Convex Integration in 1 Dimension

Why the 1 dimensional case ?

- The theory of Convex Integration is based on a 1D formula.

Aim of this talk:

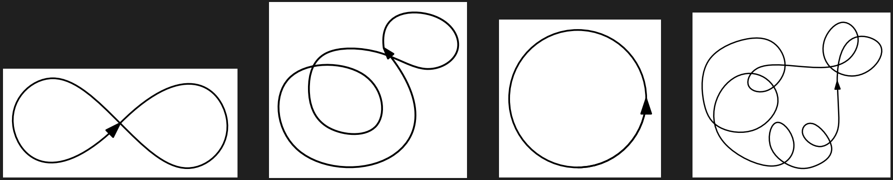
I The example of Thurston's corrugation

II The Corrugation Process (1D case)

Whitney-Graustein theorem (1937)

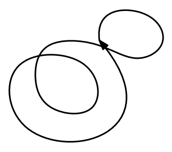
If two closed immersed curves f, $g : [0,1] \to \mathbb{R}^2$ have the same turning number, there exists a regular homotopy in $\mathbb{R}^2$ from f to g.

- The turning number is the number of turns of the derivative.



0 +1 +1 +3

So, it's possible to deform continuously this two curves without singular point!



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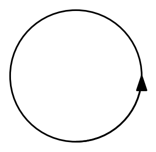

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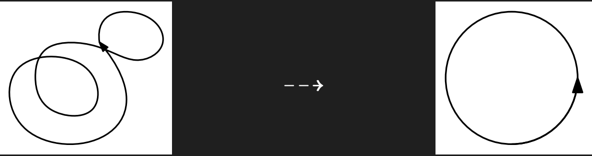
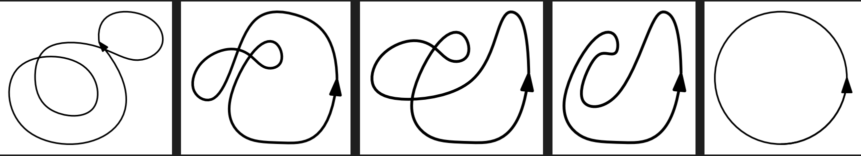


A possibility



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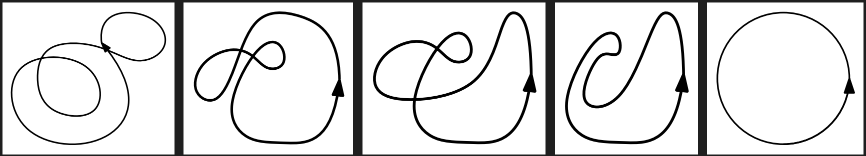


Ok, but..., **is there an algorithm to do this easily?**

Definition (Thurston's corrugations)

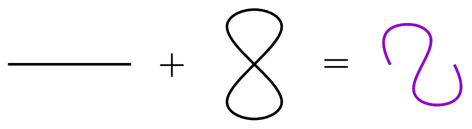
Let consider

- an initial curve $f_0 : [0,1] \to \mathbb{C}$
- a number of corrugation/oscillations $N \in \mathbb{N}^*$

then Thurston's corrugations are defined by

$$f_1(x) \quad = \quad f_0(x) + \frac{1}{N} r \left[- \sin(4\pi N x)\alpha(x) + 2\sin(2\pi N x) i\alpha(x) \right]$$

where $\overline{\gamma}_x$ is the average of γ_x, $r > 0$ an amplitude, and $\alpha(x)$ a vector.

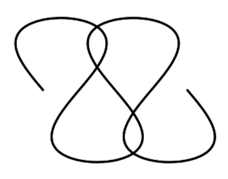


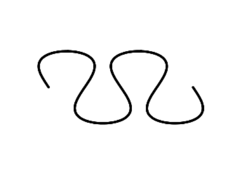
picture from *Making waves*, Lévy, Thurson.

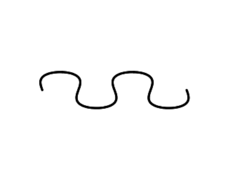
Let

$$f_0(x) = x, \quad N = 2$$

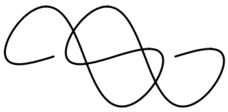
Then f_1 is







$r = \frac{1}{2},\ \alpha(x) = \frac{f_0'(x)}{\|f_0'(x)\|}$, $\quad r = \frac{1}{4},\ \alpha(x) = \frac{f_0'(x)}{\|f_0'(x)\|}$, $\quad r = \frac{1}{8},\ \alpha(x) = \frac{f_0'(x)}{\|f_0'(x)\|}$,

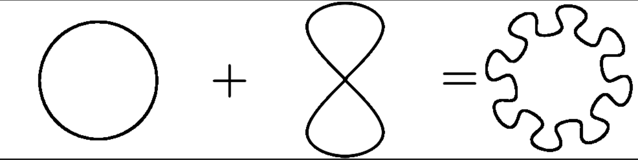


$$r = \frac{1}{4},\ \alpha(x) = \exp(i2\pi/6)\frac{f_0'(x)}{\|f_0'(x)\|}.$$

Let

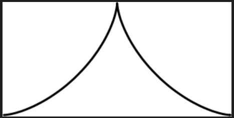
$$f_0(x) = e^{i2\pi x}, \quad N = 8, \quad r = 2, \quad \alpha(x) = \frac{f_0'(x)}{\|f_0'(x)\|}.$$

Then f_1 is



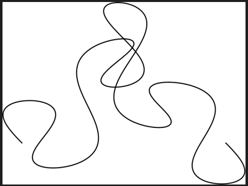
Let

$$f_0(x) = \cos^3(\pi x) + i \sin^3(\pi x)$$

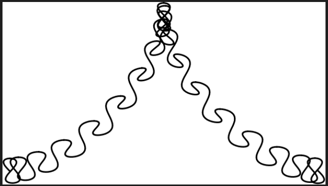


which is singular at $x = \frac{1}{2}$, $x \in]0, 1[$.

For $r = 0.8$, $\alpha(x) = -1 \in \mathbb{C}$, then f_1 is



$N = 4$



$N = 20$.

We have succeeded in **removing a singular point** !

What's happening ?

For curves from $[0,1]$ to $\mathbb{R}^2 \simeq \mathbb{C}$, the 1-jet space is

$$J^1([0,1],\mathbb{C}) = [0,1] \times \mathbb{C} \times \mathbb{C}.$$

A curve f is an immersion if

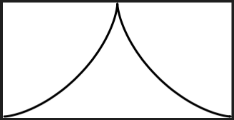
$$f'(x) \neq 0, \quad \forall x \in [0,1]$$

The differential relation associated to this constraint is

$$\mathscr{R}_{im} \;:=\; \{(x,y,v_1) \,|\, v_1 \neq 0\} = [0,1] \times \mathbb{C} \times (\mathbb{C}\backslash\{0\})$$

Let consider the formal solution

$$\sigma(x) = (\ x,\ f_0(x) = \cos^3(\pi x) + i\sin^3(\pi x),\ v_1(x) = -1\ )$$



With $\alpha(x) = v_1(x)$, we set

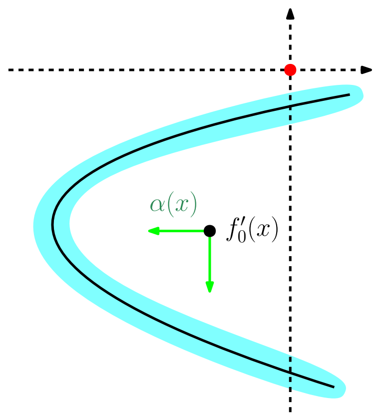
$$f_1(x) \quad = \quad f_0(x) + \frac{1}{N} r \Big[-\sin(4\pi N x)\alpha(x) + 2\sin(2\pi N x) i\alpha(x) \Big]$$

whose derivative is

$$f_1'(x) \quad = \quad f_0'(x) + r4\pi[-\cos(4\pi N x)\alpha(x) + \cos(2\pi N x) i\alpha(x)] + O(\tfrac{1}{N})$$

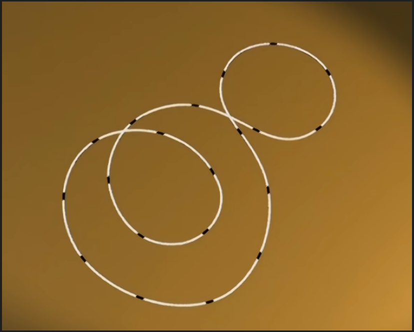
Let's draw f_1'

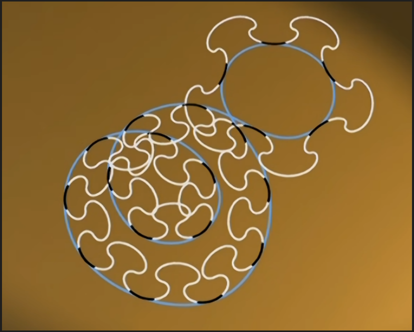
$$f_1'(x) = f_0'(x) + r4\pi[-\cos(4\pi Nx)\alpha(x) + \cos(2\pi Nx)i\alpha(x)] + O(\tfrac{1}{N})$$
$$= f_0'(x) + r\,[\text{ an arc of parabola }] + O(\tfrac{1}{N})$$



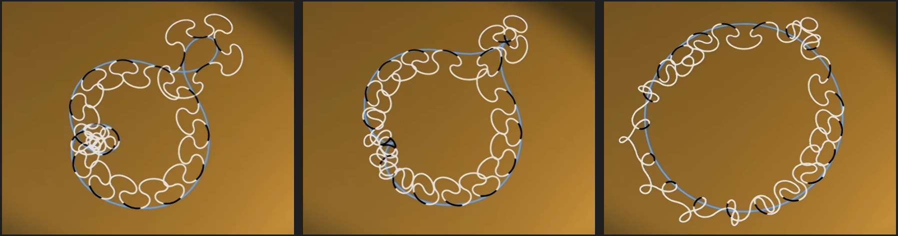
So, if the parameter r is large enough, f_1' **is an immersion !**

We add corrugations

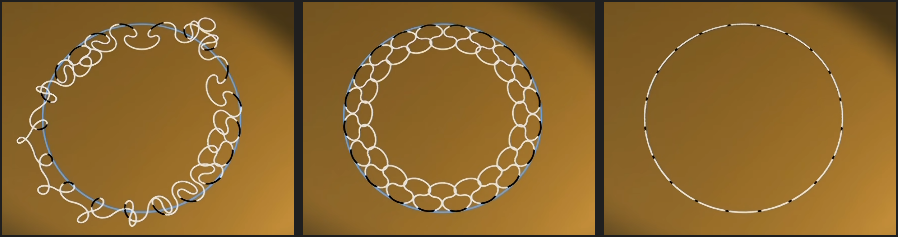




we do a linear homotopy (between the two blue curves)

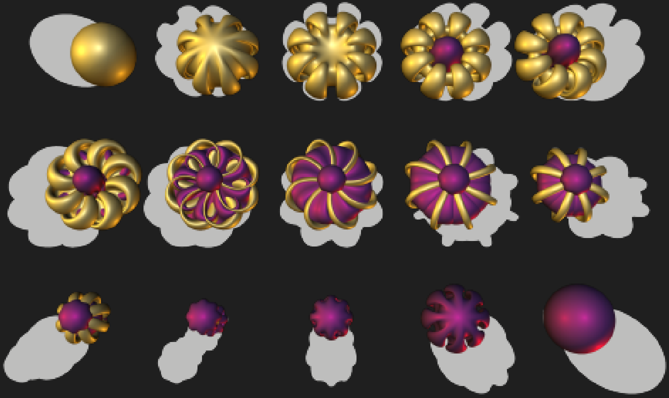


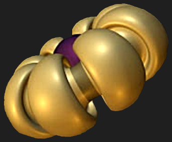
we unravel the corrugations, and then remove them.



video *Outside In*

Thurston’s sphere eversion





video *Outside In*

article *Making Waves*, S. Lévy, W. Thurston

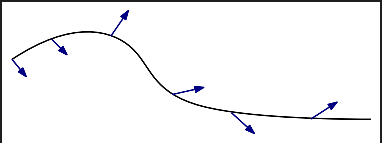
II - The Corrugation process

Let $J^1([0,1],\mathbb{R}^n)$ be the 1-jet space of curves from $[0,1]$ to $\mathbb{R}^n$. We consider a relation

$$\mathscr{R} \subset J^1([0,1],\mathbb{R}^n)$$

and a formal solution

$$\sigma : x \mapsto (x, f_0(x), v_1(x)) \in \mathscr{R}.$$



We want to generalize the former approach to build an holonomic solution of $\mathscr{R}$ from a formal one.

Definition (Corrugation process for curves, T. 2022)
Let

- $f_0 : [0,1] \to \mathbb{R}^n$ be an initial curve,
- $N \in \mathbb{N}^*$ be a number,
- $(\gamma_x(\cdot))_{x\in[0,1]}$ be a family of loops,

then the map given by

$$f_1(x) := f_0(x) + \frac{1}{N}\int_{s=0}^{Nx}\left(\gamma_x(s) - \overline{\gamma_x}\right)ds$$

where $\overline{\gamma}_x = \int_0^1 \gamma_x(s)ds$ is the average of $\gamma_x(\cdot)$, is said to be obtained from f_0 by a **Corrugation Process**.

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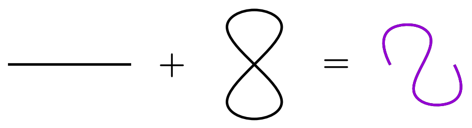
The number $N \in \mathbb{N}^*$ is called the number of corrugations.

Note that the choice of

$$\gamma_x(t) = r4\pi[-\cos(4\pi t)\alpha(x) + \cos(2\pi t)i\alpha(x)]$$

in the formula of the Corrugation process leads to Thurston's corrugations.



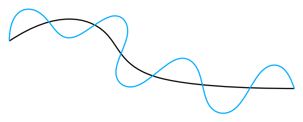
picture from *Making waves*, Lévy, Thurson.

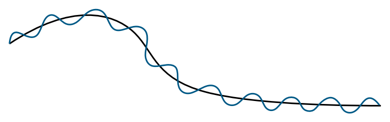
Properties of

$$f_1(x) := f_0(x) + \frac{1}{N}\int_{s=0}^{Nx} \Big(\gamma_x(s) - \overline{\gamma_x}\Big)\, ds$$

- **C^0-closeness**: for any $x \in [0,1]$, we have

$$f_1(x) = f_0(x) + O(1/N) \quad \Leftrightarrow \quad \|f_1 - f_0\|_{C^0} = O(1/N)$$





- **derivative**: If $\overline{\gamma}_x = f_0'(x)$ for any x, we have

$$f_1'(x) = \gamma_x(Nx) + O(1/N)$$

$$\left(\quad f_1(x) := f_0(x) + \frac{1}{N}\int_{s=0}^{Nx}\left(\gamma_x(s) - \overline{\gamma_x}\right)ds \quad \right)$$

Proof of the C^0-closeness.

- $\gamma_x(\cdot)$ is 1-periodic,
- $\gamma_x(\cdot) - \overline{\gamma_x}$ is 1-periodic of average $= 0$,

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- $f_1(x) = f_0(x) + O(\frac{1}{N})$

$$\left(\quad f_1(x) := f_0(x) + \frac{1}{N}\int_{s=0}^{Nx} \Big(\gamma_x(s) - \overline{\gamma_x}\Big)\, ds \quad \right)$$

Computation of the derivative:

- $$f_1'(x) \;=\; f_0'(x) + \gamma_x(Nx) - \overline{\gamma_x} + \frac{1}{N}\int_{s=0}^{Nx} \partial_x \Big(\gamma_x(s) - \overline{\gamma_x}\Big)\, ds$$

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- By assumption, we have $\overline{\gamma}_x = f_0'(x)$ for any x,

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Let assume that the relation is open. Since

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if N is large enough, then

"the image of $\gamma_x(\cdot)$ satisfies the constraint $\Rightarrow$ f_1' satisfies the constraint"

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Precisely,

$$(x, f_0(x), \gamma_x(s)) \in \mathscr{R} \quad \Rightarrow \quad (x, f_0(x), f_1'(x)) \in \mathscr{R}$$

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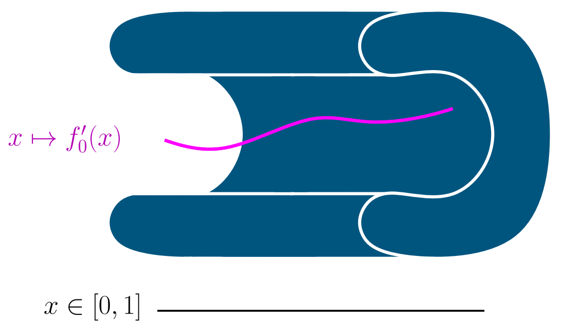
Recall that, to use properties of the Corrugation process, we need the average of $\gamma_x(\cdot)$ to be equal to $f_0'(x)$.

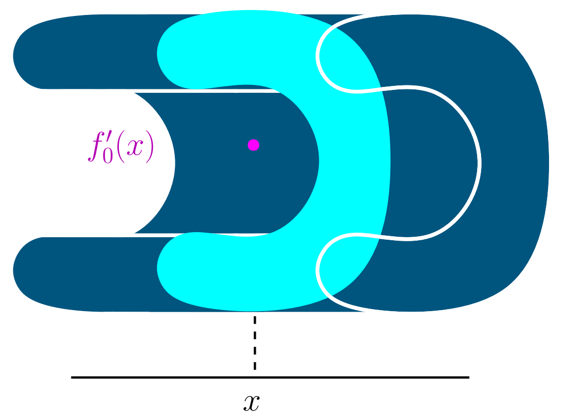
How to build the loop family γ_x?

Let $\mathscr{R}$ be a relation and f_0 be a curve whose derivative lies in the convex hull of $\mathscr{R}$



How to build the loop family γ_x?

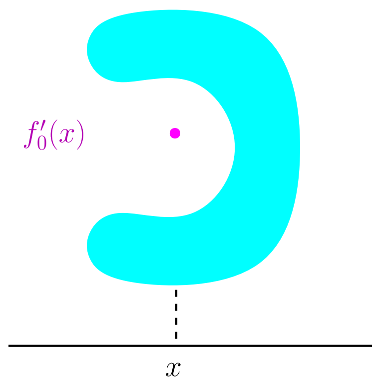
Let $x \in [0, 1]$



How to build the loop family γ_x?

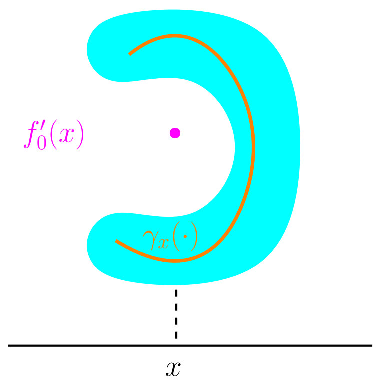
Let's define the slice of $\mathscr{R}$ over x by

$$\mathscr{R}_x := \{v_1 \mid (x, f_0(x), v_1) \in \mathscr{R}\}$$



How to build the loop family γ_x?

We require the image of γ_x to lie inside $\mathscr{R}_x$ and $\overline{\gamma}_x = f_0'(x)$



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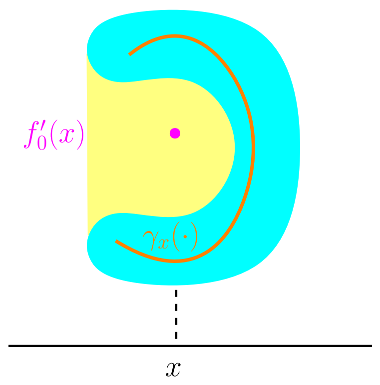
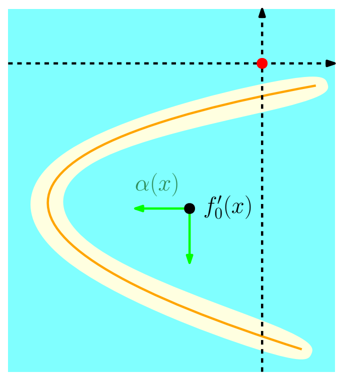


Observe that the condition $\overline{\gamma}_x = f_0'(x)$ implies that **$f_0'(x)$ has to belong to the convex hull of $\mathscr{R}_x$.**

Let's go back to the relation of immersions. Let $f_0 : [0,1] \to \mathbb{C}$ be any curve.



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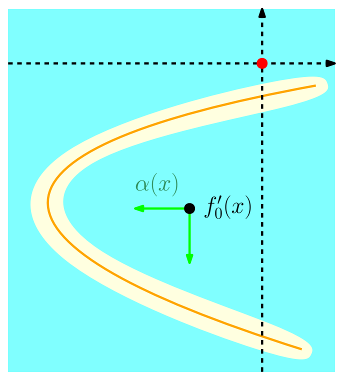

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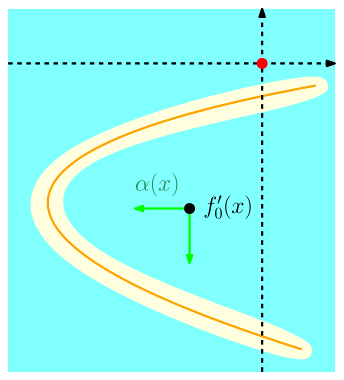
Thus the convex hull of $\mathscr{R}_x$ is $\mathbb{C}$, so $f_0'(x)$ is in the convex hull!

Let's go back to the relation of immersions. Let $f_0 : [0,1] \to \mathbb{C}$ be any curve.



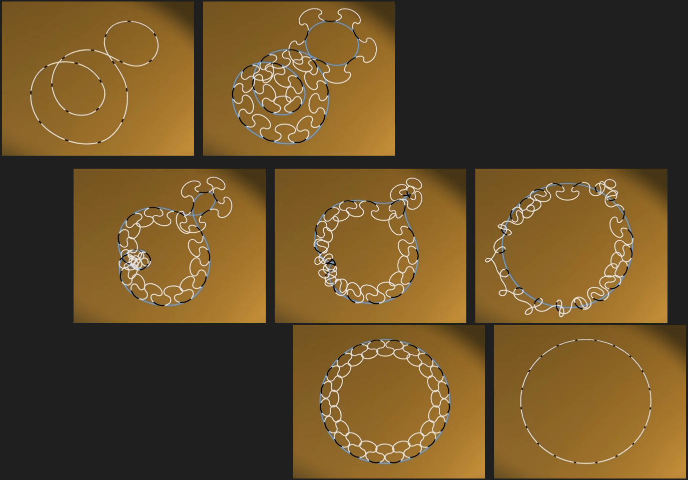
We have $\mathscr{R}_x = \{v_1 \mid (x, f_0(x), v_1) \in \mathscr{R}\} = \mathbb{C} \setminus \{0\}$

Thus the convex hull of $\mathscr{R}_x$ is $\mathbb{C}$, so $f_0'(x)$ is in the convex hull!

- Such a relation is called **ample**.

Back to the Thurston's homotopy:

$f_{(0,0)} =$

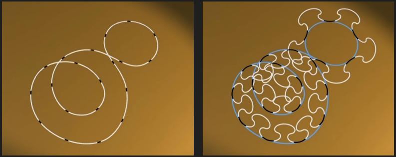


$= f_{(0,1)}$

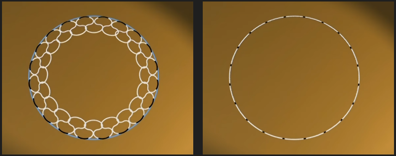
corrugations are directed by a family of vector $(\alpha(x))_t$ such that

$$(x, f_{(0,t)}(x), \alpha_t(x)) \quad \text{lie in the relation } \mathscr{R}.$$

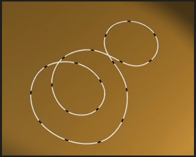
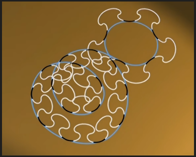
Here t is the parameter of the homotopy $f_{(0,t)}$.



(... intermediary steps with corrugations ...)



At the beginning we must have $\alpha_0(x) = f'_{(0,0)}(x)$ and at the end $\alpha_1(x) = f'_{(0,1)}(x)$.

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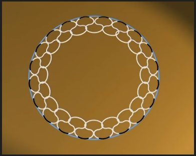
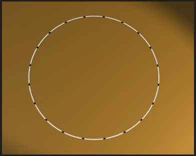
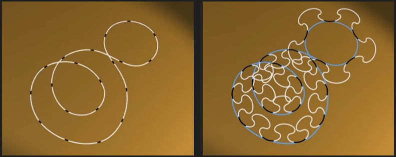
 

At the beginning we must have $\alpha_0(x) = f'_{(0,0)}(x)$ and at the end $\alpha_1(x) = f'_{(0,1)}(x)$.

So a regular homotopy is possible if there exists a continuous family of vector $\alpha_t(x)$ that never vanishes,



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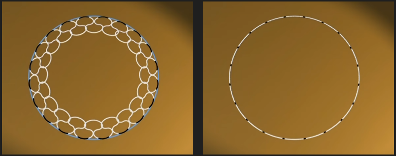

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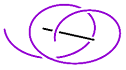
So a regular homotopy is possible if there exists a continuous family of vector $\alpha_t(x)$ that never vanishes,

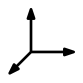
i.e. if there exists a homotopy of formal solutions between the two holonomic solutions $j^1 f_0$ and $j^1 f_1$.

Appendix: Other formulas in the literature (given in 1D)

1954 - Nash's formula (relation of isometric maps in codim 2)

$$f_1(x) := f_0(x) + \frac{1}{N} r \left[\Gamma_1(Nx)\mathbf{n}_1(x) + \Gamma_2(Nx)\mathbf{n}_2(x)\right]$$





with $\Gamma_1(Nx) = \cos(Nx)$, $\Gamma_2(Nx) = \sin(Nx)$, r a parameter of the problem, $\mathbf{n}_1$, $\mathbf{n}_2$ two unit normal vectors and $N \in \mathbb{N}$.

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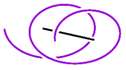

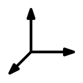

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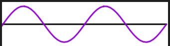
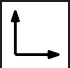
1955 - Kuiper's formula (relation of isometric maps in codim 1)

$$f_1(x) := f_0(x) + \frac{1}{N} r \left[\Gamma_1(Nx)\mathbf{t}(x) + \Gamma_2(Nx)\mathbf{n}(x)\right]$$

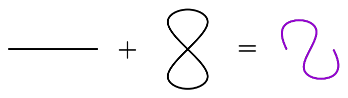




with $\Gamma_1(Nx) = \frac{-a^2 \sin(2Nx)}{8}$, $\Gamma_2(Nx) = a\sin(Nx - \frac{a^2 \sin(2Nx)}{8})$, r and a parameters of the problem, $\mathbf{t}$ a unit tangent vector and $\mathbf{n}$ a unit normal vector.

1995 - Thurston's formula (Sphere eversion, relation of immersion in codim 1)

$$f_1(x) := f_0(x) + r\left[\Gamma_1(Nx) + i\Gamma_2(Nx)\right]$$



with $r \in \mathbb{R}$, $\Gamma_1(Nx) = -\sin(4\pi Nx)$, $\Gamma_2(Nx) = 2\sin(2\pi Nx)$ and $N \in \mathbb{N}$.

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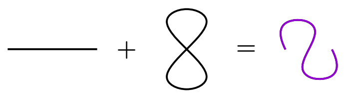

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2009 - Conti-De Lellis-Székelyhidi's formula (codim 1, relation of isometric maps in codim 1)

$$f_1(x) := f_0(x) + \frac{1}{N}\left[\Gamma_1(x, Nx)\mathbf{t}(x) + \Gamma_2(x, Nx)\mathbf{n}(x)\right]$$

with $\Gamma_1(Nx) = \int_0^{Nx} r\cos(a\sin(2\pi s)) - 1 ds$,
$\Gamma_2(Nx) = \int_0^{Nx} r\sin(a\sin(2\pi s))ds$, r and a parameters of the problem, $\mathbf{t}$ a unit tangent vector and $\mathbf{n}$ a unit normal vector.

Appendix: Other formulas in the literature (given in 1D)

1986 - Gromov's formula (Convex Integration Theory)

$$f_1(x) := f_0(0) + \int_{s=0}^{x} \gamma_s(Ns)ds$$

with a family of loops $(\gamma_t)_t$ and $N \in \mathbb{N}$.

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2022 - Corrugation process, a variant of Gromov's formula

$$f_1(x) := f_0(x) + \frac{1}{N}\int_{s=0}^{Nx} \left(\gamma_x(s) - \overline{\gamma}_x\right)ds$$

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To use both of these formula, we need

- the average of γ_x satisfies $\overline{\gamma}_x = f_0'(x)$, for any x,
- the image of γ_x satisfies the differential constraint.

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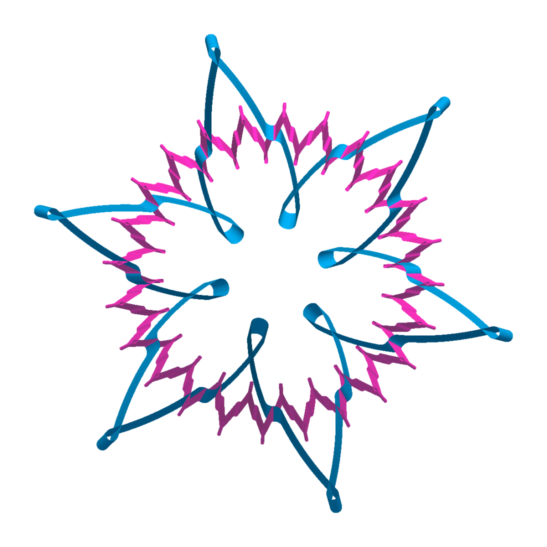

Merci !