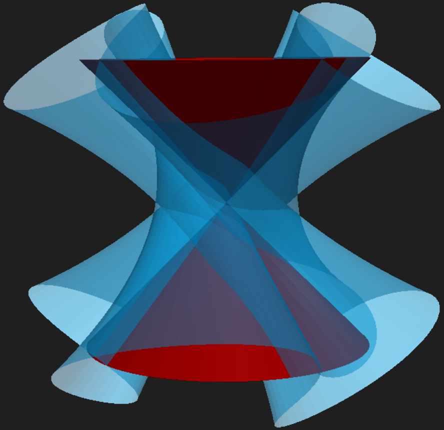
Convex integration theory



I - Convex integration theory

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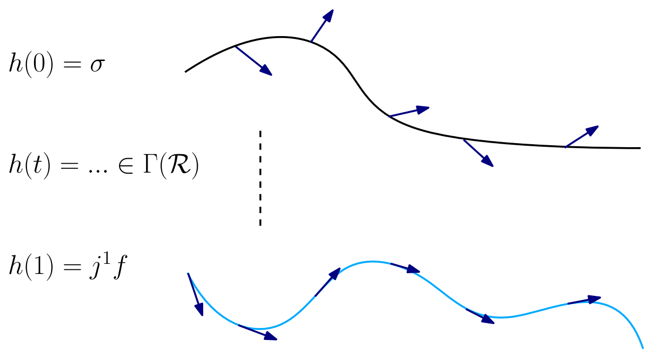
The Convex integration theory provides a tool to prove that a *h*-principle holds for a relation $\mathscr{R}$.

For that, Convex integration introduces an explicit way to deform a formal solution to a holonomic solution.



Definition (Corrugation process)

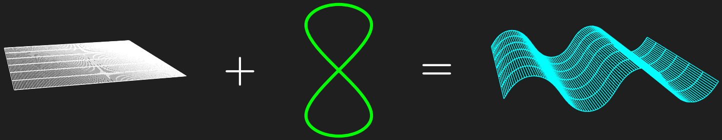
Let

- $f_0 : [0,1]^m \to \mathbb{R}^n$ be a map,
- $N \in \mathbb{N}^*$ be a number of corrugations,
- $(\gamma_x(\cdot))_{x\in[0,1]^m}$ be a family of loops,
- ∂_j be a direction of corrugation

then the map given by

$$f_1(x) := f_0(x) + \frac{1}{N}\int_{s=0}^{Nx_j}\left(\gamma_x(s) - \overline{\gamma_x}\right)ds$$

where $\overline{\gamma}_x = \int_0^1 \gamma_x(s)ds$, is said to be obtained from f_0 by a Corrugation process in the direction ∂_j.



Coordinate-free expression of the Corrugation process.

Let

- $f_0 : U \to (W, h)$ be a map from an open set $U \subset M$ to a complete manifold W endowed with a Riemannian metric,
- $N \in \mathbb{N}^*$ be a number of corrugations,
- $\gamma : U \times \mathbb{R}/\mathbb{Z} \to f_0^* TW$ be a family of loops,
- $\pi : U \to \mathbb{R}$ be a submersion,

then the Corrugation process writes

$$f_1(x) := \exp_{f_0(x)} \frac{1}{N} \int_{s=0}^{N\pi(x)} \Big(\gamma(x, s)) - \overline{\gamma}(x) \Big) ds$$

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If $M = [0, 1]^m$, $W = \mathbb{E}^n$ and $\pi(x) = x_j$, this formula reduces to the previous one.

Properties of

$$f_1(x) := f_0(x) + \frac{1}{N}\int_{s=0}^{Nx_j} \left(\gamma_x(s) - \overline{\gamma_x}\right) ds$$

- **C^0-closeness**: for any $x \in [0,1]^m$, we have

$$\begin{cases} \|f_1 - f_0\|_{C^0} = O(1/N) \\ \|\partial_i f_1 - \partial_i f_0\|_{C^0} = O(1/N), \quad \text{for } i \neq j \end{cases}$$

- **derivative along ∂_j**: If $\overline{\gamma}_x = \partial_j f_0(x)$ for any x, we have

$$\partial_j f_1(x) = \gamma_x(Nx_j) + O(1/N)$$

- Assume that we have a formal solution of the form

$$\sigma_{j-1} = (x,\ f_{j-1},\ \partial_1 f_{j-1},\ \dots,\ \partial_{j-1} f_{j-1},\ v_j, \dots, v_n) \in \mathscr{R}$$

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Now setting

$$\widetilde{\sigma_{j-1}} = (x,\ f_{j-1},\ \partial_1 f_{j-1},\ \ldots,\ \partial_{j-1} f_{j-1},\ \gamma_x(Nx_j),\ v_{j+1},\ \ldots, v_n)$$

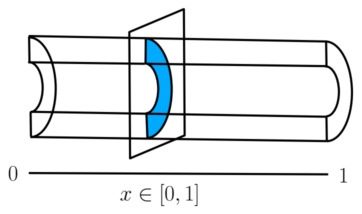
we first look for a condition on the loop γ_x to have

$$\widetilde{\sigma_{j-1}} \in \mathscr{R}$$

Definition

Let $S = (x, Y, V_1, \ldots, V_m)$ be a point in $\mathscr{R}$. The slice of $\mathscr{R}$ in the direction ∂_j over S is the subset

$$\mathscr{R}_{j,S} := \{w \in \mathbb{R}^n \,|\, (x, Y, V_1, \ldots, V_{j-1}, w, V_{j+1}, \ldots, V_m) \in \mathscr{R}\}$$



- So $\widetilde{\sigma_{j-1}} \in \mathscr{R}$ iff the image of $\gamma_x(\cdot)$ lies inside $\mathscr{R}_{j,S}$ with $S = \sigma_{j-1}(x)$.

Now we have

$$\widetilde{\sigma_{j-1}} = (x,\ f_{j-1},\ \partial_1 f_{j-1},\ \dots,\ \partial_{j-1} f_{j-1},\ \gamma_x(Nx_j),\ v_{j+1},\ \dots,\ v_n) \in \mathscr{R}$$

Let us recall that, to have

$$\partial_j f_j(x) = \gamma_x(Nx_j) + O(\frac{1}{N})$$

we have required that

$$\overline{\gamma}_x = \partial_j f_{j-1}(x).$$

Now we have

$$\widetilde{\sigma_{j-1}} = (x,\, f_{j-1},\, \partial_1 f_{j-1},\, \dots,\, \partial_{j-1} f_{j-1},\, \gamma_x(Nx_j),\, v_{j+1},\, \dots,\, v_n) \in \mathscr{R}$$

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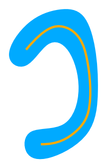
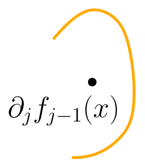
To sum up, we have two conditions on γ_x:

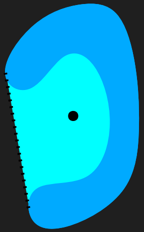
- its image lies inside $\mathscr{R}_{j,S}$



- its average is $\partial_j f_{j-1}(x)$



- So the existence of such a loop $\gamma_x(\cdot)$ is possible iff $\partial_j f_{j-1}(x)$ belongs to the convex hull of a path-connected component of the slice $\mathscr{R}_{j,S}$.



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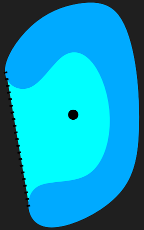

Rk. To obtain a *h*-principle, the choice of the path-connected component is constrained by the formal solution.

- Finally , from

$$\sigma_{j-1} = (x,\ f_{j-1},\ \partial_1 f_{j-1},\ \ldots,\ \partial_{j-1} f_{j-1},\ v_j, \ldots, v_n) \in \mathscr{R}$$

we have built a map f_j and a section

$$\widetilde{\sigma_{j-1}} = (x,\ f_{j-1},\ \partial_1 f_{j-1},\ \ldots,\ \partial_{j-1} f_{j-1},\ \gamma_x(Nx_j),\ v_{j+1},\ \ldots, v_n) \in \mathscr{R}$$

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As

$$\begin{cases} f_j(x) & = & f_{j-1}(x) + O(1/N) \\ \partial_i f_j(x) & = & \partial_i f_{j-1}(x) + O(1/N), \quad i \neq j \\ \partial_j f_j(x) & = & \gamma_x(Nx_j) + O(1/N) \end{cases}$$

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if N is large enough and if the relation $\mathscr{R}$ is open, then

$$\sigma_j = (x,\ f_j, \partial_1 f_j, \dots, \partial_j f_j,\ v_{j+1}, \dots, v_n) \in \mathscr{R}$$

Iterating in the directions $\partial_1, \ldots, \partial_m$, we build a sequence of formal solutions:

$$\begin{array}{rclllll}
\sigma_0(x) & = & (x, & f_0(x), & v_1(x), & v_2(x), & \ldots, & v_m(x)) & \in \mathscr{R} \\
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 & & & \vdots & & & & & \\
\sigma_m(x) & = & (x, & f_m(x), & \partial_1 f_m(x), & \partial_2 f_m(x), & \ldots, & \partial_m f_m(x)) & \in \mathscr{R}
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\end{array}$$

At the step m, we obtain a holonomic solution of $\mathscr{R}$.

$$\sigma_m(x) = j^1 f_m(x) \in \mathscr{R}$$

Definition (ample)

Let $\mathscr{R}$ be a relation. If the convex hull of each path-connected component of each slice of $\mathscr{R}$ is $\mathbb{R}^n$ (or if the slice is empty), then we say that $\mathscr{R}$ is ample.

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Convex Integration Theorem, Gromov

Let $\mathscr{R}$ be open and ample. Then $\mathscr{R}$ satisfies the full h-principle.

II - Examples

1 - The relation of immersions

▶ **The relation.** The map $f_0 : [0,1]^m \to \mathbb{R}^n$ is an immersion if the image of $j^1 f_0$ is in

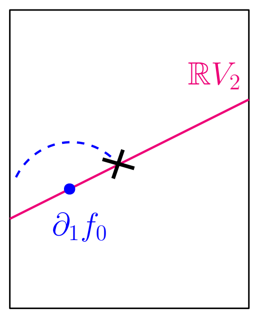
$$\mathscr{R}_{im} := \{(x, y, v_1, \dots, v_m) \mid v_1, \dots, v_m \text{ linearly independant}\}$$

▶ **The slice.** Let $S = (x, Y, V_1, \dots, V_m)$ be a point in $\mathscr{R}_{im}$. The slice in the direction ∂_1 is:

$$\mathscr{R}_{im,1,S} := \{w \in \mathbb{R}^n \mid w \notin Span(V_2, \dots, V_m)\}$$

Codimension 0

$$m = n = 2$$
$$\{w \in \mathbb{R}^2 \mid w \notin Span(V_2)\}$$
$$\{w \in \mathbb{R}^2 \mid w \in \mathbb{R}^2 \setminus \mathbb{R}V_2\}$$



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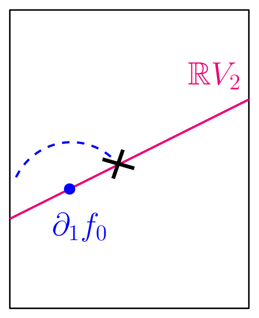

Codimension 1

$$m = 2,\ n = 3$$
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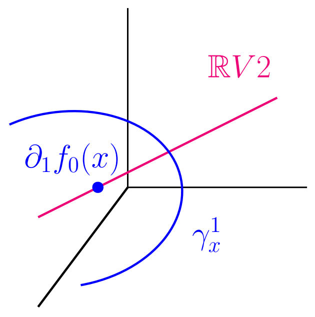

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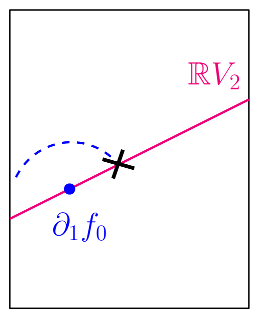

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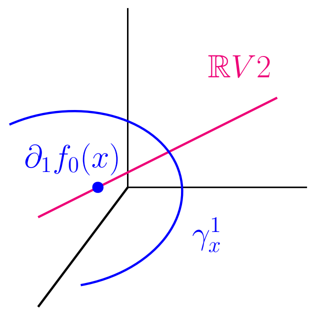

- The relation of immersions in codim. 0 is not ample.
- The relation of immersions in codim. ≥ 1 is ample.

Let f_0 be a parametrization of a cone

$$f_0 : \quad [0,1]^2 \longrightarrow$$

$$\subset \mathbb{R}^3$$

$$(t,\theta) \longmapsto \Big((2t-1)\cos(2\pi\theta),\, (2t-1)\sin(2\pi\theta)\Big)$$

Let f_0 be a parametrization of a cone

$$\begin{aligned} f_0 : \quad [0,1]^2 &\longrightarrow \quad [\text{cone}] \subset \mathbb{R}^3 \\ (t,\theta) &\longmapsto \Big((2t-1)\cos(2\pi\theta), (2t-1)\sin(2\pi\theta)\Big) \end{aligned}$$

For $x = (t,\theta)$, let v_2 be a vector field such that

$$\sigma_0(x) = (x, f_0(x), \partial_t f_0(x), v_2(x))$$

is a formal solution.

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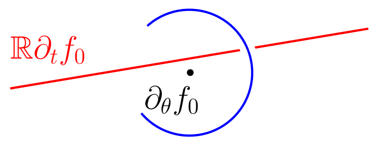
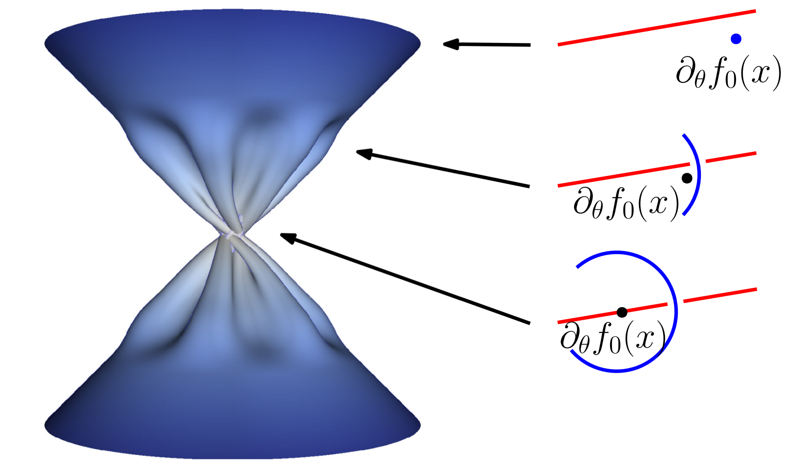
is a formal solution. We choose the loop γ_x to be an arc of circle in the plane normal to $\mathbb{R}\partial_t f_0(x)$ and of average $\partial_\theta f_0(x)$




$\partial_\theta f_0(x)$
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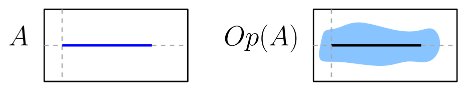
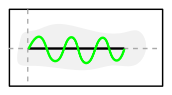
Holonomic Approximation theorem of order 1 and $A = [0,1]^m \times \{0\}$

Let $A = [0,1]^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}$, $Op(A)$ an open neighborhood of A.



Let $\sigma = ((x,t), f_0, L) : Op(A) \to J^1(Op(A), \mathbb{R}^n)$ be a section. For any $\epsilon > 0$, there exists

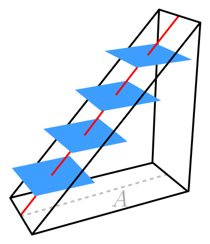
- a function $\delta : \mathbb{R}^m \to \mathbb{R}$ such that $\|\delta\| < \epsilon$, and we set

$$A_\delta := \{(x, \delta(x)) \,|\, x \in [0,1]^m\} =$$ 

- a map f_1 defined near A_δ such that $\|j^1 f_1 - \sigma\|_{C^0} < \epsilon$ on a sufficiently small open neighborhood of A_δ.

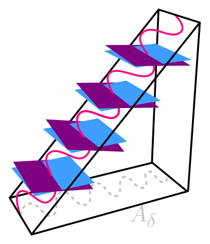
Example of the mountain path. Let

$$\sigma((x,t)) = ((x,t), f_0(x,t) = x, L_{(x,t)} = 0)$$



Holonomic Approximation Theorem

$\dashrightarrow$



The associated relation is

$$\mathscr{R} = \{((x,t), y, v_1, v_2) \mid \|y - x\| < \epsilon, \; \|v_i - 0\| < \epsilon\}$$

and the slice in the direction ∂_1 is $\mathscr{R}_1 = \{w \in \mathbb{R} \mid \|w\| < \epsilon\}$ so it's not ample!

Theorem (Massot-T. 2021)

We can rewrite the Holonomic Approximation of order 1 as an open ample relation, so solvable by Convex integration.

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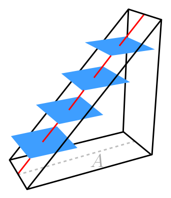

Corrugation process

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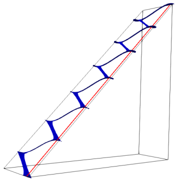

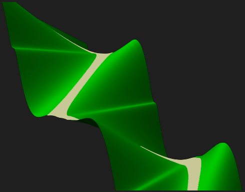

3 - Thickened relations

Let consider the toy example

$$\mathscr{R}_{orth} = \{(x, y, v_1, v_2) \,|\, \text{angle}(v_1, v_2) = \frac{\pi}{2},\ v_1, v_2 \neq 0\}$$

and a slice in ∂_2 is $\mathscr{R}_{orth,2,S} = \{v_2 \,|\, \text{angle}(V_1, v_2) = \frac{\pi}{2},\ v_2 \neq 0\}$

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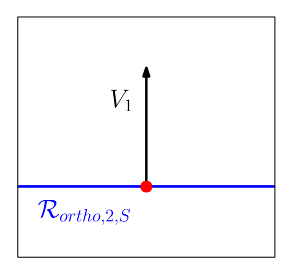
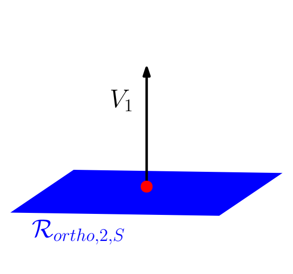
in $\mathbb{R}^2$, **not ample**



in $\mathbb{R}^3$, **not ample**



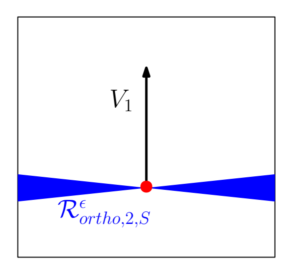
Let $\epsilon > 0$, and now consider

$$\mathscr{R}^{\epsilon}_{orth} = \{(x, y, v_1, v_2) \,|\, |\text{angle}(v_1, v_2) - \frac{\pi}{2}| < \epsilon,\ v_1, v_2 \neq 0\}$$

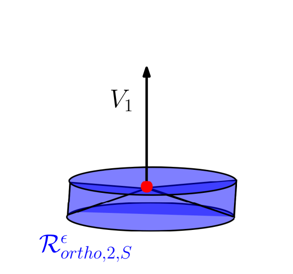
and a slice in the direction ∂_2 is

$$\mathscr{R}^{\epsilon}_{orth,2,S} = \{v_2 \,|\, |\text{angle}(V_1, v_2) - \frac{\pi}{2}| < \epsilon,\ v_2 \neq 0\}$$

in $\mathbb{R}^2$, **not ample**



in $\mathbb{R}^3$, **ample**



With the idea of thickening,

- the relation of Lagrangian immersions $\mathscr{R}_{lag}$ is not ample,
- the relation of ϵ-Lagrangian immersions $\mathscr{R}^{\epsilon}_{lag}$ is ample.

and also

- the relation of isotropic immersions $\mathscr{R}_{isot}$ is not ample,
- the relation of ϵ-isotropic immersions $\mathscr{R}^{\epsilon}_{isot}$ is ample.

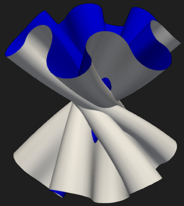
but

- the relation of ϵ-coisotropic immersions $\mathscr{R}^{\epsilon}_{coisot}$ is not ample.

(see *Introduction to the h-principle*, Eliashberg, Mishachev, §19.2)

Appendix -

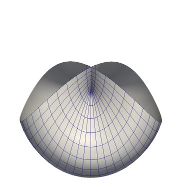
We saw that the relation of immersions (in codim.≥ 1) is ample, and we managed to remove the singularity of a cone.

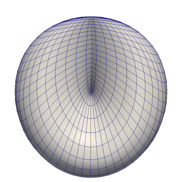


What about a more complicated surface ? Like the crosscap ? (a representation $\mathbb{R}P^2$)

Appendix -

The crosscap is obtained by gluing two opposite points on a disk, and closing it like described on the picture:

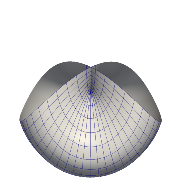
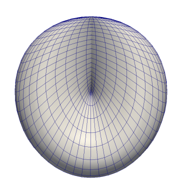
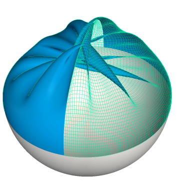




This surface has 2 singular points.

Appendix -

The crosscap is obtained by gluing two opposite points on a disk, and closing it like described on the picture:

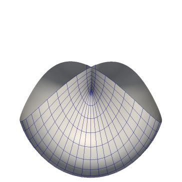
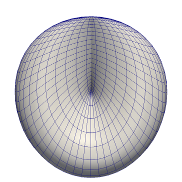
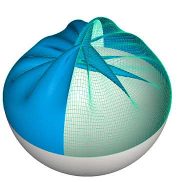




This surface has 2 singular points.

Let's go!

Appendix -

The crosscap is obtained by gluing two opposite points on a disk, and closing it like described on the picture:

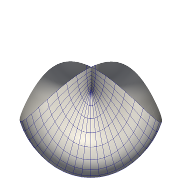
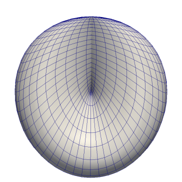
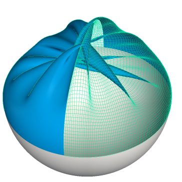




This surface has 2 singular points. Let's go!

▶ Note that on this new surface, corrugations are localized a neighborhood of the segment connecting the two singular points.

Appendix -

The crosscap is obtained by gluing two opposite points on a disk, and closing it like described on the picture:

This surface has 2 singular points. Let's go!

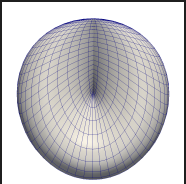
▶ Note that on this new surface, corrugations are localized a neighborhood of the segment connecting the two singular points.

▶ Is it possible to localize the corrugations on a neighborhood of each singular point ?

Appendix -

If it was possible to remove only one singular point,

this would mean that we could build an intermediary closed surface with only one singular point...

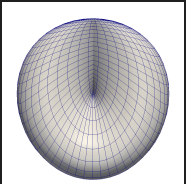


but...

Appendix -

If it was possible to remove only one singular point,

this would mean that we could build an intermediary closed surface with only one singular point...



but... from a theorem of Whitney, it is impossible.

- There is an obstruction to find such a formal solution.

- Tomorrow, we'll see the relation of isometric maps which is closed and not ample, but still satisfies a convex condition.

References

Books

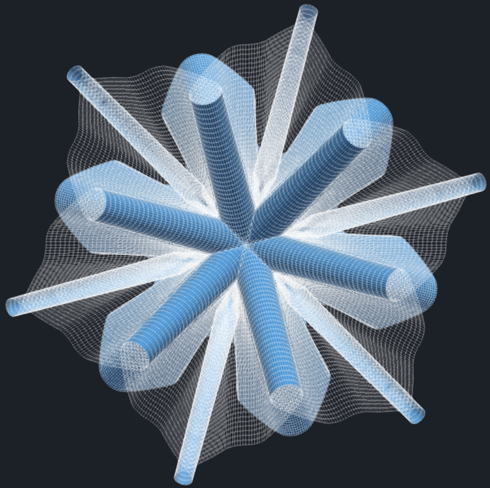
- *Partial differential relation*, Gromov, 1986
- *Convex integration theory. Solutions to the h-principle in geometry and topology*, Spring, 1998
- *Introduction to the h-principle*, Eliashberg, Mishachev, 2002

Articles

- *The singularities of a smooth n-manifold in* $(2n-1)$*-space*, Whitney, 1944
- *Holonomic approximation through convex integration*, Massot, Theillière, 2021
- *Convex integration theory without integration*, Theillière, 2022

Additional references

- *Applications of Convex integration to symplectic and contact geometry*, McDuff, 1987
- Loose legendrian embeddings in high dimensional contact manifolds, Murphy, 2019
- *Convex integration with avoidance and hyperbolic* $(4, 6)$ *distributions*, Martinez-Aguinaga, del Pino, 2021



Merci !