(Chapter 20.) Kissing numbers

Optional course: Discrete and polyhedral geometry

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The Kepler Problem - Sphere packing

Kepler: "There must be some definite cause why, whenever snow begins to fall, its initial formations invariably display the shape of a six-cornered starlet. For if it happens by chance, why do they not fall just as well with five corners or with seven? Why always with six...?"
(*The Six-Cornered Snowflake.*)

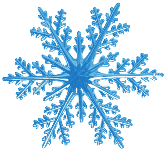


Figure : A typical dendritic snowflake.

Newton and Gregory problem

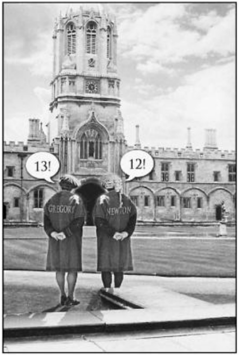


Figure : The Oxford discussion of David Gregory (1659-1708) and Isaac Newton (1643-1272) on "kissing spheres".

Problem

How many unit spheres can simultaneously touch a given sphere of the same size?

Definition - Kissing numbers

K_d is the d-dimensional *kissing numbers*, defined as the largest number of unit balls touching a fixed unit ball in $\mathbb{R}^d$.

Note that K_d can be also viewed as the largest number of points on a unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ with pariwise distances ≥ 1.

Example - "Kissing spheres" in *n*-dimensional space.

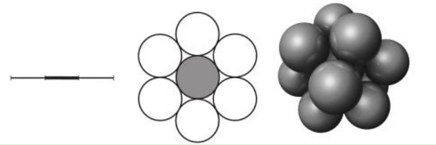


Figure : $K_1 = 2, K_2 = 6$ and $K_3 = 12$.

Proposition 20.1.

$K_1 = 2$ and $K_2 = 6$.

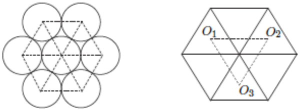


Figure : Six touching circles and centers of twelve touching sphere.

Proposition 20.2.

$12 \leq K_3 \leq 14.$

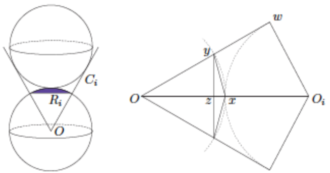


Figure : Computing the height of a cap C_i on a sphere.

Euler's formula

Theorem 25.1. (Euler's formula)

Let $P \subset \mathbb{R}^3$ be an convex polytope, and let V, E, F denote the set of vertices, edges, and faces of P, respectively. The classical Euler's formula is: $|V| - |E| + |F| = 2$.

Corollary 25.1.

Let P be a simplicial polytope with n vertices. Then P has $3n - 6$ edges and $2n - 4$ faces.

The area of spherical polygons.

We consider a unit sphere $\mathbb{S}^2$ with center at the origin O.
Recall that area($\mathbb{S}^2$)$= 4\pi$.

Theorem 41.1. (Girard's formula)

Let T be a spherical triangle with angles α, β and γ. Then area(T)$= \alpha + \beta + \gamma - \pi$.

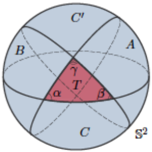


Figure : The area for spherical triangles.

Proof.

Let A, B and C be the triangular regions attached to the triangle as in Figure. Observe that area$(A \cup T)$=$\frac{\alpha}{2\pi} \times \text{area}(\mathbb{S}^2) = 2\alpha$, area$(B \cup T)$=$2\beta$, and area$(C \cup T)$=$2\gamma$. Now for the upper hemisphere H we have:

$$\begin{aligned}
\text{area}(H) &= \text{area}(A) + \text{area}(B) + \text{area}(C') + \text{area}(T) \\
&= (\text{area}(A) + \text{area}(T)) + (\text{area}(B) + \text{area}(T)) \\
&\quad +(\text{area}(C) + \text{area}(T)) - 2\text{area}(T)) \\
&= 2(\alpha + \beta + \gamma - \text{area}(T)).
\end{aligned}$$

On the other hand, area$(H) = \text{area}(S)/2 = 2\pi$, which implies the result. $\square$

Corollary 1.7.

Let $z_1, \ldots, z_n \in \mathbb{R}^2$ be points in the plane, such that every three of them can be covered by a circle of radius r. Then all points can be covered by a circle of radius r.

Corollary 1.8.

Let $z_1, \ldots, z_n \subset \mathbb{R}^2$ be points in the plane, such that all pairwise distances $|z_i z_j|$ are at most 1. Then all points can be covered by a circle of radius $\frac{1}{\sqrt{3}}$.

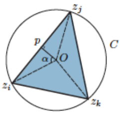


Figure : Computing the raduis of the cricumscribed circle C.

Proof.

By Corollary 1.7., it suffices to show that every 3 points z_i, z_j and z_k can be covered with a circle of radius $\frac{1}{\sqrt{3}}$. There are 3 cases: triangle $T = (z_i z_j z_k)$ is either acute, right or obtuse. In the last two case take a circle C centered at the midpoint of the longest edge. Clearly, the radius of C is at most $\frac{1}{2} < \frac{1}{\sqrt{3}}$.

When T is acute, the proof is apparent from Figure. Let O be the center of the circumscribed circle C. Suppose $\angle z_i O z_j$ is the largest angle in T and let $\alpha = \angle z_i O p$, be as in the figure. Then $\alpha = \frac{1}{2}(\angle z_i O z_j) \geq \frac{\pi}{3}$. On the other hand, $|z_i p| = \frac{1}{2}|z_i z_j| \leq \frac{1}{2}$, by assumption. Therefore, radius$(C) = |z_i O| = \frac{|z_i p|}{\sin(\alpha)|} \leq \frac{1}{\sqrt{3}}$, as desired. □

Kissing's numbers in third dimensions

Theorem 20.3.

$K_3 \leq 13.$

Lemma 20.4.

For the caps $R_1, \ldots, R_{14}$ defined as above, we have:

(1) $\text{area}(R_i) > 0.918\sigma \quad \forall 1 \leq i \leq 14;$

(2) $\text{area}(R_i \cap R_j) < 0.0068\sigma \quad \forall 1 \leq i < j \leq 14;$

(3) $\text{area}(R_i \cap R_j \cap R_k) = 0 \quad \forall 1 \leq i < j < k \leq 14;$

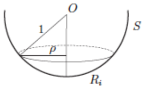


Figure : Computing the area of spherical caps R_i.

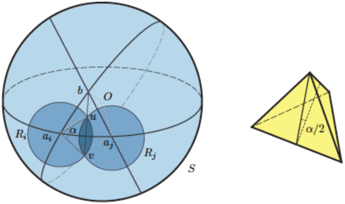


Figure : Computing the area of lenses $R_i \cap R_j$.

Lemma 20.5.

For all faces $(a_i a_j a_k)$ in P, the radius$(C_{ijk}) \geq \rho$, where $\rho = \frac{1}{\sqrt{3}}$.

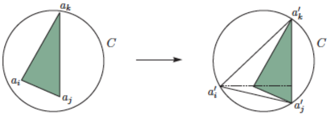


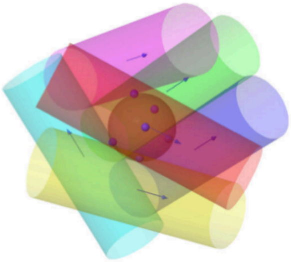
Figure : Expanding triangle $(a_i a_j a_k) \to (a'_i a'_j a'_k)$.

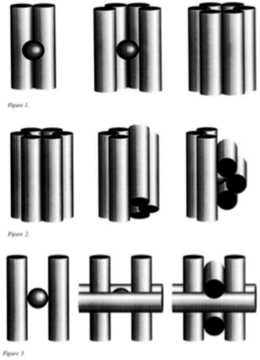
Dimension	Best known lower bound	Best know upper bound
3	**12**	**12**
4	**24**	**24**
5	40	45
6	72	78
7	126	134
8	**240**	**240**
9	306	364
10	500	554
11	582	870
12	840	1357
13	1154	2069
14	1606	3183
15	2564	4866
16	4320	7355
17	5346	11072
18	7398	16572
19	10668	24812
20	17400	36764
21	27720	54584
22	49896	82340
23	93150	124416
24	**196560**	**196560**

Table : A survey on the kissing numbers.

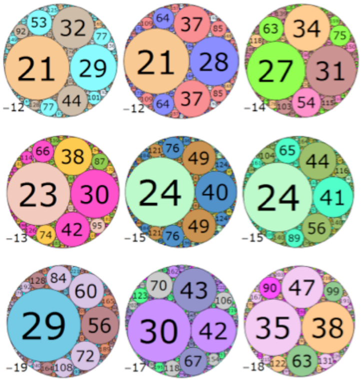
Examples and Applications.

- Kissings cylinders,





- Descartes' theorem - circles,



- Theory of digital codes (Cryptographie and number theory),
  - Block code is any member of the large and important family of error-correcting codes that encode data in blocks.
    Increase the dimensions: the number of near neighbors increases very rapidly, this value is given by the kissing numbers => the number of ways for noise to make the receiver choose a neighbor (hence an error) grows as well. This is a fundamental limitation of block codes,
  - (Binary) Golay codes is a type of linear error-correcting code used in digital communications.
    Practical application : NASA deep space missions, Radio communications, etc.,

Thank you for your attention.