Combined with the assumption of real-analyticity, this leads to what is called a *polarization* of the conformal structure. In Section 6, global dynamical arguments will then show that a second polarization must appear, which ultimately implies conformal flatness and completes the proof of Theorem 1.2.

3. Conformal Cartan connections and curvature

3.1. **Cartan's viewpoint on conformal structures.** The *normal Cartan connection* associated with a conformal Lorentzian structure will play an important role in our proofs. We recall below some basic facts about this perspective.

3.1.1. *Equivalence problem for Lorentzian conformal structures.* Let $\mathbf{R}^{2,n}$ denote $\mathbf{R}^{n+2}$ with standard basis $\{e_0, \ldots, e_{n+1}\}$, equipped with the quadratic form

$$Q_{2,n}(x) = 2x_0x_{n+1} + 2x_1x_n + x_2^2 + \cdots + x_{n-1}^2.$$

The Lorentzian Einstein Universe, denoted $\mathbf{Ein}^{1,n-1}$, is the projectivization of the null cone

$$\mathcal{N}^{2,n} \setminus \{0\} = \{\mathbf{x} \in \mathbf{R}^{n+2} \setminus \{0\} \mid Q_{2,n}(\mathbf{x}) = 0\}.$$

It is a smooth quadric hypersurface of $\mathbf{RP}^{n+1}$ that naturally inherits a conformal class $[g_{1,n-1}]$ of Lorentzian signature from the ambient quadratic form on $\mathbf{R}^{2,n}$. It is diffeomorphic to $\mathbf{S}^1 \times_\iota \mathbf{S}^{n-1}$, where ι is inversion on both factors. By construction, there is a transitive conformal action of the group $G = \mathrm{PO}(2,n)$ on $\mathbf{Ein}^{1,n-1}$, and in fact $\mathrm{Conf}(\mathbf{Ein}^{1,n-1}, [g_{1,n-1}]) \cong G$.

Thus, $\mathbf{Ein}^{1,n-1}$ is a compact, conformally homogeneous space, identified with G/P, where P is the parabolic subgroup of G stabilizing an isotropic line in $\mathbf{R}^{2,n}$. It has been known since É. Cartan that in dimension $n \geq 3$, each conformal Lorentzian structure defines a unique Cartan geometry infinitesimally modeled on $\mathbf{Ein}^{1,n-1}$ in the following sense.

Theorem 3.1 (É. Cartan; see [Sha96], Ch. V, and [ČS09], Sec. 1.6)**.** *Let M be a connected manifold of dimension $n \geq 3$. A conformal Lorentzian structure $[g]$ on M canonically determines:*

- *a principal P-bundle $\pi : \widehat{M} \to M$; and*
- *a regular, normal Cartan connection, i.e., a one-form $\omega : \widehat{M} \to \mathfrak{g}$ satisfying, for all $\hat{x} \in \widehat{M}$,*
  (1) $\omega_{\hat{x}} : T_{\hat{x}}\widehat{M} \xrightarrow{\sim} \mathfrak{g}$ is a linear isomorphism;
  (2) $(R_p)^*\omega = \mathrm{Ad}(p^{-1}) \circ \omega$ for all $p \in P$;
  (3) $\omega\big(\frac{d}{dt}(\hat{x} \cdot e^{tY})\big) \equiv Y$ for all $Y \in \mathfrak{p}$.

Regularity and normality are technical conditions on the curvature of ω (see Section 3.1.2 below) ensuring uniqueness. For their detailed definitions, refer to [Sha96] and [ČS09].

It is useful to clarify the meaning of Theorem 3.1 in the case of the conformal structure on the model space $\mathbf{Ein}^{1,n-1}$. As seen above, $\mathbf{Ein}^{1,n-1}$ can be identified with the homogeneous space G/P, where $G = \mathrm{PO}(2,n)$. In this situation, the Cartan bundle is simply the group G itself, which naturally fibers over G/P. The regular, normal connection associated with the conformal structure is, in this case, nothing but the Maurer–Cartan connection ω^G.

Let us now explain how to recover the conformal structure $[g]$ from the pair $(\widehat{M}, \omega)$. Let $x \in M$ and $\hat{x} \in \widehat{M}_x$. For any $u \in T_xM$ and any $\hat{u} \in T_{\hat{x}}\widehat{M}$ such that $\pi_*(\hat{u}) = u$, define $\iota_{\hat{x}}(u)$ to be the projection of $\omega_{\hat{x}}(\hat{u})$ to $\mathfrak{g}/\mathfrak{p}$. This yields a well-defined linear isomorphism $\iota_{\hat{x}} : T_xM \to \mathfrak{g}/\mathfrak{p}$.

Equivariance of ω with respect to the right P-action on $\widehat{M}$ (see condition (2) in Theorem 3.1) implies that for every $\hat{x} \in \widehat{M}$ and every $p \in P$:

$$\iota_{\hat{x}\cdot p} = \mathrm{Ad}(p^{-1}) \circ \iota_{\hat{x}}. \tag{2}$$

Let $\mathbb{I}_{2,n}$ denote the inner product determined by the quadratic form $Q_{2,n}$, and let $\mathbb{I}_{1,n-1}$, or simply $\mathbb{I}$, be its restriction to the Minkowski subspace $e_0^\perp \cap e_{n+1}^\perp$.

The Lie algebra $\mathfrak{g} = \mathfrak{so}(2,n)$ can be parametrized as follows:

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & \xi & 0 \\ v & X & -\mathbb{I}^t\xi \\ 0 & -^t v\mathbb{I} & -a \end{pmatrix},\ a \in \mathbf{R},\ v \in \mathbf{R}^n,\ \xi \in \mathbf{R}^n,\ X \in \mathfrak{so}(\mathbf{R}^n, \mathbb{I}) \cong \mathfrak{so}(1,n-1) \right\}.$$

The *grading decomposition* $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ is given by the subalgebras parametrized by v, (a, X), and ξ, respectively; note that $\mathfrak{p} = \mathfrak{g}_0 \ltimes \mathfrak{g}_{+1}$. The subgroup $G_0 < P$ with Lie algebra $\mathfrak{g}_0$ preserves a unique similarity class of Lorentzian scalar products on $\mathfrak{g}/\mathfrak{p}$. If $\langle\, ,\, \rangle$ is one of them, then for every $x \in M$ and $\hat{x}$ in the fiber over x:

$$[g_x] = [\langle \iota_{\hat{x}}(.), \iota_{\hat{x}}(.)\rangle]. \tag{3}$$

3.1.2. *Conformal curvature.* Let $[g]$ be a Lorentzian conformal structure on a manifold M, and let ω denote the normal Cartan connection associated with $[g]$ (see Theorem 3.1). The curvature of ω, denoted K, is the $\mathfrak{g}$-valued two-form on $\widehat{M}$ defined by

$$K(X,Y) = d\omega(X,Y) + [\omega(X), \omega(Y)], \tag{4}$$

for every pair X, Y of vector fields on $\widehat{M}$. Property (3) of ω implies that K vanishes when one of its arguments is vertical (i.e., tangent to the fibers of $\pi : \widehat{M} \to M$). Moreover, the normalization condition imposed on ω implies that the two-form K actually takes values in $\mathfrak{p}$ (torsion-freeness condition).

It is often convenient to view K as a function

$$\kappa : \widehat{M} \to \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}.$$

Let P act on the module $\wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$ by $(p.\gamma)(u,v) = \mathrm{Ad}(p)\gamma(\mathrm{Ad}(p^{-1})u, \mathrm{Ad}(p^{-1})v)$ for all $u, v \in \mathfrak{g}/\mathfrak{p}$. Then the curvature function κ satisfies the natural equivariance property:

$$\kappa_{\hat{x}\cdot p} = p^{-1}.\kappa_{\hat{x}},$$

for every $\hat{x} \in \widehat{M}$ and $p \in P$.

Vanishing of the Cartan curvature on an open subset $U \subset M$ is equivalent to the conformal flatness of U. We can make the link between κ and the Weyl $(3,1)$-tensor W associated with the conformal class $[g]$ more explicit. If κ^0 denotes the projection of κ onto the $\mathfrak{g}_0$ factor of $\mathfrak{p} \cong \mathfrak{g}_0 \ltimes \mathfrak{g}_{+1}$, then for every $x \in M$ and $u, v, w \in T_xM$:

$$W_x(u,v,w) = [\kappa^0_{\hat{x}}(\iota_{\hat{x}}(u), \iota_{\hat{x}}(v)), \iota_{\hat{x}}(w)].$$

Thanks to the equivariance properties of ι and κ, this expression is independent of the choice of $\hat{x}$ in the fiber over x.

3.1.3. *Lifting conformal transformations and conformal vector fields.* Conformal transformations of M lift naturally and uniquely to bundle automorphisms of $\widehat{M}$ leaving ω invariant; conformal vector fields similarly lift to P-invariant vector fields on $\widehat{M}$ whose Lie derivative annihilates ω. Conversely, any bundle automorphism of $\widehat{M}$ preserving ω (resp. any vector field on $\widehat{M}$ preserved by the P-action and preserving ω) induces a conformal diffeomorphism of M (resp. a conformal vector field on M). The lifted action of $\mathrm{Conf}(M,[g])$ preserves the parallelization of $\widehat{M}$ determined by ω (see property (1) of ω in Theorem 3.1) and is therefore free. Similarly, lifts of conformal vector fields to $\widehat{M}$ are nonvanishing, showing that such lifts are entirely determined by their value at a point.

Another way to understand this fact is the following. Let X be a conformal vector field on M, and let us lift it to $\widehat{M}$ as explained above. For any vector field Z on $\widehat{M}$, the relation $L_X\omega(Z) = 0$ reads

$$X.\omega(Z) - \omega([X,Z]) = 0.$$

Together with the definition of the Cartan curvature:

$$K(X,Z) = d\omega(X,Z) + [\omega(X), \omega(Z)] = X.\omega(Z) - Z.\omega(X) - \omega([X,Z]) + [\omega(X), \omega(Z)],$$

we obtain

$$Z.\omega(X) = [\omega(X), \omega(Z)] - K(X,Z). \tag{5}$$

If $t \mapsto \hat{\alpha}(t)$ is any smooth path in $\widehat{M}$, and if we put $\xi(t) := \omega(X(\hat{\alpha}(t)))$, then equation (5) leads to the following first-order linear ODE for ξ:

$$\xi'(t) = [\xi(t), \omega(\hat{\alpha}'(t))] - \kappa(\xi(t), \omega(\hat{\alpha}'(t))). \tag{6}$$

We call this the *Killing transport equation* along $\hat{\alpha}$.

3.1.4. *Conformal exponential map.* Any $Z \in \mathfrak{g}$ naturally defines a vector field $\hat{Z}$ on $\widehat{M}$ by the relation $\omega(\hat{Z}) \equiv Z$. Let $\{\varphi^t_{\hat{Z}}\}$ denote the local flow on $\widehat{M}$ generated by $\hat{Z}$. At each $\hat{x} \in \widehat{M}$, define $\mathcal{W}_{\hat{x}} \subset \mathfrak{g}$ as the set of $Z \in \mathfrak{g}$ such that φ^t_Z is defined for $t \in [0,1]$ at $\hat{x}$. Then the exponential map at $\hat{x}$ is

$$\exp(\hat{x}, \cdot) : \mathcal{W}_{\hat{x}} \to \widehat{M}, \qquad \exp(\hat{x}, Z) = \varphi^1_{\hat{Z}} \cdot \hat{x}.$$

It is standard that $\mathcal{W}_{\hat{x}}$ is a neighborhood of 0, and the map $\xi \mapsto \exp(\hat{x}, \xi)$ determines a diffeomorphism from an open set $\mathcal{V}_{\hat{x}} \subset \mathcal{W}_{\hat{x}}$ containing 0 onto a neighborhood of $\hat{x}$ in $\widehat{M}$. Moreover, the map $\pi \circ \exp_{\hat{x}}$ induces a diffeomorphism from a neighborhood of 0 in $\mathfrak{g}_{-1}$ to a neighborhood of $x = \pi(\hat{x})$ in M.

Let f be a conformal transformation of M. Then $f_*(\hat{Z}) = \hat{Z}$, and if $p \in P$, $(R_p)_*(\hat{Z}) = \hat{Z}_p$, where $Z_p = (\operatorname{Ad} p^{-1})Z$. This implies the important equivariance property:

$$f(\exp(\hat{x}, \xi)) \cdot p^{-1} = \exp(f(\hat{x}) \cdot p^{-1}, (\operatorname{Ad} p)\xi). \tag{7}$$

3.1.5. *Development of curves and null geodesic segments.* Recall that a *pregeodesic* of a Lorentzian metric g is a parametrized curve $\gamma : I \to M$ satisfying a differential equation of the form

$$\frac{D}{dt}\dot{\gamma}(t) = f(t)\dot{\gamma}(t),$$

for some smooth function $f : I \to \mathbf{R}$. After a suitable reparametrization, a pregeodesic which is an immersion becomes an affinely parametrized geodesic.

A remarkable fact of Lorentzian conformal geometry is that all metrics within the same conformal class $[g]$ *admit the same null pregeodesics*. In this case, we say that $\gamma(I)$ is a *null geodesic segment*. This property can be verified directly by computation, but the normal Cartan connection associated with a conformal structure provides a more conceptual approach, which we briefly present below.

The key point is that the Cartan connection defines a *development map*, which associates to any curve traced on M a curve in the model space $\mathbf{Ein}^{1,n-1}$. Recall from Subsection 3.1.1 that $\mathbf{Ein}^{1,n-1}$ can be viewed as the homogeneous space $\mathrm{PO}(2,n)/P$, that its Cartan bundle is simply given by the projection $\pi_G : \mathrm{PO}(2,n) \to \mathbf{Ein}^{1,n-1}$, and that the normal connection associated with the conformal structure of $\mathbf{Ein}^{1,n-1}$ is the Maurer–Cartan form ω^G on $\mathrm{PO}(2,n)$.

Let $\gamma : I \to M$ be a smooth curve, and consider a lift $\hat{\gamma} : I \to \widehat{M}$. We denote by $\hat{\gamma}_*$ the unique curve in $\mathrm{PO}(2,n)$ satisfying the differential equation

$$\omega^G(\hat{\gamma}_*'(t)) = \omega(\hat{\gamma}'(t)),$$

with initial condition $\hat{\gamma}_*(0) = 1_G$. Observe that locally, the ODE defining $\hat{\gamma}_*$ is linear, so that this curve is defined on I. We set $\gamma_*(t) := \pi_G(\hat{\gamma}_*(t))$, and call γ_* the *developed curve* of γ. One checks that if $\hat{\beta} : I \to \widehat{M}$ is another lift of the same curve γ, then there exists $p \in P$ such that $\beta_* = p \cdot \gamma_*$. Thus, any P-invariant family $\mathcal{F}$ of curves on $\mathbf{Ein}^{1,n-1}$ defines a distinguished class of curves on M: namely, those curves whose developments lie in representatives of $\mathcal{F}$.

A *photon* of the Einstein universe $\mathbf{Ein}^{1,n-1}$ is defined as the projection onto $\mathbf{Ein}^{1,n-1}$ of a totally isotropic two-plane in $\mathbf{R}^{2,n}$. It turns out that the null pregeodesics are precisely those curves whose developments parametrize portions of photons. This is the content of the following statement, which also shows the conformal invariance of null pregeodesics.

Theorem 3.2 ([Fr14], Thm. 5.3.3)**.** *A curve $\gamma : I \to M$ parametrizes a null geodesic segment if and only if its development $\gamma_* : I \to \mathbf{Ein}^{1,n-1}$ parametrizes a photon.*

As an example of application of Theorem 3.2, let us pick $\hat{x}$ in the Cartan bundle $\widehat{M}$. Let $u \in \mathfrak{g}_{-1}$ be such that $\langle u, u\rangle = 0$. Then, $s \mapsto \pi \circ \exp(\hat{x}, su)$ is an immersive parametrization of a null geodesic segment on M. We shall make implicit use of this fact several times in what follows.

3.2. **Algebraic features of $\mathfrak{so}(2,n)$.** We recall here further algebraic information about the Lie algebra $\mathfrak{so}(2,n)$, which will be used extensively in our proofs below. Recall the description of $\mathfrak{g} = \mathfrak{so}(2,n)$ given in Section 3.1.1:

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & \xi & 0 \\ v & X & -\mathbb{I}^t\xi \\ 0 & -{}^tv\mathbb{I} & -a \end{pmatrix},\ a \in \mathbf{R},\ v \in \mathbf{R}^n,\ \xi \in \mathbf{R}^n,\ X \in \mathfrak{so}(\mathbf{R}^n, \mathbb{I}) \cong \mathfrak{so}(1, n-1) \right\}.$$

Similarly, the Lie algebra $\mathfrak{so}(1, n-1)$ can be decomposed as:

$$\mathfrak{so}(1, n-1) = \left\{ \begin{pmatrix} b & U_+ & 0 \\ U_- & R & -{}^tU_+ \\ 0 & -{}^tU_- & -b \end{pmatrix},\ b \in \mathbf{R},\ U_- \in \mathbf{R}^{n-2},\ U_+ \in (\mathbf{R}^{n-2})^*,\ R \in \mathfrak{so}(n-2) \right\}.$$

An $\mathbf{R}$-split Cartan subalgebra in $\mathfrak{g}$ is

$$\mathfrak{a} = \left\{ \begin{pmatrix} a & & & & \\ & b & & & \\ & & \mathbf{0} & & \\ & & & -b & \\ & & & & -a \end{pmatrix},\ a, b \in \mathbf{R} \right\}, \qquad \mathbf{0} = (0, \ldots, 0)\ n-2 \text{ times.} \tag{8}$$

Let $\alpha, \beta \in \mathfrak{a}^*$ be defined by $\alpha(a, b) = a$ and $\beta(a, b) = b$. The following diagram represents the full restricted root-space decomposition of $\mathfrak{so}(2, n)$ with respect to $\mathfrak{a}$:

$$\begin{pmatrix} \mathfrak{a} & \mathfrak{g}_{\alpha-\beta} & \mathfrak{g}_\alpha & \mathfrak{g}_{\alpha+\beta} & 0 \\ \mathfrak{g}_{\beta-\alpha} & \mathfrak{a} & \mathfrak{g}_\beta & 0 & \mathfrak{g}_{\alpha+\beta} \\ \mathfrak{g}_{-\alpha} & \mathfrak{g}_{-\beta} & \mathfrak{m} & \mathfrak{g}_\beta & \mathfrak{g}_\alpha \\ \mathfrak{g}_{-\alpha-\beta} & 0 & \mathfrak{g}_{-\beta} & \mathfrak{a} & \mathfrak{g}_{\alpha-\beta} \\ 0 & \mathfrak{g}_{-\alpha-\beta} & \mathfrak{g}_{-\alpha} & \mathfrak{g}_{\beta-\alpha} & \mathfrak{a} \end{pmatrix} \qquad \begin{array}{l} \mathfrak{m} \cong \mathfrak{so}(n-2) \\ \dim \mathfrak{g}_\beta = n-2 = \dim \mathfrak{g}_\alpha \\ \dim \mathfrak{g}_{\alpha-\beta} = 1 = \dim \mathfrak{g}_{\alpha+\beta} \end{array} \tag{9}$$

The factor $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in a maximal compact subalgebra of $\mathfrak{so}(2, n)$ and is parametrized by R in the decomposition of $\mathfrak{so}(1, n-1)$. In accordance with this decomposition, we denote the root space $\mathfrak{g}_{-\beta}$ (resp. $\mathfrak{g}_\beta$) by $\mathfrak{u}^-$ (resp. $\mathfrak{u}^+$). The corresponding unipotent subgroups of G are U^- and U^+.

For later use, we fix a basis $E_1, \ldots, E_n$ of $\mathfrak{g}_{-1}$ such that $E_1 \in \mathfrak{g}_{-\alpha+\beta}$, $E_n \in \mathfrak{g}_{-\alpha-\beta}$, and $E_i \in \mathfrak{g}_{-\alpha}$ for $2 \leq i \leq n-1$. We also impose the normalization $\langle E_1, E_n \rangle = 1$ and $\langle E_i, E_j \rangle = \delta_{ij}$ for $2 \leq i, j \leq n-1$.

3.3. **Gromov stratification for real-analytic structures.** Following [Gro88], a conformal class $[g]$ in dimension ≥ 3 is a rigid geometric structure. In analytic regularity, it follows that the orbits of local conformal vector fields are the fibers of a certain surjective analytic map onto a stratified analytic space. The same is true if instead of the orbits of all local conformal vector fields, we consider only those that centralize a given global analytic vector field Y. This amounts to say that the enhanced geometric structure $[g] \cup \{Y\}$ is rigid and analytic and therefore the stratification result applies to it. We will denote by $\mathrm{Kill}_Y^{loc}(x)$ the Lie algebra of local conformal vector fields defined in a neighborhood of x and that centralize Y. In the context of pseudo-Riemannian geometry, below are important consequences of Gromov's stratification theorem.

Theorem 3.3 ([Gro88], §3). *Let $(M, [g])$ be a closed manifold of dimension ≥ 3 endowed with an analytic pseudo-Riemannian conformal class. Let Y be an analytic vector field defined on M. Then,*

1. *For all $x \in M$, the Kill_Y^{loc}-orbit $\mathcal{O}(x)$ of x is a locally closed, analytic submanifold of M.*
2. *For every x, the closure $\overline{\mathcal{O}(x)}$ is semi-analytic and locally connected.*
3. *For all $y \in \overline{\mathcal{O}(x)} \setminus \mathcal{O}(x)$, $\dim \mathcal{O}(y) < \dim \mathcal{O}(x)$.*

Remark 3.4. *These statements can also be deduced from several analogous results formulated within the framework of Cartan geometries. If we were considering orbits of all local conformal vector fields, the stratification result is proved in* [Mel11, Section 4]. *A Frobenius theorem, the key part in this theory, is also proved in* [Pec16] *for enhanced Cartan geometries, which include geometric structures such as $[g] \cup \{Y\}$. The setting is in $\mathcal{C}^\infty$ regularity and the conclusions are valid over an open-dense subset. However, the method* *of* [Mel11] *are easily adaptable to this context, and yield a Frobenius theorem for compact, real-analytic, enhanced Cartan geometries, valid on entire manifold, which is exactly what is needed here.*

3.4. **Algebraic group structure on isotropy subgroups.** Another useful consequence of Frobenius theorem is the following proposition. We formulate it using the formalism of Cartan geometries, but it is originally stated in [Gro88, §3] for rigid geometric structures.

Let $(M, [g])$ be a manifold of dimension ≥ 3 endowed with a pseudo-Riemannian conformal structure. For $x \in M$, let Conf_x^{loc} denote the group of germs at x of local conformal diffeomorphisms fixing x. Let $(M, \hat{M}, \omega)$ denote the normalized Cartan geometry modeled on (G, P) associated to $[g]$, and let $\hat{x} \in \hat{M}_x$. For all $f \in \mathrm{Conf}_x^{loc}$, there is a unique $p \in P$ such that $\hat{f}(\hat{x}).p^{-1} = \hat{x}$. The correspondence $\{f \in \mathrm{Conf}_x^{loc} \mapsto p \in P\}$ identifies Conf_x^{loc} with a subgroup $P^{\hat{x}} < P$.

Proposition 3.5. *In analytic regularity,* $\mathrm{Ad}_{\mathfrak{g}}(P^{\hat{x}})$ *is an algebraic subgroup of* $\mathrm{Ad}_{\mathfrak{g}}(P)$*.*

Proof. Let $\Phi : \hat{M} \to W$ be the map as defined in [Mel11, Section 4]. By Proposition 3.8 of the same article, $P^{\hat{x}}$ coincides with the stabilizer in P of $\Phi(\hat{x}) \in W$. The result follows since $\mathrm{Ad}_{\mathfrak{g}}(P)$ acts algebraically on W. □

4. Linear mixed singularity implies conformal flatness

Having gathered the necessary elements concerning the Cartan-geometric viewpoint on conformal structures, we can now pursue the program outlined in Section 2.6, with the aim of proving Theorem 1.2. In particular, we shall study the consequences of the existence of a mixed-type singularity for the conformal vector field Y, and establish the following:

Proposition 4.1. *Let* (M, g) *be a closed, real-analytic Lorentzian manifold. Let* Y *be a conformal vector field on* M*. If* Y *admits a singularity of mixed type, then* (M, g) *is conformally flat.*

4.1. **A general result about zeros of conformal vector fields.** The idea behind the proof of Proposition 4.1 is to show that the existence of a mixed-type singularity necessarily entails the existence of contracting/expanding singularities, which in turn implies conformal flatness. Underlying this property is a more general phenomenon, which we wish to isolate in the form of the following proposition:

Proposition 4.2. *Let* (M, g) *be a smooth Lorentzian manifold. Let* Y *be a conformal vector field. Let* $\gamma : (-\epsilon, 1+\epsilon) \to M$*, with* $\epsilon > 0$*, be a smooth immersion parametrizing a null geodesic segment. We assume that* Y *vanishes at* $\gamma(0)$ *and* $\gamma(1)$*, and that* $Y(\gamma(t))$ *is collinear with* $\gamma'(t)$ *for all* t*. If* $\gamma(0)$ *is a singularity of mixed type for* Y*, then there exists* $0 < t_0 \leq 1$ *such that* $\gamma(t_0)$ *is a contracting/expanding singularity of* Y*.*

Proof. The proof will use intensively the notions and notations introduced in Sections 3.1 and 3.2.

We work in the Cartan bundle $\widehat{M}$, endowed with its normal Cartan connection ω, and we lift Y to a vector field $\hat{Y}$ on $\widehat{M}$ satisfying $L_{\hat{Y}}\omega = 0$. By hypothesis, $\gamma(0)$ is a singularity of mixed type for Y, and in particular it is isolated. Hence there exists $0 < t_0 \leq 1$ such that Y vanishes at $\gamma(t_0)$, but $Y(\gamma(t)) \neq 0$ for $t \in (0, t_0)$. We now lift γ to an immersion $\hat{\gamma} : (-\epsilon, 1+\epsilon) \to \widehat{M}$. Because $\gamma(0)$ is a singularity of mixed type, we may choose $\hat{\gamma}$ such that

$$\omega(\hat{Y}(\hat{\gamma}(0))) = \mathrm{diag}(a, b, R, -b, -a) =: D,$$

with $|b| > |a| > 0$ and $R \in \mathfrak{o}(n-2)$.

Let us denote $\xi(t) := \omega(\hat{Y}(\hat{\gamma}(t)))$. Our hypothesis that Y is collinear with γ' means that

$$\xi(t) = \lambda(t)\, \omega(\hat{\gamma}'(t)) + v(t),$$

where $\lambda : (-\epsilon, 1+\epsilon) \to \mathbf{R}$ is a function and $v(t) \in \mathfrak{p}$ for all $t \in (-\epsilon, 1+\epsilon)$. Recall the Killing transport equation established in Section 3.1.3:

$$\xi'(t) = [\xi(t), \omega(\hat{\gamma}'(t))] - \kappa(\xi(t), \omega(\hat{\gamma}'(t))). \tag{10}$$

Since κ is antisymmetric and vanishes as soon as one of its arguments belongs to $\mathfrak{p}$, we are left with

$$\xi'(t) = [\xi(t), \omega(\hat{\gamma}'(t))]. \tag{11}$$

We now consider the analogous picture in the flat model. Namely, we identify $\mathbf{Ein}^{1,n-1}$ with the homogeneous space $\mathrm{PO}(2,n)/P$, and look at the bundle $\pi_G : \mathrm{PO}(2,n) \to \mathbf{Ein}^{1,n-1}$. As already mentioned in Section 3.1, the normal Cartan connection for $\mathbf{Ein}^{1,n-1}$ is simply the Maurer–Cartan form ω^G on $\mathrm{PO}(2,n)$. Let $\hat{\gamma}_*$ denote the development of $\hat{\gamma}$ (see Section 3.1.5), i.e. the unique curve in $\mathrm{PO}(2,n)$ satisfying

$$\omega^G(\hat{\gamma}_*'(t)) = \omega(\hat{\gamma}'(t)) \quad \text{for all } t \in (-\epsilon, 1+\epsilon),$$

with $\hat{\gamma}_*(0) = 1_G$. Since γ is assumed to be an immersion, the vector $\hat{\gamma}'(t)$ is never vertical. The same is therefore true for $\hat{\gamma}_*'(t)$, and $\gamma_* := \pi_G \circ \hat{\gamma}_*$ is itself an immersion. Theorem 3.2 ensures that $t \mapsto \gamma_*(t)$ parametrizes a segment of a photon $\Delta \subset \mathbf{Ein}^{1,n-1}$. This photon contains $\gamma_*(0) = [e_0]$.

Let us now consider on $\mathrm{PO}(2,n)$ the right-invariant vector field $\hat{Y}_*$ such that $\hat{Y}_*(1_G) = D$. Because $L_{\hat{Y}_*}\omega^G = 0$, the function

$$\xi_*(t) := \omega^G(Y_*(\hat{\gamma}_*(t)))$$

satisfies equation (10). But ω^G has zero Cartan curvature, so equation (10) reduces to

$$\xi_*'(t) = [\xi_*(t), \omega^G(\hat{\gamma}_*'(t))] = [\xi_*(t), \omega(\hat{\gamma}'(t))].$$

In other words, ξ and ξ_* satisfy the very same first-order ODE and take the same value at $t = 0$, so $\xi(t) = \xi_*(t)$ for all $t \in (-\epsilon, 1+\epsilon)$. Let us call Y_* the the projection of $\hat{Y}$ on $\mathbf{Ein}^{1,n-1}$. Then $Y_*(\gamma_*(t))$ is tangent to $\gamma_*'(t)$ for all $t \in (-\epsilon, 1+\epsilon)$, and $Y_*(\gamma_*(t)) = 0$ if and only if $Y(\gamma(t)) = 0$. In particular $Y_*(\gamma_*(t)) \neq 0$ for $t \in (0, t_0)$, and $Y_*(\gamma_*(t_0)) = 0$. To determine the nature of the singularity $\gamma(t_0)$, we want to understand the P-conjugacy class of $\xi(t_0)$, which amounts to determining that of $\xi_*(t_0)$.

To do this, we first observe that there are only two photons containing $[e_0]$ which are invariant under the matrix D, namely $\Delta_1 = [\mathrm{span}(e_0, e_1)]$ and $\Delta_n = [\mathrm{span}(e_0, e_n)]$. Hence Δ must be equal to either Δ_1 or Δ_n. Let us treat the case $\Delta = \Delta_1$ (the other case is analogous). The vector field Y_* has exactly two singularities on Δ_1, namely $[e_0]$ and $[e_1]$, so $\gamma_*(t_0)$ is either $[e_0]$ or $[e_1]$. But since $Y_*(\gamma_*(t)) \neq 0$ for $t \in (0, t_0)$ and γ_* is an immersion, we must have $\gamma_*(t_0) = [e_1]$.

Let r_1 be the element of $\mathrm{PO}(2,n)$ that switches e_0 and e_1, as well as e_n and e_{n+1}, and fixes the other basis vectors $e_2, \ldots, e_{n-1}$. Then $\pi(r_1) = [e_1]$, hence there exists $p_1 \in P$ such that $\gamma_*(t_0) \cdot p_1 = r_1$. As a consequence,

$$\omega^G(\hat{Y}_*(\hat{\gamma}_*(t_0))) = \mathrm{Ad}(p_1)\, \omega^G(\hat{Y}_*(r_1)).$$

By left-invariance of ω^G, we get

$$\omega^G(\hat{Y}_*(r_1)) = \mathrm{Ad}(r_1^{-1}) \cdot D = \mathrm{diag}(b, a, R, -a, -b).$$

Hence

$$\omega(\hat{Y}(\gamma(t_0))) = \mathrm{Ad}(p_1) \cdot \mathrm{diag}(b, a, R, -a, -b).$$

Since $\mathrm{diag}(b, a, R, -a, -b) \in \mathfrak{g}_0$, the field Y is linearizable at $\gamma(t_0)$. Because $|b| > |a|$, the singularity $\gamma(t_0)$ is contracting/expanding, and Proposition 4.2 is proved.

When $\Delta = \Delta_n$, we proceed along the same lines, replacing r_1 with r_n, the element of $\mathrm{PO}(2,n)$ that switches e_0 and e_n, e_1 and e_{n+1}, and leaves $e_2, \ldots, e_{n-1}$ fixed. In this case one obtains

$$\omega(Y(\gamma(t_0))) = \mathrm{Ad}(p_n) \cdot \mathrm{diag}(-b, -a, R, a, b),$$

for some $p_n \in P$, which again yields the contracting/expanding property for the singularity $\gamma(t_0)$. $\square$

4.2. **Proof of Proposition 4.1.** We call x_0 the *mixed-type singularity* of Y. The vector field Y is linearizable in a neighborhood of x_0, and we denote by $\{\varphi_Y^t\}$ its flow. By assumption, replacing Y with $-Y$ if necessary, there exists a frame $(E_1, \dots, E_n)$ in which the differential $D_{x_0}\varphi_Y^t$ takes the form:

$$D_{x_0}\varphi_Y^t = \begin{pmatrix} e^{(a+b)t} & & \\ & e^{at}R^t & \\ & & e^{(a-b)t} \end{pmatrix},$$

where $b > a > 0$, and $\{R^t\}$ is a one-parameter subgroup of $\mathrm{SO}(n-2)$. In particular, x_0 is an isolated singularity of Y, and the flow $\{\varphi_Y^t\}$ admits a local stable manifold of dimension 1. More precisely, we saw in Lemma 2.2 that φ_Y^t acts by homothetic transformations for some metric g_0, defined on a neighborhood of x_0 and belonging to the local conformal class $[g]$. A local parametrization of the stable manifold is therefore given by $s \mapsto \exp_{x_0}(sE_n)$, where the exponential map is that of the metric g_0. We deduce that the local stable manifold is a *null geodesic segment* (see Subsection 3.1.5). Moreover, any conformal vector field defined locally around x_0 and commuting with Y must vanish at x_0 (since x_0 is isolated) and be tangent to the local stable manifold of φ_Y^t.

We now consider the geometric structure $[g] \cup \{Y\}$. Let $0 < s_0$ be small enough so that $x_{s_0} := \exp_{x_0}(s_0 E_n)$ is well defined, and introduce:

$$\alpha := \left\{ \varphi_Y^t \cdot x_{s_0} \;\middle|\; t \in \mathbf{R} \right\}.$$

By the previous discussion, this is a null geodesic segment, which in fact coincides with the Kill_Y^{loc}-orbit of x_{s_0}. Since this Kill_Y^{loc}-orbit is one-dimensional, the remarkable topological properties of Kill_Y^{loc}-orbits in the analytic setting, as stated in Theorem 3.3, allow us to fully understand the closure $\overline{\alpha}$ of α in M. We refer the reader to the proof of Lemma 5.4 in [MP22b] to see how Theorem 3.3 leads to:

Lemma 4.3. *There exists a point $x_1 \in M$ such that:*

$$\overline{\alpha} = \{x_0\} \cup \alpha \cup \{x_1\}.$$

Note that, *a priori*, x_1 may coincide with x_0. Also note that $\lim_{t\to-\infty} \varphi_Y^t \cdot x_{s_0} = x_1$, and therefore x_1 is a singularity of Y. We shall now show that there exists $\epsilon > 0$ and an immersion $\gamma : (-\epsilon, 1+\epsilon) \to M$ that parametrizes a null geodesic segment such that $\gamma(0) = x_0$, $\gamma(1) = x_1$, and $\gamma([0,1]) = \{x_0\} \cup \alpha \cup \{x_1\}$. We can then apply Proposition 4.2, which ensures that the field Y must admit another singularity of contracting/expanding type. The conformal flatness of $(M, [g])$ will then follow from Corollary 2.4.

For $T \in \mathbf{R}$, introduce the following sets:

$$\alpha_{\geq T} := \left\{ \varphi_Y^t \cdot \exp_{x_0}(s_0 E_n) \;\middle|\; t \geq T \right\},$$

and

$$\alpha_{\leq T} := \left\{ \varphi_Y^t \cdot \exp_{x_0}(s_0 E_n) \;\middle|\; t \leq T \right\}.$$

Proving the existence of γ amounts to showing that, for large $|T|$, the sets $\{x_0\} \cup \alpha_{\geq T}$ and $\alpha_{\leq -T} \cup \{x_1\}$ are each contained in the interior of an open null geodesic segment. This has already been verified at the beginning of the proof for $\{x_0\} \cup \alpha_{\geq T}$. We now study the case of $\alpha_{\leq -T} \cup \{x_1\}$.

Fix an arbitrary metric g in the conformal class $[g]$, as well as an auxiliary Riemannian metric h on M. There exist constants $R > 0$ and $r > 0$, and a neighborhood U of x_1, such that if $z \in U$, the exponential map $\exp_z$ (with respect to g) is defined and injective on $B(0_z, R)$ (the ball in the metric h_z), and $\exp_z(B(0_z, R))$ contains the closed ball $\overline{B}(z, r)$ (with respect to the Riemannian distance d_h). Let $z_T := \varphi^{-T}(x_{s_0})$. For large $T > 0$, we have $z_T \in U$ and $\alpha_{\leq -T} \cup \{x_1\} \subset B(z_T, r)$. Let u be a tangent vector to α at z_T, of unit h-norm. Then the map

$$s \mapsto \exp_{z_T}(su), \quad s \in (-R, R),$$

(where the exponential is taken with respect to g) is an immersion that parametrizes a null geodesic segment containing $\alpha_{\leq -T} \cup \{x_1\}$ in its interior.

5. Linear balanced singularity implies local gravitational pp-wave metric in conformal class

In this section, we begin to investigate the geometric consequences implied by the presence of a blanced singularity for a conformal vector field (see Subsection 2.2 regarding this notion). These remarkable properties, which are of independent interest, are stated in Proposition 5.1 below. They will be used in Section 6 to complete the proof of Theorem 1.2.

Recall that *a Brinkmann metric* is a Lorentzian metric admitting a parallel lightlike vector field X. Consequently, its orthogonal distribution $X^\perp$ is invariant by the Levi-Civita connection of the Lorentz metric and therefore integrable. In the case where the foliation defined by $X^\perp$ has flat leaves, with respect to the restriction of the Levi-Civita connection, the Brinkmann metric is a *pp-wave metric*. When a pp-wave metric is moreover Ricci-flat, we say that it is a *gravitational pp-wave*.

We also introduce the notion of *polarization* for a Lorentzian manifold (M, g). This notion is interesting only for manifolds of dimension ≥ 4. For such a manifold, let us denote by W the Weyl tensor on M. Let $\mathcal{D}$ be a lightlike one-dimensional smooth distribution on M. We say that M is *polarized with respect to* $\mathcal{D}$ if $W(\mathcal{D}^\perp, \mathcal{D}^\perp, \mathcal{D}^\perp) = 0$ and $\operatorname{Im} W \subset \mathcal{D}^\perp$.

The aim of this section is to prove the following result, part of which was previously obtained in [MP22b, Sec 4]. We provide a self-contained proof in the next three subsections for the reader's convenience.

Proposition 5.1. *Let (M, g) be a real-analytic Lorentzian manifold of dimension ≥ 3. Assume that Y is a local conformal vector field, defined on an open subset of M, admitting a linearizable singularity x_0 which is balanced. Then*

(1) If the dimension of M is 3, then (M, g) is conformally flat.
(2) In any dimension ≥ 4, there exists a neighborhood V of x_0 such that the conformal class $[g|_V]$ contains a gravitational pp-wave metric g_0, for which Y is a homothetic vector field.
(3) If moreover (M, g) is not conformally flat (hence if M is of dimension ≥ 4), there exists on M a $\mathrm{Conf}^{loc}(M)$-invariant, analytic lightlike line field $\mathcal{D}$, with the following properties:
   (a) The distribution $\mathcal{D}^\perp$ is integrable, and Y is tangent to the leaves of $\mathcal{D}^\perp$.
   (b) (M, g) is polarized with respect to $\mathcal{D}$.
   (c) In a neighborhood of each point of M, there exists a nonvanishing conformal vector field tangent to $\mathcal{D}$. Two conformal vector fields with this property differ by scalar multiplication.
   (d) At x_0, the direction $\mathcal{D}$ is transverse to the local one-dimensional singular locus of Y.

Observe that in the statement above, M is not assumed to be closed, hence Y may not be complete. However, under the hypotheses of the proposition, there exists a neighborhood U of x_0 and a coordinate system $(x_1, \ldots, x_n)$ on U, in which the local flow $\{\varphi_Y^t\}$ of Y is linear equal to:

$$\bar{h}^t = \begin{pmatrix} e^{-2t} & & \\ & e^{-t}R^t & \\ & & 1 \end{pmatrix} \tag{12}$$

where $\{R^t\}$ is a one-parameter group in $\mathrm{SO}(n-2)$. It follows that, shrinking U if necessary, φ_Y^t is *well defined for all times* $t \geq 0$. The semi-group $\{\varphi_Y^t\}_{t\geq 0}$ admits a one-dimensional manifold of fixed points on U, and a foliation by codimension one strongly stable manifolds. In particular, for every $x \in U$, $\lim_{t\to+\infty} \varphi_Y^t(x)$ exists in U.

We begin by examining point (1) of the proposition, dealing with the three-dimensional case, since it follows quickly from previous results. Let us consider the linearizing coordinate system (x_1, x_2, x_3) for U. For every $x \in U$, the matrix of $D_x\varphi_Y^t$, $t \geq 0$, relatively to the frame field $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ is just $\operatorname{diag}(e^{-2t}, e^{-t}, 1)$.

It follows that, given a sequence $t_k \to +\infty$, $\varphi_Y^{t_k}$ is stable at x, in the sense of [Fra07, Def. 3]. Now [Fra07, Prop. 5] yields that the Cotton tensor vanishes on U. By analyticity, (M, g) is conformally flat.

5.1. **Parallel submanifolds in the Cartan bundle and special metrics in the conformal class.** Throughout this section, $(M,[g])$ denotes a Lorentzian conformal structure of dimension $n \geq 3$. We interpret this conformal structure as a Cartan geometry modeled on $\mathbf{Ein}^{1,n-1}$ (see Theorem 3.1), and we denote by $(\hat{M},\omega)$ the associated normal Cartan bundle. We refer the reader to Section 3.1, and especially to Section 3.2 for the notations used below.

In what follows, we shall be interested in the *parallel submanifolds* of the bundle $\hat{M}$. Given a vector subspace $\mathfrak{h} \subset \mathfrak{g}$, we say that a submanifold $\hat{\Sigma} \subset \hat{M}$ is *parallel with respect to* $\mathfrak{h}$ when $\hat{\Sigma}$ is an integral leaf of the distribution $\omega^{-1}(\mathfrak{h})$. Equivalently, for every $\hat{x} \in \hat{\Sigma}$, one has $\omega_{\hat{x}}(T_{\hat{x}}\hat{\Sigma}) = \mathfrak{h}$. The most interesting situation occurs when $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, for in that case ω induces a Cartan subgeometry on $\hat{\Sigma}$.

The presence of parallel submanifolds of dimension > 1 often reflects special geometric properties of the conformal structure $(M,[g])$. For instance, if the distribution $\omega^{-1}(\mathfrak{g}_{-1})$ admits an integral leaf through $\hat{x} \in \hat{M}$, then $x = \pi(\hat{x})$ admits of a conformally flat neighborhood. Similarly, an integral leaf of $\omega^{-1}(\mathfrak{g}_{-1}\oplus\mathfrak{g}_0)$ yields, locally, a Ricci-flat metric in the conformal class around x (see the proof of Proposition 5.2 below). Here, we refine those results and prove :

Proposition 5.2. *Let $(M,[g])$ be a conformal Lorentzian manifold. Assume that there exists, in $\hat{M}$, an integral leaf $\hat{\Sigma}$ of the distribution $\omega^{-1}(\mathfrak{g}_{-1} \oplus \mathfrak{u}^+)$, and let $\hat{x} \in \hat{\Sigma}$. Then there exists, in a neighborhood V containing $x = \pi(\hat{x})$, a Ricci-flat Brinkmann metric in the conformal class $[g|_V]$ with parallel lightlike vector field equal to the projection of $\omega|_{\hat{\Sigma}}^{-1}(E_1)$, up to scalar.*

Proof. We recall the definition of the conformal exponential map given in Section 3.1.4.

Because $\hat{\Sigma}$ is parallel with respect to $\mathfrak{g}_{-1} \oplus \mathfrak{u}^+$, there exists a small neighborhood $\mathcal{V}$ of 0 in $\mathfrak{g}_{-1} \oplus \mathfrak{u}^+$, for which $\exp(\hat{x},\mathcal{V})$ is defined and such that we have the inclusion $\exp(\hat{x},\mathcal{V}) \subset \hat{\Sigma}$. We may choose $\mathcal{V}$ small enough so that $\hat{V}_{-1} = \exp(\hat{x},\mathcal{V} \cap \mathfrak{g}_{-1})$ is transverse to the fibers of π and projects diffeomorphically onto its image V, an open neighborhood of x. We then consider $\hat{V} = \hat{V}_{-1} \cdot U^+$, the saturation of $\hat{V}_{-1}$ by the U^+-action. It is a U^+-principal bundle over V. We observe that $\hat{V}$ is still a parallel submanifold with respect to $\mathfrak{g}_{-1} \oplus \mathfrak{u}^+$. Indeed, $\omega(T\hat{V}_{-1}) \subset \omega(T\hat{\Sigma}) \subset \mathfrak{g}_{-1} \oplus \mathfrak{u}^+$, so that actually $\omega(T\hat{V}) \subset \mathfrak{g}_{-1} \oplus \mathfrak{u}^+$ by U^+-equivariance of ω and $\mathrm{Ad}(U^+)$-invariance of $\mathfrak{g}_{-1} \oplus \mathfrak{u}^+$.

The data of the bundle $\hat{V}$ defines a metric g_0 in the conformal class $[g|_V]$ in the following way. For any choice of $y \in V$, and u,v in T_yV, we define:

$$g_0(u,v) := \langle \iota_{\hat{y}}(u), \iota_{\hat{y}}(v)\rangle,$$

where $\hat{y}$ is any point of $\hat{V}$ in the fiber of y, and $\langle\, ,\, \rangle$ is the Lorentzian scalar product on $\mathfrak{g}/\mathfrak{p}$ introduced in Section 3.1.1 (here we identify $\mathfrak{g}/\mathfrak{p}$ with $\mathfrak{g}_{-1}$). The definition of g_0 is consistent because, by equation (2), we have for any $p \in U^+$:

$$\langle \iota_{\hat{y}.p}(u), \iota_{\hat{y}.p}(v)\rangle = \langle \mathrm{Ad}(p^{-1})\iota_{\hat{y}}(u), \mathrm{Ad}(p^{-1})\iota_{\hat{y}}(v)\rangle = \langle \iota_{\hat{y}}(u), \iota_{\hat{y}}(v)\rangle.$$

By restriction, the Cartan connexion ω induces on $\hat{V}$ a Cartan connexion with values in the Lie algebra $\mathfrak{g}_{-1} \oplus \mathfrak{u}^+$, that we will denote $\overline{\omega}$. We observe that the curvature $\overline{K}$ of $\overline{\omega}$ is merely the restriction of K to $T\hat{V}$.

One says that a vector field on $\hat{V}$ is *horizontal* when it is tangent to $\overline{\omega}^{-1}(\mathfrak{g}_{-1})$. For any vector field X on V, there exists a unique lift $\hat{X}$ on $\hat{V}$ which is horizontal and U^+-invariant. Conversely, any horizontal U^+-invariant vector field $\hat{X}$ on $\hat{V}$ projects to a well-defined vector field X on V.

The formula

$$\widehat{\nabla_X Y} = \overline{\omega}^{-1}\big(\hat{X} \cdot \overline{\omega}(\hat{Y})\big) \tag{13}$$

defines a connection ∇ on V. This connection is torsion-free, because for any vector fields X,Y on V, $\overline{K}(\hat{X},\hat{Y}) = K(\hat{X},\hat{Y})$ takes values in $(\mathfrak{g}_{-1} \oplus \mathfrak{u}^+) \cap \mathfrak{p} = \mathfrak{u}^+$.

Finally, for any three vector fields X, Y, Z on V, we have

$$\hat{Z} \cdot \langle \omega(\hat{X}), \omega(\hat{Y}) \rangle = \langle \hat{Z} \cdot \omega(\hat{X}), \omega(\hat{Y}) \rangle + \langle \omega(\hat{X}), \hat{Z} \cdot \omega(\hat{Y}) \rangle,$$

and thus $Z \cdot g_0(X,Y) = g_0(\nabla_Z X, Y) + g_0(X, \nabla_Z Y)$. Therefore, ∇ is the Levi-Civita connection of g_0.

Let us now consider the vector field $\hat{E}_1$ on $\hat{V}$ defined by $\overline{\omega}(\hat{E}_1) \equiv E_1$. Because $\mathrm{Ad}(U^+)$ fixes E_1, the field $\hat{E}_1$ is U^+-invariant and hence defines a vector field X on V. Since $\langle E_1, E_1 \rangle = 0$, the field X is lightlike. Moreover, for any horizontal and U^+-invariant vector field $\hat{Y}$, we trivially have $\hat{Y} \cdot \omega(\hat{E}_1) = 0$. This shows that $\nabla X = 0$: the field X is parallel with respect to g_0, and thus g_0 is a Brinkmann metric.

Finally, let R denote the Riemann curvature tensor of the metric g_0. If $y \in V$ and $u, v, w \in T_y V$, then (see Section 3.1.2 for the link between the Cartan curvature and the Weyl tensor)

$$R_y(u,v,w) = [\overline{\kappa}(\iota_{\hat{y}}(u), \iota_{\hat{y}}(v)), \iota_{\hat{y}}(w)] = [\kappa^0(\iota_{\hat{y}}(u), \iota_{\hat{y}}(v)), \iota_{\hat{y}}(w)] = W_y(u,v,w).$$

The Riemann tensor of g_0 is thus equal to its Weyl tensor, proving that g_0 is Ricci-flat. □

5.2. **Second point of Proposition 5.1: Existence of a local pp-wave metric.** By hypothesis, Y is linearizable around x_0. This means that there exists $\hat{x}_0 \in \widehat{M}$ in the fiber of x_0, and a one-parameter group $\{h^t\}$ in G_0, such that for every $t \geq 0$, $\varphi_Y^t(\hat{x}_0).h^{-t} = \hat{x}_0$ (see for instance [Fra12b], Prop. 4.2). The linear isomorphism $\iota_{\hat{x}_0}$ conjugates the action of $D_{x_0}\varphi_Y^t$ on $T_{x_0}M$ to the action of $\mathrm{Ad}(h^t)$ on $\mathfrak{g}_{-1}$, after G_0-equivariant identification of $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{g}_{-1}$. It follows that up to conjugating h^t in G_0, which amounts to right translating $\hat{x}_0$ in the fiber, $\mathrm{Ad}(h^t)$ acts on $\mathfrak{g}_{-1}$, endowed with the basis $(E_1, \ldots, E_n)$, by the matrix $\bar{h}^t$ (see (12)). In other words, h^t takes the following block-diagonal form in $\mathrm{PO}(2,n)$: $h^t = \mathrm{diag}(e^t, e^{-t}, R^t, e^t, e^{-t})$, where $\{R^t\}$ is the same one-parameter group as in (12).

5.2.1. *Parallel submanifolds determined by the dynamics.* The adjoint action of h^t on $\mathfrak{g} = \mathfrak{so}(2,n)$ preserves individually every root space. More precisely, the space

$$\mathfrak{s}_0 = \mathfrak{g}_{-\alpha-\beta} + \mathfrak{a} + \mathfrak{m} + \mathfrak{g}_{\alpha+\beta} = \{\xi \in \mathfrak{g} \ : \ \mathrm{Ad}(h^t)\xi = \xi \ \forall t\} \tag{14}$$

comprises the fixed points of $\mathrm{Ad}(h^t)$. On the spaces $\mathfrak{s}_{+1} = \mathfrak{g}_\alpha + \mathfrak{g}_{-\beta}$ and $\mathfrak{s}_{-1} = \mathfrak{g}_{-\alpha} + \mathfrak{g}_\beta$, the action is by Euclidean similarities

$$\xi \mapsto e^{\pm t}\,\mathrm{Ad}(R^t)\xi$$

Denote $\mathfrak{s}_{-2} = \mathfrak{g}_{\beta-\alpha}$ (resp. $\mathfrak{s}_{+2} = \mathfrak{g}_{\alpha-\beta}$), the eigenspaces associated to the eigenvalue e^{-2t} (resp. e^{2t}). Now consider

$$\mathfrak{s}^{<} = \mathfrak{s}_0 + \mathfrak{s}_{-1} + \mathfrak{s}_{-2} = \mathfrak{g}_{-1} + \mathfrak{a} + \mathfrak{m} + \mathfrak{u}^+ + \mathfrak{g}_{\alpha+\beta},$$

the *stable space* for $\mathrm{Ad}(h^t)$, $t \geq 0$; it comprises all vectors $\xi \in \mathfrak{g}$ such that $\mathrm{Ad}(h^t)\xi$ remains bounded for $t \geq 0$. The *strongly stable space*

$$\mathfrak{s}^{<<} = \mathfrak{s}_{-1} + \mathfrak{s}_{-2} = E_1^\perp + \mathfrak{u}^+$$

comprises all vectors $\xi \in \mathfrak{g}$ such that $\lim_{t \to +\infty} \mathrm{Ad}(h^t)\xi = 0$. The distributions $\omega^{-1}(\mathfrak{s}^{<})$ and $\omega^{-1}(\mathfrak{s}^{<<})$ can be integrated into parallel submanifolds in $\widehat{M}$, by the following proposition. (See also [MP22b, Sec 4].)

Proposition 5.3 ([Fra12a], Prop. 4.8)**.** *There exists a neighborhood $\mathcal{V}$ of 0 in $\mathfrak{g}$ such that $\Sigma^{<} = \exp(\hat{x}_0, \mathcal{V} \cap \mathfrak{s}^{<})$ and $\Sigma^{<<} = \exp(\hat{x}_0, \mathcal{V} \cap \mathfrak{s}^{<<})$ are integral leaves of the distributions $\omega^{-1}(\mathfrak{s}^{<})$ and $\omega^{-1}(\mathfrak{s}^{<<})$.*

In the sequel, it will be useful to choose $\mathcal{V}$ as a product of convex neighborhoods of the origin in each rootspace of $\mathfrak{a}$.

5.2.2. *Special values of the curvature function.* Recall from Proposition 5.2 that a local integral leaf of the distribution $\omega^{-1}(\mathfrak{g}_{-1} + \mathfrak{u}^+)$ corresponds to a local Brinkmann metric in the conformal class. We will show that this distribution is integrable on the manifold $\Sigma^<$. Involutivity will follow from the special form of the Cartan curvature:

Lemma 5.4. *For every $\hat{x} \in \hat{\Sigma}^<$, the curvature $\kappa_{\hat{x}}$ has the following properties:*

(1) $\mathrm{Im}\,\kappa_{\hat{x}} \subset \mathfrak{u}^+$.
(2) For every ξ, η in $E_1^\perp$, $\kappa_{\hat{x}}(\xi, \eta) = 0$.

Proof. Let $\hat{x} \in \hat{\Sigma}^<$. It is of the form $\hat{x} = \exp(\hat{x}_0, \xi)$, for $\xi \in \mathcal{V} \cap \mathfrak{s}^<$. Because $\xi \in \mathfrak{s}^< = \mathfrak{s}_0 + \mathfrak{s}_{-1} + \mathfrak{s}_{-2}$, we have $\lim_{t \to +\infty} \mathrm{Ad}(h^t)\xi = \xi_\infty$, where ξ_∞ is the component of ξ along $\mathfrak{s}_0$. Equation (7) yields:

$$\hat{x}_t := \varphi_Y^t(\hat{x}).h^{-t} = \exp(\hat{x}_0, \mathrm{Ad}(h^t)\xi).$$

It follows that $\lim_{t \to +\infty} \hat{x}_t = \hat{x}_\infty$ with $\hat{x}_\infty = \exp(\hat{x}_0, \xi_\infty)$. In particular, $\lim_{t \to +\infty} \kappa_{\hat{x}_t} = \kappa_{\hat{x}_\infty}$. By equation (4), $\kappa_{\hat{x}_t} = h^t.\kappa_{\hat{x}}$, from which we deduce that $h^t.\kappa_{\hat{x}}$ remains bounded for $t \geq 0$.

Now, recall that $\mathfrak{g}_{-1} = \mathfrak{g}_{-\alpha+\beta} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-(\alpha+\beta)}$. Recall also the description of the adjoint action of h^t on each rootspace given in Section 5.2.1. Let ξ, η be two linearly independent vectors picked in the set $\{\mathfrak{g}_{-\alpha+\beta}; \mathfrak{g}_{-\alpha}; \mathfrak{g}_{-(\alpha+\beta)}\}$. We have:

$$\kappa_{\hat{x}}(\mathrm{Ad}(h^{-t})\xi, \mathrm{Ad}(h^{-t})\eta) = e^{\lambda t} \kappa_{\hat{x}}(\mathrm{Ad}(R^{-t})\xi, \mathrm{Ad}(R^{-t})\eta),$$

where $\lambda = 1, 2$ or 3. Since $h^t.\kappa_{\hat{x}}$ is bounded, there must exist a constant $C \geq 0$ such that for all $t \geq 0$, we have:

$$|\,\mathrm{Ad}(h^t).\kappa_{\hat{x}}(\mathrm{Ad}(R^{-t})\xi, \mathrm{Ad}(R^{-t})\eta)| \leq Ce^{-\lambda t}. \tag{15}$$

Because $\{R^t\}$ stays in a compact subgroup, we may consider a sequence $t_k \to \infty$ such that $R^{-t_k} \to R_\infty$. Equation (15) then shows that the components of $\kappa_{\hat{x}}(\mathrm{Ad}(R_\infty)\xi, \mathrm{Ad}(R_\infty)\eta)$ along rootspaces on which $\mathrm{Ad}(h^t)$ does not act as a contraction must vanish. However, $\kappa_{\hat{x}}(\mathrm{Ad}(R_\infty)\xi, \mathrm{Ad}(R_\infty)\eta)$ belongs to $\mathfrak{p}$, and the only rootspace in $\mathfrak{p}$ which is contracted by $\mathrm{Ad}(h^t)$ is $\mathfrak{u}^+$. We get $\kappa_{\hat{x}}(\mathrm{Ad}(R_\infty)\xi, \mathrm{Ad}(R_\infty)\eta) \in \mathfrak{u}^+$ and the first point of Lemma 5.4 follows, because $\mathrm{Ad}(R_\infty)$ is an isomorphism on each rootspace.

To prove the second point of the lemma, we observe that $E_1^\perp = \mathfrak{g}_{-\alpha+\beta} \oplus \mathfrak{g}_{-\alpha}$. Taking two linearly independent vectors ξ, η in $\{\mathfrak{g}_{-\alpha}; \mathfrak{g}_{-(\alpha+\beta)}\}$, equation (15) is still satisfied, but the possible values of λ are just 2 and 3. On the other hand, the action of $\mathrm{Ad}(h^t)$ on each rootspace of $\mathfrak{p}$ is by a transformation of the form $\xi \mapsto e^{\mu t}\,\mathrm{Ad}(R^t)\xi$, where $\mu \geq -1$. Thus, considering the same sequence (t_k) as above, if a rootspace component of $\kappa_{\hat{x}}(\mathrm{Ad}(R_\infty)\xi, \mathrm{Ad}(R_\infty)\eta)$ were nonzero, equation (15) would be violated for k large. This finishes the proof of Lemma 5.4. ☐

5.2.3. *Construction of the gravitational pp-wave.* Let X and Z be two ω-constant vector fields on $\hat{\Sigma}^<$. By equation (4) defining the curvature, they satisfy the identity:

$$\omega([X, Z]) = [\omega(X), \omega(Z)] - K(X, Z). \tag{16}$$

Assume now that X and Z are tangent to $\omega^{-1}(\mathfrak{g}_{-1} + \mathfrak{u}^+)$. If $\omega(X)$ or $\omega(Z)$ belongs to $\mathfrak{u}^+$, then $K(X, Z) = 0$ and $\omega([X, Z]) \in \mathfrak{g}^{-1} + \mathfrak{u}^+$. If both $\omega(X)$ and $\omega(Z)$ belong to $\mathfrak{g}_{-1}$, then the first point of Lemma 5.4 ensures that $\omega([X, Z]) \in \mathfrak{u}^+$. Therefore $\omega^{-1}(\mathfrak{g}_{-1} + \mathfrak{u}^+)$ is involutive on $\hat{\Sigma}^<$, as claimed. By Proposition 5.2 and its proof, there exists an integral leaf $\hat{V}$ through $\hat{x}_0$ that is U^+-invariant and projects onto an open neighborhood V containing x_0, and such that the conformal class $[g|_V]$ contains a Ricci-flat Brinkmann metric g_0.

We recall that for any $y \in V$, and $u, v \in T_yM$, the metric g_0 is defined by the formula:

$$g_0(u, v) := \langle \iota_{\hat{y}}(u), \iota_{\hat{y}}(v) \rangle, \tag{17}$$

where $\hat{y}$ is any point of $\hat{V}$ projecting on y.

Let us check that g_0 is actually a gravitational pp-wave. By Proposition 5.2, the projection to V of the vector field $\omega|_{\hat{V}}^{-1}(E_1)$ on $\hat{V}$ is lightlike and parallel for the Levi-Civita connection of g_0. It defines a one-dimensional lightlike distribution $\mathcal{D}$ on V such that $\mathcal{D}^\perp$ is integrable. For any two vector fields X and Z on $\hat{V}$ tangent to $\omega|_{\hat{V}}^{-1}(E_1^\perp)$, the formula (16) and the second point of Lemma 5.4 give:

$$\omega([X,Z]) = [\omega(X), \omega(Z)] = 0.$$

The distribution $\omega|_{\hat{V}}^{-1}(E_1^\perp)$ is then integrable on $\hat{V}$. Each leaf projects on an integral leaf of $\mathcal{D}^\perp$ on V. Recall from (13) the link between the restriction of ω to $\hat{V}$ and the Levi-Civita connection ∇ of g_0. Because the leaves of $\omega|_{\hat{V}}^{-1}(E_1^\perp)$ are ω-parallel by definition, (13) shows that their projections to V are ∇-parallel, hence totally geodesic. The curvature of the induced connection vanishes by the second point of Lemma 5.4. Therefore the integral leaves of $\mathcal{D}^\perp$ are flat and totally geodesic. The metric g_0 is thus a Ricci-flat pp-wave.

5.2.4. *The field Y is homothetic for g_0.* From the beginning of the proof of Proposition 5.2, we recall that $\hat{V}$ is obtained as $\exp(\hat{x}_0, \mathcal{V}_1) \cdot U^+$, where $\mathcal{V}_1$ is a small neighborhood of 0 in $\mathfrak{g}_{-1}$. The adjoint action of h^t on $\mathfrak{g}_{-1}$ is given by $\mathrm{diag}(e^{-2t}, e^t R^t, 1)$, so that by taking $\mathcal{V}_1$ as a product of balls in $\mathfrak{g}_{-\alpha+\beta}$, $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-\alpha-\beta}$ respectively, we can ensure that $\mathrm{Ad}(h^t)(\mathcal{V}_1) \subset \mathcal{V}_1$ for all $t \geq 0$.

We then claim that if $\hat{y} \in \hat{V}$, then $\varphi_Y^t(\hat{y}) \cdot h^{-t} \in \hat{V}$. Indeed, we can write $\hat{y} = \exp(\hat{x}_0, \xi) \cdot u^+$ for some $\xi \in \mathcal{V}_{-1}$ and $u^+ \in U^+$. Then

$$\varphi_Y^t(\exp(\hat{x}_0, \xi) \cdot u^+) \cdot h^{-t} = \exp(\hat{x}_0, \mathrm{Ad}(h^t)\xi) \cdot h^t u^+ h^{-t},$$

which is an element of $\hat{V}$ because $\mathrm{Ad}(h^t)\xi \in \mathcal{V}_{-1}$ and $h^t u^+ h^{-t} \in U^+$.

Let $y \in V$ and $\hat{y} \in \hat{V}$ projecting onto y. For $u, v \in T_y M$, we set $g_0(u,v) := \langle \iota_{\hat{y}}(u), \iota_{\hat{y}}(v) \rangle$. Since $\varphi_Y^t(\hat{y}) \cdot h^{-t} \in \hat{V}$, we also have by (17):

$$g_0(D_y \varphi_Y^t(u), D_y \varphi_Y^t(v)) = \langle \iota_{\varphi_Y^t(\hat{y}) \cdot h^{-t}}(D_y \varphi_Y^t(u)), \iota_{\varphi_Y^t(\hat{y}) \cdot h^{-t}}(D_y \varphi_Y^t(v)) \rangle.$$

Now, (2) yields $\iota_{\varphi_Y^t(\hat{y}) \cdot h^{-t}}(D_y \varphi_Y^t(u)) = \mathrm{Ad}(h^t) \iota_{\varphi_Y^t(\hat{y})}(D_y \varphi_Y^t(u))$. Moreover $\iota_{\varphi_Y^t(\hat{y})}(D_y \varphi_Y^t(u)) = \iota_{\hat{y}}(u)$ because φ_Y^t preserves ω.

We therefore obtain

$$g_0(D_y \varphi_Y^t(u), D_y \varphi_Y^t(v)) = \langle \mathrm{Ad}(h^t) \iota_{\hat{y}}(u), \mathrm{Ad}(h^t) \iota_{\hat{y}}(v) \rangle = e^{2t} \langle \iota_{\hat{y}}(u), \iota_{\hat{y}}(v) \rangle = e^{2t} g_0(u,v).$$

The second point of Proposition 5.1 is now fully proved.

5.3. **Third point of Proposition 5.1: Existence of a global polarization.** We assume that (M,g) is not conformally flat, which, by the first point of Proposition 5.1, implies that M has dimension ≥ 4.

The following lemma states that a non–conformally flat, real-analytic Lorentzian manifold can be polarized with respect to at most one lightlike line field.

Lemma 5.5 ([Pec17], Lemma 9)**.** *Let (M,g) be a real-analytic Lorentzian manifold of dimension ≥ 4. Assume that M is polarized with respect to distinct lightlike distributions $\mathcal{D}$ and $\mathcal{D}'$. Then (M,g) is conformally flat.*

In fact, Lemma 9 of [Pec17] establishes a pointwise result: if a Lorentzian manifold is polarized with respect to two distinct lightlike directions $\mathcal{D}_x$ and $\mathcal{D}'_x$ at a point x, then the Weyl tensor vanishes at x. Being polarized with respect to two distinct smooth lightlike line fields therefore implies the vanishing of the Weyl tensor on an open set. The analyticity of the structure then yields conformal flatness, and the lemma follows.

5.3.1. *Local conformal vector field and polarization.* We stick here to the notations introduced in 5.2.3. We still denote by X the projection of $\omega|_{\hat{V}}^{-1}(E_1)$ to V, and by $\mathcal{D}$ the one-dimensional lightlike distribution defined by X.

Let $x \in V$ and $\hat{x} \in \hat{V}_x$. The first point of Lemma 5.4 ensures that whenever u, v are in T_xM, then $\kappa_{\hat{x}}(\iota_{\hat{x}}(u), \iota_{\hat{x}}(v)) \in \mathfrak{u}^+$. Thus, for any $w \in T_xM$, the bracket $[\kappa_{\hat{x}}(\iota_{\hat{x}}(u), \iota_{\hat{x}}(v)), \iota_{\hat{x}}(w)] \in E_1^\perp$. In other words, according to the interpretation of the Weyl curvature in terms of the Cartan curvature κ from Section 3, the image of W_x is contained in $\mathcal{D}^\perp$ for all $x \in V$, and the conformal structure is polarized with respect to $\mathcal{D}$ on V.

Because X is parallel for g_0, it is a Killing vector field of g_0, and in particular a conformal vector field on V. Let $\tilde{V}$ be a connected component of the preimage of V in the universal cover $\tilde{M}$. By analyticity and a theorem of Amores [Amo79], the lift of X to $\tilde{V}$ extends to a nontrivial global conformal vector field on $\tilde{M}$, which will be denoted by $\tilde{X}$. As X is lightlike on the open set $\tilde{V}$, $\tilde{X}$ is lightlike on all of $\tilde{M}$ by analyticity. Since $\tilde{M}$ is not conformally flat, Theorem 1 of [Fra07] implies that $\tilde{X}$ has no singularities.

Denote now by $\mathcal{D}$ the analytic lightlike line distribution on $\tilde{M}$ generated by X. By the work done in Section 5.2.3, $\mathcal{D}^\perp$ is integrable on V, hence on $\tilde{M}$ by analyticity. The conformal structure is polarized with respect to $\mathcal{D}$ on $\tilde{V}$. Now, $\mathcal{D}^\perp$ is an analytic distribution on $\tilde{M}$, and the conditions

$$W(\mathcal{D}^\perp, \mathcal{D}^\perp, \mathcal{D}^\perp) = 0 \qquad \operatorname{Im} W \subseteq \mathcal{D}^\perp$$

correspond locally to analytic equations on $\tilde{M}$. Therefore, the conformal structure is polarized with respect to $\mathcal{D}$ on all of $\tilde{M}$.

Let φ be any local conformal transformation of $\tilde{M}$ defined on an open subset U. Let $\mathcal{D}' = \varphi_*\mathcal{D}$, defined on $\varphi(U)$. Now the conformal structure on $\varphi(U)$ is polarized with respect to $\mathcal{D}$ and $\mathcal{D}'$. By Lemma 5.5, and by the assumption that $\tilde{M}$ is not conformally flat, these must be equal. Therefore, $\mathcal{D}$ is $\mathrm{Conf}^{loc}(\tilde{M})$-invariant, and in particular $\pi_1(M)$-invariant. It follows that $\mathcal{D}$ descends to a well-defined and $\mathrm{Conf}^{loc}(M)$-invariant line field on M, extending $\mathcal{D}$ as originally defined on V, with respect to which the conformal structure is polarized.

5.3.2. *The field Y is contained in $\mathcal{D}^\perp$.* We still use the notations of 5.2.3, and now prove that the vector field Y is contained in the distribution $\mathcal{D}^\perp$. Since this condition is locally given by analytic equations, it suffices to check it on the open subset V. For $|s| < \epsilon$ small enough, we define $\hat{x}_s := \exp(\hat{x}_0, sE_n)$. Because $\hat{V}$ is parallel with respect to $\mathfrak{g}_{-1} + \mathfrak{u}^+$, we have $\hat{x}_s \in \hat{V}$ for all $s \in (-\epsilon, \epsilon)$. Let us now choose $\mathcal{W}$ to be a small open neighborhood of 0 in $E_1^\perp$, so that $\exp(\hat{x}_s, \xi)$ is well defined for all $s \in (-\epsilon, \epsilon)$ and $\xi \in \mathcal{W}$. Since $\omega|_{\hat{V}}^{-1}(E_1^\perp)$ is integrable on $\hat{V}$, the sets $\hat{F}_s := \exp(\hat{x}_s, \mathcal{W})$ are integral leaves of $\omega|_{\hat{V}}^{-1}(E_1^\perp)$ contained in $\hat{V}$. The inverse mapping theorem implies that, after shrinking ϵ and $\mathcal{W}$ if necessary, the map $\Phi : (-\epsilon, \epsilon) \times \mathcal{W} \to M$ defined by $\Phi(s, \xi) := \pi(\exp(\hat{x}_s, \xi))$ is a diffeomorphism onto an open subset of V, which we may assume to be V to simplify the notation. The submanifolds $F_s := \pi(\hat{F}_s)$ foliate V, and are integral leaves of $\mathcal{D}^\perp$. We want to show that for any $y \in F_s$, we have $\varphi_Y^t(y) \in F_s$ for all sufficiently small t. We lift y to $\hat{y} = \exp(\hat{x}_s, \xi)$, with $\xi \in \mathcal{W}$. The equivariance of the exponential map (7), and the property $\mathrm{Ad}(h^t)E_n = E_n$ yield

$$\varphi_Y^t(\hat{x}_s).h^{-t} = \exp(\hat{x}_0, s\,\mathrm{Ad}(h^t)E_n) = \hat{x}_s.$$

Now $\varphi_Y^t(\hat{y}).h^{-t} = \exp(\hat{x}_s, \mathrm{Ad}(h^t)\xi)$, and for small t, $\mathrm{Ad}(h^t)\xi \in \mathcal{W}$ since $E_1^\perp$ is preserved by $\mathrm{Ad}(h^t)$. We conclude that $\varphi_Y^t(\hat{y}).h^{-t} \in \hat{F}_s$, which implies $\varphi_Y^t(y) \in F_s$, as desired.

This subsection, together with 5.3.1 completes the proof of points 3(a) and 3(b) of Proposition 5.1.

5.3.3. *Local conformal vector fields tangent to $\mathcal{D}$.* We now turn to point 3(c) of the proposition. Given a point $x \in M$, we may choose a small neighborhood U of x which is evenly covered by the covering map $\tilde{M} \to M$. Restricting the field $\tilde{X}$ constructed in 5.3.1 to a connected component of the preimage of U, and projecting on U, yields a conformal vector field X_U on U which is nonsingular and collinear to $\mathcal{D}$. To complete the proof of 3(c), it remains to show that two local conformal vector fields that belong to the line field $\mathcal{D}$ differ by a multiplicative constant. This is a direct consequence of the following result.

Lemma 5.6. [Pec23, Lemma 2.5] *Let X, Y be two conformal vector fields of a connected pseudo-Riemannian manifold (M, g) of dimension ≥ 3. If for every $x \in M$, X_x and Y_x are proportional, then X and Y are collinear.*

Finally, the transversality claimed in point 3(d) is clear. Indeed, the line $\mathcal{D}$ evaluated at x_0 is spanned by the projection $\pi_*(\omega|_{\hat{V}}^{-1}(E_1))$ while the tangent to the singular locus is spanned by $\pi_*(\omega|_{\hat{V}}^{-1}(E_n))$.

6. Linear balanced singularity implies conformal flatness

The aim of this section is to prove the

Proposition 6.1. *Let (M, g) be a closed, real-analytic, Lorentzian manifold. Let Y be a conformal vector field on M. If Y admits a singularity which is balanced, then (M, g) is conformally flat.*

This will be the next step in the strategy outlined in Section 2, and will complete the proof of Theorem 1.2.

Our proof of Proposition 6.1 will be by contradiction. So, we assume that (M, g) is not conformally flat, and Proposition 5.1 applies.

Denote by $\mathcal{D}$ the lightlike line field on M given by the third point of this proposition. Pick X a local conformal vector field on U given by point 3(c) of the same proposition. By lifting X to a connected component of the preimage of U in $\tilde{M}$ and extending this lift to the whole universal cover by analytic continuation, we obtain a global lightlike conformal vector field $\tilde{X}$ on $\tilde{M}$. This vector field may not descend to M, however, by construction of $\mathcal{D}$, at any point of $\tilde{x} \in \tilde{M}$ the projection of $\tilde{X}_{\tilde{x}}$ to M always belong to $\mathcal{D}_x$, where x is the projection of $\tilde{x}$. If $V \subset M$ is another trivialization neighborhood of the universal covering map, choosing a connecting component of its preimage, we can project $\tilde{X}$ onto V, and a different choice will yield a constant multiple of this projection (by Lemma 5.6) . We will call " representative of $\tilde{X}$" any such projections of $\tilde{X}$ on such open subsets. Abusively, we will always denote by X any such representative when the context in unambiguous.

6.1. A Kill_Y^{loc}-orbit of dimension 2 near the singularity.

Let Y be as in the statement of Proposition 6.1 with singularity x_0. The derivative $D_{x_0}\varphi_Y^t$ has the form

$$D_{x_0}\varphi^t = \begin{pmatrix} e^{-2t} & & \\ & e^{-t}R^t & \\ & & 1 \end{pmatrix}$$

as in the hypothesis of Proposition 5.1, and $\{\varphi_Y^t\}$ is conjugate to $\{D_{x_0}\varphi_Y^t\}$ in a neighborhood U of x_0.

Lemma 6.2. *If $\tilde{Y}$ denotes the lift of Y to $\tilde{M}$, then $[\tilde{Y}, \tilde{X}] = 2\tilde{X}$. In particular, if x is any other singularity of Y, then $D_x\varphi_Y^t$ preserves $\mathcal{D}_x$ and acts on it as the multiplication by e^{-2t}.*

Proof. As above, let X denote a representative of $\tilde{X}$ on U. The lightlike tangent vector X_{x_0} is the fastest eigenvector of $D_{x_0}\varphi_Y^t$, with eigenvalues e^{-2t}. It follows that $[Y, X] = 2X$ on U. Indeed, the flow of Y preserves $\mathcal{D}$ all over M by Proposition 5.1. So, $[Y, X]$ is everywhere collinear to X on U, hence there is a constant λ such that $[Y, X] = \lambda X$ thanks to Lemma 5.6. And since $[Y, X]_{x_0} = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(D_{x_0}\varphi_Y^t)^{-1}X_{x_0} = 2X_0$, we get $\lambda = 2$. The same bracket relation is then true for $\tilde{Y}$ and $\tilde{X}$ over some connected component of the preimage of U in the universal cover, hence on all of $\tilde{M}$ by rigidity.

If $x \in M$ is another singularity of Y, considering an evenly covered neighborhood V of x and picking X a representative of $\tilde{X}$ defined on V, we have $[Y, X] = 2X$ over V, from which the second claim follows. $\square$

Let $\hat{x}_0 \in \hat{M}_{x_0}$ be such that

$$\omega_{\hat{x}_0}(\hat{Y}) = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & A & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} =: Y_0 \tag{18}$$

with $A \in \mathfrak{so}(n-2)$ the infinitesimal generator of R^t. Denote by $h^t = e^{tY_0}$, so that $\varphi_Y^t(\hat{x}_0).h^{-t} = \hat{x}_0$. Recall the notation of Section 3.2. Let $\hat{\alpha}(s) = \exp_{\hat{x}_0}(sE_n)$ and $\alpha = \pi \circ \hat{\alpha}$. Because $\operatorname{Ad} h^t(E_n) = E_n$, the equivariance of the exponential map (7) implies that α is a null geodesic on which Y vanishes, and the isotropy of φ_Y^t with respect to each $\hat{\alpha}(s)$ is h^t for all $t \in \mathbf{R}$. Moreover, because E_n is transverse to $E_1^{\perp}$ in $\mathbf{R}^{1,n-1}$, the curve α is transverse to $\mathcal{F}$ along its length.

Choose a representative X on $\tilde{X}$ on the neighborhood U of x_0. Let $\epsilon > 0$ be such that $(s,t) \mapsto \varphi_X^t(\alpha(s))$ is defined on $(-\epsilon,\epsilon)^2$ and parametrizes a surface $S \subset U$.

Denote by $\mathcal{C} = \{x \in M \mid Y_x \in \mathcal{D}_x\}$. Note that $\mathcal{C}$ is a closed, Kill_Y^{loc}-invariant subset of M, since $\mathcal{D}$ is $\mathrm{Conf}^{loc}(M,g)$-invariant. Let $V \subset M$ be an open subset where a representative X of $\tilde{X}$ can be defined. Since $[X,Y] = -2X$ over V, it follows that for all $x \in V$, and small enough t, $D_x\varphi_X^t Y_x = Y_{\varphi_X^t(x)} + 2tX_{\varphi_X^t(x)}$. Applying this to U and points x in the curve α, we obtain that Y is collinear to X on the surface S, *i.e.* $S \subset \mathcal{C}$.

Lemma 6.3. *There is a neighborhood V of x_0 such that $S \cap V = \mathcal{C} \cap V$.*

Proof. Let $z \in U \setminus \alpha$. Due to the linearizability of φ_Y^t on U and the explicit form of $D_{x_0}\varphi_Y^t$, $Y_z \neq 0$ and we get that $\varphi_Y^t(z) \to z_\infty$ as $t \to +\infty$ for a certain point $z_\infty \in \alpha$. Let $W \subset U$ be a flow box for φ_X^t. Let $T \subset W$ be a transversal, and let $\eta > 0$ be such that $(y,s) \in T \times (-\eta,\eta) \mapsto \varphi_X^s(y) \in W$ is a diffeomorphism, and with $\alpha \subset T$. Let now $V \subset W$ be a cylindrical neighborhood around α (in the linear coordinates), invariant under $\{\varphi_Y^t\}_{t \geq 0}$.

Suppose now that $z \in \mathcal{C} \cap V$. Then, the full integral curve $\{\varphi_Y^t(z)\}$ is also contained in $\mathcal{C}$. This implies that X and Y are collinear along the future curve $\{\varphi_Y^t(z)\}_{t\geq 0}$. So, $\{\varphi_Y^t(z)\}_{t\geq 0}$ is contained in an integral curve of X in V. So, it is contained in a single φ_X^t-orbit in V, which must be the φ_X^t-orbit of z_∞. Hence, $z = \varphi_X^s(z_\infty)$ sor some s, which yields the conclusion. □

Let $x_1 = \varphi_X^t(x_0) \in S$ for small enough t, and consider its Kill_Y^{loc}-orbit $\mathcal{O}(x_1)$. Since $\mathcal{O}(x_1) \subset \mathcal{C}$, we get $\mathcal{O}(x_1) \cap V \subset S$ where V is as in Lemma 6.3, implying that $\dim \mathcal{O}(x_1) \leq 2$.

Lemma 6.4. *Let $x \in \mathcal{C}$ be such that $Y_x \neq 0$ and suppose that its Kill_Y^{loc}-orbit is one-dimensional. Then, x is a periodic point of $\{\varphi_Y^t\}$.*

Proof. Observe that it amounts to prove that the Kill_Y^{loc}-orbit $\mathcal{O}(x)$ is closed, hence a circle on which Y does not vanish. So, let us suppose by contradiction that $\mathcal{O}(x)$ is not closed. By Theorem 3.3, any $y \in \partial\mathcal{O}(x)$ is fixed by all local conformal vector fields centralizing Y, in particular $Y_y = 0$.

Since Y does not vanish on $\mathcal{O}(x)$, it follows that $\mathcal{O}(x) = \{\varphi_Y^t(x)\}_{t\in\mathbf{R}}$ by connectedness. Therefore, any point y in the α-limit set (for the flow of Y) of x is a singularity of Y. Let U be a neighborhood of y on which a representative of X can be defined. Shrinking U several times if necessary, we may assume that U is a flow-box for the local flow of X, with a transversal $T \subset U$ containing y so that we have $\epsilon > 0$ such that $\Phi : (s,z) \in (-\epsilon,\epsilon) \times T \mapsto \varphi_X^s(z) \in U$ realizes a diffeomorphism. Let $\gamma(s) = \Phi(s,y)$. Since $[Y,X] = 2X$, we have $0 = (\varphi_X^s)_*Y_y = Y_{\gamma(s)} + 2sX_{\gamma(s)}$, and in particular $\gamma(s) \in \mathcal{C}$. Hence, for $0 < |s| < \epsilon$ and $t \geq 0$, $\varphi_Y^t(\gamma(s)) = \gamma(e^{-2t}s)$.

By assumption, we have a decreasing sequence $(t_k) \to -\infty$ such that $\varphi_Y^{t_k}(x) \to y$. The fact that $\varphi_Y^t(\gamma(s)) = \gamma(e^{-2t}s)$ for $t \geq 0$ implies that $\varphi_Y^{t_k}(x) \in U \setminus \gamma(-\epsilon,\epsilon)$ for k large enough. Hence, up to an extraction, if

$(s_k, y_k) := \Phi^{-1}(\varphi_Y^{t_k}(x))$, we may assume that (y_k) is a sequence of pairwise distinct elements converging to y. Since $\{\varphi_Y^t(x)\}_{t\in\mathbf{R}}$ is contained in $\mathcal{C}$, each connected component of $\{\varphi_Y^t(x)\} \cap U$ is contained in an integral curve of φ_X^t in U. Moreover, except eventually for $k = 0$, we have $\{\Phi(s, y_k),\ s \in (-\epsilon, \epsilon)\} \subset \{\varphi_Y^t(x)\}$. Indeed, the contrary implies that Y vanishes at some point $z = \Phi(s', y_k)$. This point being in $\mathcal{C}$, the same reasoning as above implies that for all $t \geq 0$, and for all $\varphi_Y^t(\Phi(s, y_k)) = \Phi(s' + e^{-2t}(s - s'), y_k)$. Applying this at $s = s_k$, the sequence (t_k) being decreasing, we get $y_0 = \cdots = y_k$, hence $k = 0$. So, for any $\sigma \in (-\epsilon, \epsilon)$ fixed in advance, we can adjust the times t_k such that for all $k \geq 1$, $s_k = \sigma$. Hence, for any choice of σ, there is a sequence (δ_k) such that $\varphi_Y^{t_k+\delta_k}(x) \to \gamma(\sigma)$.

For $\sigma \neq 0$, since Y does not vanish at $\gamma(\sigma)$, the dimension of the Kill_Y^{loc}-orbit of $\gamma(\sigma)$ is at least 1. Since we have just seen that $\gamma(\sigma) \in \overline{\mathcal{O}(x)}$, Theorem 3.3 implies that $\gamma(\sigma) \in \mathcal{O}(x)$. In particular, $\gamma(\sigma) \in \mathcal{O}(x)$ and $\gamma(-\sigma) \in \mathcal{O}(x)$. Therefore, we have $u_1, u_2 \in \mathbf{R}$ such that $\gamma(\sigma) = \varphi_Y^{u_1}(x)$ and $\gamma(-\sigma) = \varphi_Y^{u_2}(x)$. By symmetry, we may assume $u_1 > u_2$. Then, $\gamma(\sigma) = \varphi_Y^{u_1-u_2}(\gamma(-\sigma)) = \gamma(-e^{2(u_1-u_2)}\sigma) \neq \gamma(\sigma)$, a contradiction. $\square$

Consequently, since $Y_{x_1} \neq 0$ and $\varphi_Y^t(x_1) \to \alpha(s)$ as $t \to +\infty$ for some $s \in (-\epsilon, \epsilon)$, x_1 is not a periodic point of φ_Y^t. So, $\mathcal{O}(x_1)$ is not one-dimensional by Lemma 6.4, hence it must be a surface, which we will denote by Σ in the rest of this section.

6.2. **Values of $\omega(\hat{Y})$ over $\bar{\Sigma}$.** In this section, we determine the $\mathrm{Ad}(P)$-orbits corresponding to the values of $\omega_{\hat{x}}(\hat{Y})$ for all $\hat{x} \in \pi^{-1}(\bar{\Sigma})$. We still denote by x_0 and x_1 the points in the previous section. Recall the explicit form of $Y_0 = \omega_{\hat{x}_0}(\hat{Y}) \in \mathfrak{p}$ in (18).

Let X be a projection of $\tilde{X}$ defined on U. The horizontality of the curvature form of the Cartan geometry implies that $K_{\hat{x}_0}(\hat{Y}, \hat{X}) = 0$. By Lemma 2.1 of [BFM09], this implies that $2\omega_{\hat{x}_0}(\hat{X}) = \omega_{\hat{x}_0}([\hat{Y}, \hat{X}]) = -[\omega_{\hat{x}_0}(\hat{Y}), \omega_{\hat{x}_0}(\hat{X})] = -[Y_0, \omega_{\hat{x}_0}(\hat{X})]$. Hence, $\mathrm{ad}(Y_0).\omega_{\hat{x}_0}(\hat{X}) = -2\omega_{\hat{x}_0}(\hat{X})$, and considering the action of $\mathrm{ad}(Y_0)$ on the restricted root-space decomposition of $\mathfrak{o}(2, n)$, we get that $X_0 := \omega_{\hat{x}_0}(\hat{X}) \in \mathfrak{g}_{\beta-\alpha}$.

Since $[\hat{X}, \hat{Y}] = -2\hat{X}$, we have $(\varphi_{\hat{X}}^t)_* \hat{Y}_{\hat{x}_0} = \hat{Y}_{\varphi_{\hat{X}}^t(\hat{x}_0)} + 2t\hat{X}_{\varphi_{\hat{X}}^t(\hat{x}_0)}$. So, if we define $\hat{x}_1 := \varphi_{\hat{X}}^t(\hat{x}_0)$, then $\omega_{\hat{x}_1}(\hat{Y}) = Y_0 - 2tX_0$. We denote once and for all $Y_{\beta-\alpha} := -2tX_0$ so that $\omega_{\hat{x}_1}(\hat{Y}) = Y_{\beta-\alpha} + Y_0$.

By definition, the pseudo-group $\mathrm{Conf}_Y^{loc}(M, [g])$ acts transitively on Σ. Hence, for all $x \in \Sigma$, there are neighborhoods U, V of x_1 and x respectively and $f : U \to V$ conformal, such that $f^*Y = Y$ and $f(x_1) = x$. Consequently, $\omega_{\hat{f}(\hat{x}_1)}(\hat{Y}) = Y_{\beta-\alpha} + Y_0$, proving that $\omega_{\hat{x}}(\hat{Y}) \in \mathrm{Ad}(P).(Y_{\beta-\alpha} + Y_0)$ for all $\hat{x} \in \pi^{-1}(\Sigma)$.

Therefore, if $x_2 \in \bar{\Sigma}$, then for all $\hat{x}_2 \in \hat{M}_{x_2}$, $\omega_{\hat{x}_2}(\hat{Y}) \in \mathfrak{g}$ belongs to the closure of $\mathrm{Ad}(P).(Y_{\beta-\alpha} + Y_0)$ in $\mathfrak{g}$.

Lemma 6.5. *Let $Z \in \mathfrak{g}$ be an element in the closure of $\mathrm{Ad}(P)(Y_{\beta-\alpha} + Y_0)$. Then, up to conjugacy in P, Z is of the form $Z = Z_0$ or $Z = Z_{\beta-\alpha} + Z_0$, where $Z_{\beta-\alpha} \in \mathfrak{g}_{\beta-\alpha}$ and $Z_0 \in \mathfrak{a} \oplus \mathfrak{m}$ is of the form $Z_0 = \mathrm{diag}(\lambda, -\lambda, B, \lambda, -\lambda)$ with $\lambda \in \{\pm 1\}$ and $B \in \mathfrak{o}(n-2)$.*

Proof. Observe that $Y_{\beta-\alpha} + Y_0 \in \mathfrak{g}$, seen as a square matrix, is semi-simple and with characteristic polynomial $(T-1)^2(T+1)^2\chi_A(T)$, where $A \in \mathfrak{o}(n-2)$ as in (18). Therefore, the same is true for Z.

For $x \in \mathbf{R}^{2,n}$ isotropic and non-zero, we denote by $[x]$ its projection in $\mathbf{Ein}^{1,n-1}$. If $x, y \in \mathbf{R}^{2,n}$ span a totally isotropic two-plane, we denote by $\Delta(x, y)$ the projection of $\mathrm{Span}(x, y)$, *i.e.* the corresponding light-like geodesic of $\mathbf{Ein}^{1,n-1}$. We denote by $(e_0, \ldots, e_{n+1})$ the same basis of $\mathbf{R}^{2,n}$ as in Section 3.1.1.

More geometrically, $Y_{\beta-\alpha} + Y_0 \in \mathfrak{g}$ lies in the Lie algebra of the stabilizer in $O(2, n)$ of the pointed light-like geodesic $([e_1], \Delta(e_0, e_1))$. Let $p_k \in P$ such that $\mathrm{Ad}(p_k)(Y_{\beta-\alpha} + Y_0) \to Z$. Because P preserves the light-cone $\mathcal{C}_{[e_0]}$ of $[e_0]$ (i.e. the union of light-like geodesics passing through $[e_0]$), which contains $\Delta(e_0, e_1)$, the set of pointed light-like geodesics

$$K = \{([y], \Delta(e_0, [x])),\ x \in \mathcal{C}_{[e_0]} \setminus \{[e_0]\},\ [y] \in \Delta(e_0, x)\}$$

is compact and P-invariant. In fact, it contains exactly two P-orbits: the orbit of $([e_0], \Delta(e_0, e_1))$ and the orbit of $([e_1], \Delta(e_0, e_1))$.

Up to an extraction, $p_k.([e_1], \Delta(e_0, e_1))$ converges to a point that belongs to one of these two orbits. Hence, since P acts transitively on $\mathcal{C}_{[e_0]} \setminus \{[e_0]\}$, we have $p_0 \in P$ such that $Z' := \mathrm{Ad}(p_0)Z$ belongs to the Lie algebra of the stabilizer of either $([e_0], \Delta(e_0, e_1))$ (case 1) or $([e_1], \Delta(e_0, e_1))$ (case 2). For the sake of readability, we will consider that $Z = Z'$ is already in one of those two stabilizers. In both cases, Z is a $\mathbf{C}$-split matrix and with characteristic polynomial $(T-1)^2(T+1)^2\chi_A$.

- Case 1. In this situation, $Z = Z_{\mathfrak{a}} + Z_{\mathfrak{m}} + Z_u$, where $Z_u \in \mathfrak{g}_u := \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha-\beta} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha\oplus\beta}$. We let $Z_0 = Z_{\mathfrak{a}} + Z_{\mathfrak{m}}$. Concretely, Z has the upper triangular form
$$Z = \begin{pmatrix} \lambda & * & * & * & 0 \\ & \mu & * & 0 & * \\ & & B & * & * \\ & & & -\mu & * \\ & & & & -\lambda \end{pmatrix}, \text{ with } B \in \mathfrak{o}(n-2), \text{ and } Z_0 = \begin{pmatrix} \lambda & & & & \\ & \mu & & & \\ & & B & & \\ & & & -\mu & \\ & & & & -\lambda \end{pmatrix}.$$
Since the spectrum of B is purely imaginary, we have $\lambda, \mu \in \{\pm 1\}$. Since $\mathrm{ad}(Z_{\mathfrak{m}})|_{\mathfrak{g}_\beta} : \mathfrak{g}_\beta \to \mathfrak{g}_\beta$ is skew-symmetric with respect to the positive definite bilinear form $B_\theta = -B(\theta., .)$, its eigenvalues are purely imaginary. Hence, we can define $X_\beta := (\mathrm{ad}(Z_{\mathfrak{m}})|_{\mathfrak{g}_\beta} - \beta(Z_{\mathfrak{a}})\,\mathrm{id})^{-1} Z_\beta$, so that $[X_\beta, Z_0] = -Z_\beta$.
Note that $[\mathfrak{g}_u, \mathfrak{g}_u] \subset \mathfrak{g}_1 = \mathfrak{g}_{\alpha-\beta} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha\oplus\beta}$. Consequently, since $[X_\beta, Z] \in \mathfrak{g}_\beta \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\beta}$,
$$\begin{aligned}\mathrm{Ad}(e^{X_\beta})Z &= Z + [X_\beta, Z_0] + \underbrace{[X_\beta, Z_u] + \frac{1}{2}[X_\beta, [X_\beta, Z]]}_{\in \mathfrak{g}_1} \\ &= Z_0 + Z_1\end{aligned}$$
with $Z_1 \in \mathfrak{g}_1$. We decompose $Z_1 = Z_{\alpha-\beta} + Z_\alpha + Z_{\alpha+\beta}$. Applying if necessary $\mathrm{Ad}(p_1)$, where
$$p_1 = \begin{pmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & I_{n-2} & & \\ & 1 & & 0 & \\ & & & & 1 \end{pmatrix}$$
we can assume that $\lambda = \mu = \pm 1$ (remark that its action does not affect the current form of $\mathrm{Ad}(e^{X_\beta})Z$). Now, by the same argument as above, we can chose $X_\alpha \in \mathfrak{g}_\alpha$ such that $[X_\alpha, Z_0] = -Z_\alpha$ and if we set $X_{\alpha+\beta} = \frac{1}{(\alpha+\beta)(Z_{\mathfrak{a}})} Z_{\alpha+\beta}$, we get $[X_{\alpha+\beta}, Z_0] = -Z_{\alpha+\beta}$. Now,
$$\mathrm{Ad}(e^{X_\alpha + X_{\alpha+\beta}})\,\mathrm{Ad}(e^{X_\beta})Z = Z_0 + Z_1 + [X_\alpha + X_{\alpha+\beta}, Z_0] = Z_0 + Z_{\alpha-\beta}.$$
Since, $[Z_0, Z_{\alpha+\beta}] = 0$, Z_0 is semi-simple and $Z_{\alpha-\beta}$ is nilpotent, we must have $Z_{\alpha-\beta} = 0$ since $Z_0 + Z_{\alpha-\beta}$ is semi-simple. This concludes the proof in case 1.
- Case 2. In the coordinates associated with the basis $(e_0, \ldots, e_{n+1})$, Z has the upper triangular form by blocks
$$Z = \begin{pmatrix} \lambda & 0 & * & * & 0 \\ x & \mu & * & 0 & * \\ & & C & * & * \\ & & & -\mu & 0 \\ & & & -x & -\lambda \end{pmatrix} = Z_{\beta-\alpha} + Z_0 + Z_\beta + Z_\alpha + Z_{\alpha+\beta}.$$
with $Z_0 = Z_{\mathfrak{a}} + Z_{\mathfrak{m}} \in \mathfrak{a} \oplus \mathfrak{m}$. If $Z_{\beta-\alpha} = 0$ (*i.e.* $x = 0$), then we are reduced to case 1. So we may assume $x \neq 0$. By the same argument as above, $\lambda, \mu \in \{\pm 1\}$. Since Z is semi-simple, so is the block
$$\begin{pmatrix} \lambda & 0 \\ x & \mu \end{pmatrix}.$$

So, we must have $\mu = -\lambda \in \{\pm 1\}$. For the same reason as in case 1, $\mathrm{ad}(Z_0)|_{\mathfrak{g}_\alpha}$ and $\mathrm{ad}(Z_0)|_{\mathfrak{g}_\beta}$ are inversible. So, we can choose $X_\alpha \in \mathfrak{g}_\alpha$ such that $[X_\alpha, Z_0] = -Z_\alpha$. We then have

$$[X_\alpha, Z] = [X_\alpha, Z_{\beta-\alpha}] - Z_\alpha + [X_\alpha, Z_\beta]$$
$$[X_\alpha, [X_\alpha, Z]] = [X_\alpha, [X_\alpha, Z_{\beta-\alpha}]] \in \mathfrak{g}_{\alpha+\beta}$$

Higher iterations of $\mathrm{ad}(X_\alpha)$ on Z are 0, hence

$$\mathrm{Ad}(e^{X_\alpha})Z = Z_{\beta-\alpha} + Z_0 + Z'_\beta + Z'_{\alpha+\beta}$$

for some new components $Z'_\beta \in \mathfrak{g}_\beta$ and $Z'_{\alpha+\beta} \in \mathfrak{g}_{\alpha+\beta}$. Pick now $X_\beta \in \mathfrak{g}_\beta$ such that $[X_\beta, Z_0] = -Z'_\beta$. Then,

$$[X_\beta, \mathrm{Ad}(e^{X_\alpha})Z] = -Z'_\beta$$
$$[X_\beta, [X_\beta, \mathrm{Ad}(e^{X_\alpha})Z]] = 0,$$

hence

$$\mathrm{Ad}(e^{X_\beta}e^{X_\alpha})Z = Z_{\beta-\alpha} + Z_0 + Z'_{\alpha+\beta}.$$

Finally, since $\mathrm{Ad}(e^{X_\beta}e^{X_\alpha})Z$ and $Z_{\beta-\alpha} + Z_0$ are semi-simple, $Z'_{\alpha+\beta}$ is nilpotent and $[Z_{\beta-\alpha} + Z_0, Z'_{\alpha+\beta}] = 0$ (recall $\lambda + \mu = 0$), we must have $Z'_{\alpha+\beta} = 0$, concluding the proof in case 2.

□

Consequently, we obtain that for any boundary point $x_2 \in \partial\Sigma$, either x_2 is a linear singularity of Y of balanced type, or $Y_{x_2} \neq 0$ and for some $\hat{x}_2 \in \hat{M}_{x_2}$, $\omega_{\hat{x}_2}(\hat{Y}) = Z_{\beta-\alpha} + Z_0$ with Z_β and Z_0 as in Lemma 6.5. By Lemma 6.4, since $\dim \mathcal{O}(x_2) \leq 1$, we get that φ_Y^t has a periodic orbit at x_2 in this second situation.

6.3. **Periodic orbits.** We start by ruling out the case of a periodic orbit of φ_Y^t, not only in the boundary of Σ, but also inside Σ.

Proposition 6.6. *The flow $\{\varphi_Y^t\}$ has no periodic orbit on the closure $\bar{\Sigma}$.*

Proof. Note that if we have such a periodic orbit at some point $x \in \bar{\Sigma}$, then there always is $\hat{x} \in \hat{M}_x$ such that $\omega_{\hat{x}}(\hat{Y}) = Z_{\beta-\alpha} + Z_0$, according to Lemma 6.5 and the paragraph that precedes it (we can choose $Z_{\beta-\alpha} = Y_{\beta-\alpha}$ and $Z_0 = Y_0$ if $x \in \Sigma$).

We will show slightly more generally that if Y has a non-singular periodic orbit $\{\varphi_Y^t(x)\}_{t\in\mathbf{R}}$ for some $x \in \mathcal{C}$ and that $\omega_{\hat{x}}(\hat{Y})$ is of the form $Z_{\beta-\alpha} + Z_0$ as above, for some $\hat{x} \in \hat{M}_x$, then an open subset of M is conformally flat, contradicting the standing hypothesis of non-conformal flatness of M.

Let $t_0 > 0$ be such that $\varphi_Y^{t_0}(x) = x$, and let $f := \varphi_Y^{t_0}$. Note that since Y has a balanced singularity at x_0 by assumption, the topology of $\mathrm{Conf}(M,[g])$ makes that the one-parameter subgroup $\{\varphi_Y^t\}_{t\in\mathbf{R}}$ is properly embedded in $\mathrm{Conf}(M,[g])$. In fact, this is equivalent to saying that for any $\hat{x} \in \hat{M}$, the map $\{t \in \mathbf{R} \mapsto \widehat{\varphi_Y^t}(\hat{x}) \in \hat{M}\}$ is proper, and since $\widehat{\varphi_Y^t}(\hat{x}_0) = \hat{x}_0.e^{tY_0}$, the claim follows.

In particular, the sequence $\hat{f}^n(\hat{x})$ goes to infinity as $n \to \pm\infty$. So, if $q = \mathrm{hol}^{\hat{x}}(f) \in P$ (recall that it is defined as the unique element of P such that $\hat{f}(\hat{x}).q^{-1} = \hat{x}$), then $(q^n) \to \infty$ in P. We now analyse the possibilities for q. Since $\hat{f}_*\hat{Y} = \hat{Y}$, it follows that $\mathrm{Ad}(q).\omega_{\hat{x}}(Y) = \omega_{\hat{x}}(Y)$.

Lemma 6.7. *Up to conjugacy in P, any element $p \in P$ centralizing $Z_{\beta-\alpha} + Z_0$ is either of the form*

$$p = \mathrm{diag}(\lambda, \lambda, R, \lambda^{-1}, \lambda^{-1}),\ |\lambda| \neq 1 \text{ or } p = \pm\,\mathrm{diag}(1,1,R,1,1)e^{X_{\alpha+\beta}}.$$

Proof. Write $p = g_0 e^{X_1}$ with $g_0 \in G_0$ and $X_1 \in \mathfrak{g}_1 = \mathfrak{g}_{\alpha-\beta} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\beta}$.

Consider first the action of $\mathrm{Ad}(p)$ on $Z_0 + Z_{\beta-\alpha}$ modulo $\mathfrak{p}$. We get $\mathrm{Ad}(g_0)Z_{\beta-\alpha} = Z_{\beta-\alpha}$ mod.$\mathfrak{p}$, hence $\mathrm{Ad}(g_0)Z_{\beta-\alpha} = Z_{\beta-\alpha}$ since $\mathrm{Ad}(G_0)$ preserves $\mathfrak{g}_{-1}$. It follows that g_0 is of the form

$$g_0 = \begin{pmatrix} \lambda & & & & \\ & \lambda & & & \\ & & R & & \\ & & & \lambda^{-1} & \\ & & & & \lambda^{-1} \end{pmatrix} e^{X'_\beta} =: p_0 e^{X'_\beta}.$$

for some $X'_\beta \in \mathfrak{g}_\beta$.

Hence, $p = p_0 e^{X_u}$, where $X_u \in \mathfrak{g}_u := \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha-\beta} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\beta}$. We then have

$$\begin{aligned} Z_{\beta-\alpha} + Z_0 &= \mathrm{Ad}(p)(Z_{\beta-\alpha} + Z_0) \\ &= \mathrm{Ad}(p_0)\left(Z_{\beta-\alpha} + Z_0 + \sum_{k\geq 1} \frac{\mathrm{ad}(X_u)^k}{k!}(Z_{\beta-\alpha} + Z_0)\right) \\ &= Z_{\beta-\alpha} + \mathrm{Ad}(p_0)\left(Z_0 + \sum_{k\geq 1} \frac{\mathrm{ad}(X_u)^k}{k!}(Z_{\beta-\alpha} + Z_0)\right) \end{aligned}$$

Decompose now $X_u = X_\beta + X_{\alpha-\beta} + X_\alpha + X_{\alpha+\beta}$. Observe that, $[X_u, Z_{\beta-\alpha} + Z_0] = [X_{\alpha-\beta}, Z_{\beta-\alpha}]$ mod.$\mathfrak{g}_u$, implying $\frac{\mathrm{ad}(X_u)^k}{k!}(Z_{\beta-\alpha} + Z_0) \in \mathfrak{g}_u$ for all $k \geq 2$ since $[X_{\alpha-\beta}, Z_{\beta-\alpha}] \in \mathfrak{a}$. Therefore, if we decompose $Z_0 = Z_\mathfrak{a} \oplus Z_\mathfrak{m}$, then we get

$$Z_\mathfrak{a} + Z_\mathfrak{m} = Z_\mathfrak{a} + \mathrm{Ad}(p_0)Z_\mathfrak{m} + \mathrm{Ad}(p_0)\left([X_{\alpha-\beta}, Z_{\beta-\alpha}] + Z_u\right) \text{ for some } Z_u \in \mathfrak{g}_u. \tag{19}$$

Hence, considering components on $\mathfrak{m}$ and $\mathfrak{a}$ respectively, we get $Z_\mathfrak{m} = \mathrm{Ad}(p_0)Z_\mathfrak{m}$ and $[X_{\alpha-\beta}, Z_{\beta-\alpha}] = 0$, which in turn implies that $X_{\alpha-\beta} = 0$ because the bracket $\mathfrak{g}_{\alpha-\beta} \times \mathfrak{g}_{\beta-\alpha} \to \mathfrak{a}$ is non-degenerate.

Equation (19) then reads $Z_u = 0$. We now develop this term in details. By definition,

$$Z_u = \sum_{k\geq 1} \frac{(\mathrm{ad}(X_u))^k}{k!}(Z_{\beta-\alpha} + Z_0), \text{ where } X_u = X_\beta + X_\alpha + X_{\alpha+\beta}.$$

Note that $[X_{\alpha+\beta}, Z_0] = 0$ due to the form of Z_0. Developping the expression, we obtain

$$\begin{aligned} &\sum_{k\geq 1} \frac{(\mathrm{ad}(X_u))^k}{k!} Z_0 = [X_\beta, Z_0] + [X_\alpha, Z_0] + \frac{1}{2}\left([X_\beta, [X_\alpha, Z_0]] + [X_\alpha, [X_\beta, Z_0]]\right) \\ &\sum_{k\geq 1} \frac{(\mathrm{ad}(X_u))^k}{k!} Z_{\beta-\alpha} = [X_\alpha, Z_{\beta-\alpha}] + \frac{1}{2}[X_\alpha, [X_\alpha, Z_{\beta-\alpha}]]. \end{aligned}$$

Considering the components on $\mathfrak{g}_\alpha$, we get that $[X_\alpha, Z_0] = 0$. Since $\alpha(Z_\mathfrak{a}) = \pm 1$, it implies $[Z_\mathfrak{m}, X_\alpha] = \pm X_\alpha$. But since $\mathrm{ad}(Z_\mathfrak{m})|_{\mathfrak{g}_\alpha} : \mathfrak{g}_\alpha \to \mathfrak{g}_\alpha$ is skew-symmetric with respect to the positive definite form $B_\theta = -B(.,\theta.)$, it has purely imaginary eigenvalues, hence $X_\alpha = 0$. It then follows $[X_\beta, Z_0] = 0$, and $X_\beta = 0$ by the same argument because $\beta(Z_\mathfrak{a}) = \pm 1$.

Finally, $X_u = X_{\alpha+\beta} \in \mathfrak{g}_{\alpha+\beta}$. Hence, $\mathrm{Ad}(p_0)X_u = \lambda^2 X_u$. Consequently, for all $t \in \mathbf{R}$, $e^{tX_u}p_0 = p_0 e^{\lambda^{-2}tX_u}$, so

$$e^{tX_u} p e^{-tX_u} = e^{tX_u}(p_0 e^{X_u})e^{-tX_u} = p_0 e^{(t(\lambda^{-2}-1)+1)X_u}.$$

Consequently, if $|\lambda| \neq 1$, and if we set $t = \frac{\lambda^2}{\lambda^2 - 1}$, the conjugate of p by e^{tX_u} equals p_0 as announced. □

Let $\hat{x} \in \hat{M}_x$ as chosen previously. Consider $P^{\hat{x}} < P$ as defined before Proposition 3.5. Then, $q \in P^{\hat{x}}$. Moreover, $\mathrm{Ad}_\mathfrak{g}(P^{\hat{x}})$ being algebraic, the Jordan decomposition of $\mathrm{Ad}_\mathfrak{g}(q)$ is internal to $\mathrm{Ad}_\mathfrak{g}(P^{\hat{x}})$. So, since either $\mathrm{Ad}_{\mathfrak{g}}(q) = \mathrm{Ad}_{\mathfrak{g}}(q_h)\,\mathrm{Ad}_{\mathfrak{g}}(q_e)$ or $\mathrm{Ad}_{\mathfrak{g}}(q) = \mathrm{Ad}_{\mathfrak{g}}(q_u)\,\mathrm{Ad}_{\mathfrak{g}}(q_e)$ where

$$q_h := \mathrm{diag}(\lambda, \lambda, 1, \cdots, 1, \lambda^{-1}, \lambda^{-1}),\ q_e := \mathrm{diag}(1, 1, R, 1, 1),\ q_u := e^{X_{\alpha+\beta}},$$

and the products being commutative, we obtain that $\mathrm{Ad}_{\mathfrak{g}}(q_h), \mathrm{Ad}_{\mathfrak{g}}(q_u) \in \mathrm{Ad}_{\mathfrak{g}}(P^{\hat{x}})$ in the first case and second case respectively. Note that in the second case, $X_{\alpha+\beta} \neq 0$ since $\{q^n\}$ has not compact closure in P.

In the first case, the Zariski closure of $\langle \mathrm{Ad}_{\mathfrak{g}}(q_h) \rangle$ is $\{\mathrm{Ad}_{\mathfrak{g}}(\mathrm{diag}(x, x, 1 \dots, 1, x^{-1}, x^{-1})),\ x \in \mathbf{R}^*\}$, so $\mathrm{ad}_{\mathfrak{g}}(\mathrm{diag}(1, 1, 0, \dots, 0, -1, -1)) \in \mathrm{ad}_{\mathfrak{g}}(\mathfrak{p}^{\hat{x}})$ in the first case, and similarly we get $\mathrm{ad}_{\mathfrak{g}}(X_{\alpha+\beta}) \in \mathrm{ad}_{\mathfrak{g}}(\mathfrak{p}^{\hat{x}})$ in the second case. Hence, $\mathrm{ad}_{\mathfrak{g}}$ being injective, we obtain by definition of $P^{\hat{x}}$ that there exists Z a local conformal vector field defined in the neighborhood of x and such that $\omega_{\hat{x}}(\hat{Z}) = \mathrm{diag}(1, 1, 0, \dots, 0, -1, -1)$ in the first case, and $\omega_{\hat{x}}(\hat{Z}) = X_{\alpha+\beta}$ in the second case.

The second case is not possible, because $\hat{Z}$ is a conformal vector field of a neighborhood of x in M, with a second order singularity at x. It would in particular be non-linearizable at x, and Theorem 2.1 would imply conformal flatness of M by analyticity, which is excluded.

In the first case, Z would have a linear balanced dynamics near x, yielding a polarization $\mathcal{D}'$ of the conformal structure on M by point 3(b) of Proposition 5.1. On the other hand, the singular locus of Z coincides locally with the periodic orbit of x. Because, $x \in \bar{\Sigma}$, the tangent to this periodic orbit at x is $\mathcal{D}_x$. Point 3(d) of Proposition 5.1 then ensures that $\mathcal{D} \neq \mathcal{D}'$, and Lemma 5.5 implies that M is conformally flat, concluding the proof. □

6.4. **Final contradiction.** Hence, we obtain that any boundary point $x \in \partial\Sigma$ is a singularity of Y. By Lemma 6.5, there exists $\hat{x} \in \hat{M}_x$ such that $\omega_{\hat{x}}(\hat{Y}) = Z_0$ with Z_0 as in Lemma 6.5, *i.e.* x is a balanced linear singularity of Y. In particular, in some coordinates of T_xM,

$$D_x\varphi_Y^t = \begin{pmatrix} 1 & & \\ & e^{\epsilon t}R^t & \\ & & e^{\epsilon 2t} \end{pmatrix}$$

for some $\epsilon \in \{\pm 1\}$ and $\{R^t\}$ a one-parameter subgroup of $O(n-2)$. Note that the algebraic analysis of Lemma 6.5 cannot provide the value of the sign ϵ: some singularities of Y could be attracting, and others repelling. But since $[\tilde{Y}, \tilde{X}] = 2\tilde{X}$, if we pick a neighborhood of x where a representative X of $\tilde{X}$ can be defined, then $D_x\varphi_Y^t X_x = e^{-2t}X_x$, proving that $\epsilon = -1$ for every singularity $x \in \partial\Sigma$.

Hence, every $x \in \partial\Sigma$ has a fundamental system of neighborhoods which are all stable under the action of the semi-group $\{\varphi_Y^t\}_{t\geq 0}$. For instance, in the linearizing chart, any Euclidean cylindrical neighborhood around the axis corresponding to the line of fixed points of $D_x\varphi_Y^t$ is stable for positive times (and in fact collapses to the segment of fixed points in the domain of the chart).

It follows that for all $y \in \Sigma$, $\alpha(y) \cap \partial\Sigma = \emptyset$, where $\alpha(y)$ denotes the α-limit set of y with respect to the flow $\{\varphi_Y^t\}$. Indeed, if $x \in \partial\Sigma$, we can find U a neighborhood of x such that $y \notin U$ and which is stable under $\{\varphi_Y^t\}_{t\geq 0}$. It follows that $\{\varphi_Y^t(y),\ t \leq 0\} \cap U = \emptyset$, hence $\alpha(y) \cap U = \emptyset$, and $x \notin \alpha(y)$.

For an arbitrary $y \in \Sigma$, consider $K \subset \alpha(y)$ a compact, minimal, $\{\varphi_Y^t\}$-invariant subset. We have seen above that $K \subset \Sigma$. Observe that K must have empty interior (for the induced topology of Σ), because if not, its boundary would be empty by minimality, proving that $K = \Sigma$ by connectedness. But this is not possible since we have seen that any $x \in \partial\Sigma$ has a neighborhood disjoint from $\alpha(y)$, proving that $\alpha(y) \neq \Sigma$. We finally invoke the following generalization of Poincaré-Bendixson Theorem to closed surfaces, due to R. Schwartz.

Theorem 6.8 ([Sch63]). *Let S be a surface and $\{\varphi^t\}$ a $\mathcal{C}^1$ flow of S. If $K \subset S$ is a compact, minimal, invariant subset, with empty interior, then K is either a singularity or a periodic orbit of the flow.*

Actually, in the statement of the main theorem of [Sch63] there is a compactness assumption on the surface. Nevertheless, the proof only requires a compact minimal subset, so it in fact demonstrates the statement above.

Consequently, since Y has no singularity in Σ, we obtain that K is a periodic orbit of $\{\varphi_Y^t\}$ contradicting Proposition 6.6.

Alternatively, the double of $\bar{\Sigma}$ along its boundary admits a vector field agreeing with Y on each copy of $\bar{\Sigma}$. The theorem of [Sch63] applied *verbatim* to this closed surface gives a contradiction as above.

We conclude that $(M,[g])$ is conformally flat, and the proof of Proposition 6.1 is complete.

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