Making Corrugations

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Abstract

The aim of this lecture is to provide the mathematical tools for the construction of corrugated surfaces. We first present a fundamental formula, called the "Corrugation Process" (Section 1), and show how to apply it to recover the original Thurston's corrugations (Section 2). We then turn to a more general problem: given one or more differential constraints, how to construct surfaces that satisfy these constraints by means of the corrugation process? The answer requires the introduction of the 1-jet space, the notion of differential relation (Section 3) and to distinguish between formal and holonomic solutions (Section 4). Once this formalism is in place, we apply it to a concrete situation, that of the desingularization of a cone. The constructive nature of the theory allows us to provide explicit expressions and numerous visualizations (Section 6).

Contents

All definitions and constructions of this lecture are presented for the case of surfaces in $\mathbb{R}^3$. The case of curves is easily deduced from the case of surfaces.

1 The Corrugation Process

In this section, we define the Corrugation Process and we give its fundamental properties.

Definition 1. Let $f_0 : [0,1]^2 \to \mathbb{R}^3$ be a smooth map. Let $\gamma : [0,1]^2 \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a smooth loop family which associates to any point x of $[0,1]^2$ a loop $\gamma(x,\cdot)$. Let $j \in \{1,2\}$ the direction of corrugations and $N \in \mathbb{R}_+^*$ the number of corrugations. For any $x = (x_1, x_2)$, we set

$$f_1(x) \quad := \quad f_0(x) + \frac{1}{N}\int_{t=0}^{Nx_j} \big(\gamma(x,t) - \overline{\gamma}(x)\big)dt \tag{1}$$

where $\overline{\gamma}(x) := \int_0^1 \gamma(x,s)ds$ is the average of the loop $\gamma(x,\cdot)$. We say that f_1 is obtained by a Corrugation Process from f_0.

Note that we can immediately state the following property:

(P_0) for any x where the loop $\gamma(x,\cdot)$ is constant, we have $f_1(x) = f_0(x)$.

In particular, if somewhere the loop family is reduced to a point, then the map $f_1 = f_0$ at this point.

Proposition 2. *If the loop family γ satisfies the condition*

$$\forall x \in [0,1]^2, \quad \overline{\gamma}(x) = \partial_j f_0(x) \tag{2}$$

then the map f_1 satisfies the following properties

(P_1) $\|f_1 - f_0\|_\infty = O(1/N)$

(P_2) *for any $i \neq j$, we have* $\|\partial_i f_1 - \partial_i f_0\|_\infty = O(1/N)$

(P_3) $\partial_j f_1(x) = \gamma(x, Nx_j) + O(1/N)$

Note that the number of corrugations N allows to control the closeness between the new map f_1 and the source map f_0 (Property P_1), and also between the derivatives $\partial_i f_1$ and $\partial_i f_0$ if the derivative is not in the direction ∂_j of the corrugation (Property P_2). So, modulo a controlled error, we just modify one partial derivative, and this derivative is close to the image of the loop family γ (Property P_3).

Proof. To have Property P_1, we first have to note that the map

$$(x,u) \longmapsto \int_{t=0}^{u} \big(\gamma(x,t) - \overline{\gamma}(x)\big)dt$$

is 1-periodic for the variable u, so bounded and we have

$$f_1(x) \quad = \quad f_0(x) + \frac{1}{N}.O(1) = f_0(x) + O\left(\frac{1}{N}\right)$$

which is Property P_1. Now considering the partial derivatives for $\partial_i \neq \partial_j$, we have

$$\begin{aligned}\partial_i f_1(x) &= \partial_i f_0(x) + \frac{1}{N}\partial_i\left(\int_{t=0}^{Nx_j} \gamma(x,t) - \overline{\gamma}(x)dt\right)\\ &= \partial_i f_0(x) + O\left(\frac{1}{N}\right).\end{aligned}$$

This shows Property P_2. For Property P_3, by derivating we have:

$$\begin{aligned}\partial_j f_1(x) &= \partial_j f_0(x) + \frac{1}{N}\partial_j\left(\int_{t=0}^{Nx_j} \gamma(x,t) - \overline{\gamma}(x)dt\right)\\ &= \partial_j f_0(x) + \gamma(x, Nx_j) - \overline{\gamma}(x) + \frac{1}{N}\int_{t=0}^{Nx_j} \partial_j\Big(\gamma(x,t) - \overline{\gamma}(x)\Big)dt\\ &= \partial_j f_0(x) + \gamma(x, Nx_j) - \overline{\gamma}(x) + O\left(\frac{1}{N}\right).\end{aligned}$$

Since $\overline{\gamma}(x) - \partial_j f_0(x) = 0$, we obtain P_3. □

2 First example: Thurston's corrugations

In this section, we use the Corrugation Process to get the Thurston's formula of corrugation [2]. In particular in this section we set

$$\gamma(x,t) = -r4\pi\cos(4\pi t) + ir4\pi\cos(2\pi t).$$

In the following, we denote

$$\Gamma : (x,u) \longmapsto \int_{t=0}^{u} \big(\gamma(x,t) - \overline{\gamma}(x)\big)dt$$

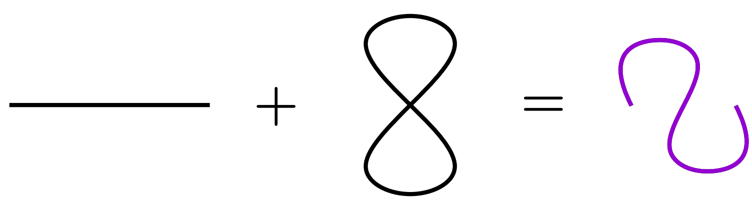
and its components $\Gamma_1 + i\Gamma_2$ given by

$$\Gamma_1(x,u) = -r\sin(4\pi u), \quad \Gamma_2(x,u) = 2r\sin(2\pi u).$$

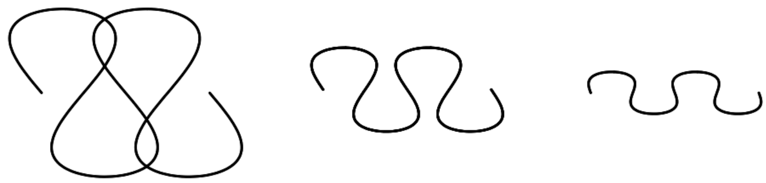
Basic example. Let $f_0 : x \mapsto x + i0$ be the map that parametrizes a segment in the complex plane. The Corrugation Process writes

$$f_1(x) = f_0(x) + \frac{1}{N}\Big[\Gamma_1(x, Nx) + i\Gamma_2(x, Nx)\Big].$$

The image of f_1 depends on N and r. In [2], Thurston summarizes this formula by the following picture



where Γ_1, Γ_2 are the component of the eight-shaped curve. By applying the Corrugation Process, we can build



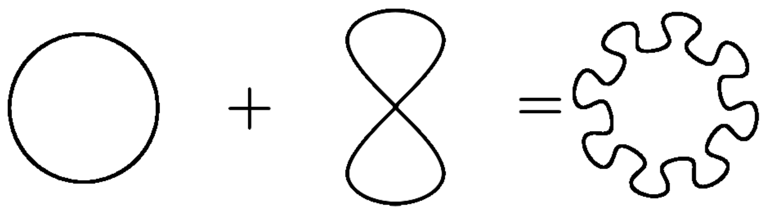
In this picture $N = 2$ and $r = 1/2, \quad 1/4, \quad 1/8$.

There is a threshold r^* such that the image has no self-intersection if $r < r^*$. Modifying N is close to zoom in or zoom out.

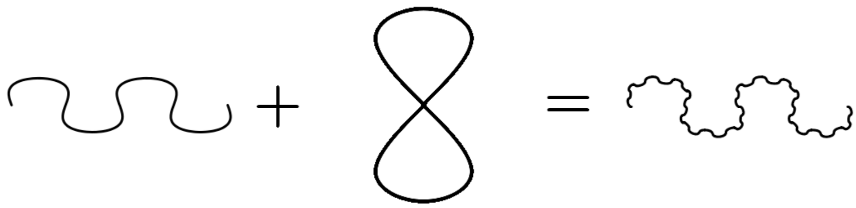
For any regular curve f_0. Let now consider a map $f_0 : [0, 1] \to \mathbb{C}$ such that for any $x \in [0, 1]$ we have $f_0'(x) \neq 0$. We add corrugations to f_0 in the following way:

$$f_1(x) \quad = \quad f_0(x) + \frac{1}{N}\Big[\Gamma_1(x, Nx)\frac{f_0'(x)}{\|f_0'(x)\|} + \Gamma_2(x, Nx)i\frac{f_0'(x)}{\|f_0'(x)\|}\Big]$$

If $f_0(x) = \exp(i2\pi x)$, $N = 8$ and $r = 2$, we obtain



If we corrugate the segment a second time, with $(N_1, r_1) = (2, \frac{1}{8})$ and $(N_2, r_2) = (20, \frac{1}{16})$ we obtain



For any curve f_0, regular or singular. One of the aims of this construction is to remove singular points, so let us now consider a singular map f_0. In particular there exists (at least) one point x_0 such that $f_0'(x_0) = 0$. This means the previous formula, whose corrugations are defined for $(\frac{f_0'(x)}{\|f_0'(x)\|}, i\frac{f_0'(x)}{\|f_0'(x)\|})$, is now not well-defined at x_0. To replace the tangent unit vector field $\frac{f_0'(x)}{\|f_0'(x)\|}$ we have to choose a unit vector field $\alpha(x)$. We now set

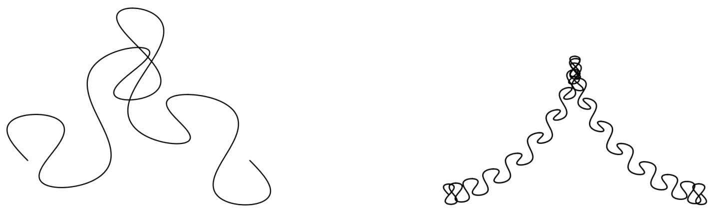
$$f_1(x) = f_0(x) + \frac{1}{N}\Big[\Gamma_1(x, Nx)\alpha(x) + \Gamma_2(x, Nx)i\alpha(x)\Big]$$

Testing it on a segment, we obtain

for $N = 2$, $r = 1/2$, $\alpha(x) = 1$ and $\alpha(x) = \exp(i2\pi/6)$.

Let us now consider $f_0(x) = \cos^3(\pi x) + i \sin^3(\pi x)$ which is singular for $x = 0, \frac{1}{2}$ and 1.

Setting $\alpha(x) = -1$, $r = 0.8$ and for $N = 4$ and 20 we obtain



3 The 1-jet space and differential relations

In this section, we introduce the basic objects needed to formalize the constraints that the surface undergoes: the 1-jet space and the notion of differential relation. As an illustration, we consider two historical constraints: the immersion and the ϵ-isometric constraints.

Definition 3. The 1-*jet space* $J^1([0,1]^2, \mathbb{R}^3)$ of maps from $[0,1]^2$ to $\mathbb{R}^3$ is the set defined to be

$$J^1([0,1]^2, \mathbb{R}^3) := [0,1]^2 \times \mathbb{R}^3 \times (\mathbb{R}^3)^2.$$

Definition 4. The 1-*jet* of a C^1-map $f : [0,1]^2 \to \mathbb{R}^3$ is the map given by

$$\begin{array}{rccc} j^1 f : & [0,1]^2 & \longrightarrow & J^1([0,1]^2, \mathbb{R}^3) \\ & x & \longmapsto & (x, f(x), \partial_1 f(x), \partial_2 f(x)) \end{array}$$

Definition 5. A *differential relation* of order 1 is a subset $\mathscr{R}$ of the 1-jet space $J^1([0,1]^2, \mathbb{R}^3)$.

A map f is *regular* or is an *immersion* if, for any x, its partial derivatives $\partial_1 f(x)$, $\partial_2 f(x)$ are linearly independents, or equivalently if, for any x, the rank of its differential df_x is maximal.

The **relation of Immersions** is given by

$$\mathscr{I} := \{(x, y, v_1, v_2) \,|\, v_1, v_2 \text{ are linearly independents}\}$$

So if f is an immersion, then for any x,

$$j^1 f(x) = (x, f(x), \partial_1 f(x), \partial_2 f(x)) \in \mathscr{I}.$$

In the case of curves, ie for $m = 1$, we have

$$\mathscr{I} \quad := \quad \{(x, y, v_1) \,|\, v_1 \neq 0\} = [0,1] \times \mathbb{R}^3 \times (\mathbb{R}^3 \backslash \{0\})$$

Recall that a metric $g : [0,1]^2 \to M_2(\mathbb{R})$ is a map which associates to any $x \in U$ a symmetric definite-positive matrix often denoted by

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

with $EG - F^2 > 0$ and $E > 0$. For the sake of clarity, in the following we denote

$$g = (g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

which is a symmetric positive-definite matrix, so $g_{12} = g_{21}$, $g_{11}g_{22} - g_{12}^2 > 0$ and $g_{11} > 0$.

A map $f : ([0,1]^2, g) \to (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$, with g a metric, is isometric if it preserves the length of curves. This condition writes

$$\forall x \in [0,1]^2,\ \forall i, j, \quad g_{ij} = \langle \partial_i f(x), \partial_j f(x) \rangle$$

The isometric relation is very constraining. Traditionally, we associate to it a less constraining relation, the one of ϵ-isometric maps.

For $\epsilon > 0$, the **relation of ϵ-Isometric Immersions** from $([0,1]^2, g)$, with g a metric, to $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ is given by

$$\mathscr{I}\mathscr{I}(\epsilon) := \{(x, y, v_1, v_2) \mid |g_{ij} - \langle v_i, v_j \rangle| < \epsilon\}.$$

For $m = 1$, ie if we consider the relation of ϵ-Isometric Immersions $\mathscr{I}\mathscr{I}$ for curves, we have

$$\mathscr{I}\mathscr{I}(\epsilon) \quad := \quad \{(x, y, v_1) \,|\, |g_{11} - \|v_1\|^2| < \epsilon\} = [0,1] \times \mathbb{R}^3 \times S$$

with S the ball of radius $\sqrt{g_{11}} + \epsilon$ minus the ball of radius $\sqrt{g_{11}} - \epsilon$. So S is a thickening of the sphere of radius $\sqrt{g_{11}}$.

4 Formal and holonomic solutions

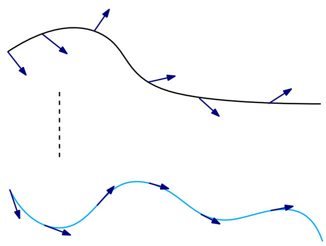
In a pioneering work, Nash used the idea of corrugations to solve the isometric relation [3]. About fifteen years later, Gromov understood how to generalize Nash's approach to solve large families of different constraints: the convex integration theory was born [1]. This theory deforms a kind of sub-solution, called formal solution, into a real solution. This deformation is built by successively applying a Corrugation Process.

Formal and holonomic solutions. Let $\mathscr{R}$ be a differential relation and let

$$\begin{array}{rccc} \sigma: & [0,1]^2 & \longrightarrow & J^1([0,1]^2,\mathbb{R}^3) \\ & x & \longmapsto & (x, f(x), v_1(x), v_2(x)) \end{array}$$

Definition 6. The map σ is a *formal solution of* $\mathscr{R}$ if its image lies in $\mathscr{R}$. If moreover there exists a map $f : [0,1]^2 \to \mathbb{R}^3$ such that $\sigma = j^1 f$ then σ is a *holonomic solution.*

The idea is to deform a formal solution to a holonomic one by using corrugations.



On the top curve, the navy blue vector field v is not the derivative of the curve, but on the bottom curve the navy blue vector field v is the derivative of the curve.

The Corrugation Process is essentially a 1-dimensional deformation. Then we restrict the relation to only one direction.

Definition 7. Let $\mathscr{R}$ be a differential relation, $\sigma = (x, y, v_1, v_2)$ be a point of this relation. The **slice of $\mathscr{R}$ over σ in the direction** 1 is the subset

$$\mathscr{R}(\sigma, \partial_1) := \{w \in \mathbb{R}^3 \,|\, (x, y, w, v_2) \in \mathscr{R}\}$$

and in the direction 2

$$\mathscr{R}(\sigma, \partial_2) := \{w \in \mathbb{R}^3 \,|\, (x, y, v_1, w) \in \mathscr{R}\}$$

Examples of slices. If $m = 1$, the slice is just the projection on the third component of the 1-jet space. Let us consider $m = 2$ and $j = 1$. The slice of the relation of immersions $\mathscr{I}$ for surfaces in the direction 1 is

$$\begin{aligned} \mathscr{I}(\sigma, \partial_1) &:= \{w \in \mathbb{R}^3 \,|\, (x, y, w, v_2) \in \mathscr{I}\} \\ &= \{w \in \mathbb{R}^3 \,|\, w \text{ and } v_2 \text{ lin. independents}\} \\ &= \mathbb{R}^3 \backslash \mathbb{R} v_2 \end{aligned}$$

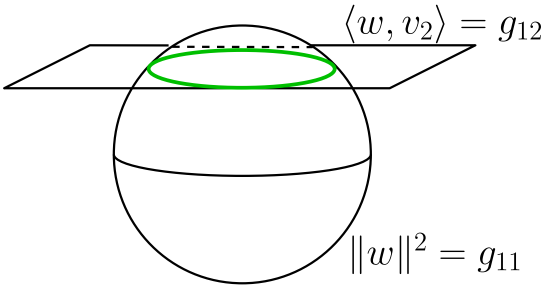
And the slice of ϵ-isometric maps $\mathscr{I}s(\epsilon)$ for surfaces in the direction 1 is

$$\begin{aligned} \mathscr{I}s(\epsilon, \sigma, \partial_1) &:= \{w \in \mathbb{R}^3 \,|\, (x, y, w, v_2) \in \mathscr{I}s(\epsilon)\} \\ &= \{w \in \mathbb{R}^3 \,|\, |g_{11} - \|w\|^2| < \epsilon \text{ and } |g_{12} - \langle w, v_2\rangle| < \epsilon\} \\ &= S \cup H(\epsilon) \end{aligned}$$

with S a thickening of the sphere of radius $\sqrt{g_{11}}$ and with $H(\epsilon)$ an ϵ-thickening of the hyperplane

$$H := \{v \mid \langle v, v_2\rangle = g_{12}\}.$$

So the slice of isometric maps for surfaces in $\mathbb{R}^3$ is



Deformation of a formal solution to a holonomic one. So to build a holonomic solution, we will deform a formal solution by using the Corrugation Process. Precisely let $\sigma : x \mapsto (x, f(x), v_1(x), v_2(x))$ be a formal solution. Assuming there exists a loop family

$$\begin{array}{rccc} \gamma_1 : & [0,1]^2 \times \mathbb{R}/\mathbb{Z} & \longrightarrow & \mathbb{R}^3 \\ & (x,t) & \longmapsto & \gamma_1(x,t) \end{array}$$

such that

(a) for any x, the image of $\gamma_1(x,\cdot)$ lies in $\mathscr{R}(\sigma(x),1)$;

(b) for any x, the average $\overline{\gamma_1}(x) = \partial_1 f(x)$;

then, by the Corrugation Process, we build a map f_1 such that

$$\sigma_1 : x \mapsto (x, f_1(x), \partial_1 f_1(x), v_2(x)) \in \mathscr{R}$$

if the relation is open and if the number N_1 of corrugations is large enough. Then, assuming the existence of a loop family γ_2 with similar properties, we can build by corrugation a map f_2 such that

$$\sigma_2 = j^1 f_2 : x \longmapsto (x, f_2(x), \partial_1 f_2(x), \partial_2 f_2(x)) \in \mathscr{R}.$$

So we have obtain a holonomic solution of $\mathscr{R}$.

The question of the properties that the relation has to satisfy to ensure the existence of such a loop family is not addressed in this lecture.

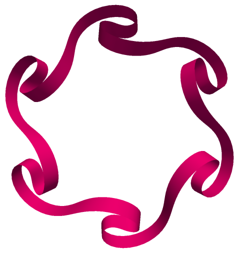
5 Influence of the choice of γ

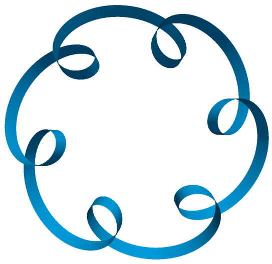
The Corrugation Process requires choosing a path γ satisfying the two properties (a) and (b). These two properties being not very constraining, there is a large degree of freedom in the choice of γ. In this paragraph we show the influence of the modification of some parameters on the shape of the resulting surface.

Modification of the image of γ. We can modify the shape of the image of γ provided its image lies in the relation and its average satisfies (b). In the two pictures below, we consider the two following choices

$$\gamma_1(t) = e^{i\alpha_0 \cos(2\pi t)}, \quad \gamma_2(t) = \cos(2\pi t) + i \sin(2\pi t)$$

where α_0 is chosen such that $\overline{\gamma_1} = 0$.





At the left: an **arc-of-circle**-shaped loop; at the right: a **circle**-shaped loop.

Modification of the parametrization of γ. Here we consider the two following parametrization of an arc of circle of angle $\pi/3$:

$$\gamma_3(t) = \exp(i\frac{\pi}{3}\cos(2\pi t)) + c_3, \quad \gamma_4(t) = \exp(i\, g(t)) + c_4$$

where c_3 and c_4 are constants such that the loops satisfy property (b), and where $g(t)$ a piecewise linear map which is equal to $\frac{\pi}{3}$ on $[0, \frac{1}{4} - \eta] \cup [\frac{3}{4} + \eta, 1]$ and is equal to $-\frac{\pi}{3}$ on $[\frac{1}{4} + \eta, \frac{3}{4} - \eta]$. The figures below show the corresponding corrugations.

At the left: corrugations obtained with γ_3, at the right: with γ_4.

6 Explicit constructions on a cone

In this paragraph, we will show the power of Convex Integration through a very simple example, that of the desingularization of a cone. This is obviously a "toy" example since such a desingularization is not a mathematical problem. But the consideration of this example allows to easily illustrate the variety of solutions offered by the Convex Integration, this variety being itself the consequence of the vast degree of freedom allowed by the choice of the loop family γ.

Initial map. Let $f_0 : [0,1] \times [-1,1] \to \mathbb{R}^3$ be the parametrization of a truncated cone given by

$$\begin{array}{rccc} f_0 : & [0,1]\times[-1,1] & \longrightarrow & \mathbb{R}^3 \\ & (x_1,x_2) & \longmapsto & (x_2\cos(2\pi x_1),\, x_2\sin(2\pi x_1),\, x_2) \end{array}$$

Computing the partial derivatives, we obtain

$$\begin{aligned} \partial_1 f_0(x_1,x_2) &= (-2\pi x_2\sin(2\pi x_1),\, 2\pi x_2\cos(2\pi x_1),\, 0) \\ \partial_2 f_0(x_1,x_2) &= (\cos(2\pi x_1),\, \sin(2\pi x_1),\, 1) \end{aligned}$$

Direction of corrugation. Note that the partial derivative $\partial_1 f_0$ vanishes for $(x_1,x_2) \in [0,1] \times \{0\}$. To have an immersion, we will build a new map f_1 adding corrugations on f_0 in the direction ∂_1 such that the new partial derivative $\partial_1 f_1$ does not vanish and is linearly independent to $\partial_2 f_0$ (which is almost unchanged by the Corrugation Process).

Formal solution. As the partial derivative $\partial_2 f_0$ never vanishes, it is enough to find a vector field v_1 which does not vanish and is linearly independent to $\partial_2 f_0$ to have a formal solution

$$\sigma = ((x_1,x_2),\, f_0,\, v_1,\, \partial_2 f_0)$$

of the relation of immersions. By modifying $\partial_1 f_0$ we can choose

$$v_1(x) := (-\sin(2\pi x_1), \cos(2\pi x_1), 0)$$

which never vanishes and which is linearly independent to $\partial_2 f_0$. In the following, we will use

$$\mathbf{n(x)} := \frac{v_1(x) \wedge \partial_2 f_0(x)}{\|v_1(x) \wedge \partial_2 f_0(x)\|}$$

which coincides with the normal vector of the cone where it is well-define.

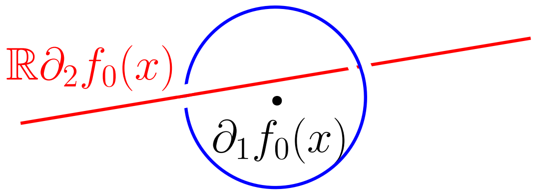
Loop family for immersions (1). To build an immersion, we have to choose a loop whose image lies in the slice

$$\begin{aligned} \mathscr{I}(\sigma,\partial_1) &:= \{w \in \mathbb{R}^3 \,|\, (x,y,w,\partial_2 f_0) \in \mathscr{I}\} \\ &= \{w \in \mathbb{R}^3 \,|\, w \text{ and } \partial_2 f_0 \text{ lin. independents}\} \\ &= \mathbb{R}^3 \backslash \mathbb{R}\partial_2 f_0 \end{aligned}$$

and whose average is $\partial_1 f_0$. Let us consider

$$\gamma : \quad (x,t) \quad \mapsto \quad \gamma(x,t) = r(x)\Big[\cos(2\pi t)\frac{v_1(x)}{\|v_1(x)\|} + \sin(2\pi t)\mathbf{n}(x)\Big] + \partial_1 f_0(x)$$

be a circle of radius $r(x)$ and of center $\partial_1 f_0(x)$.



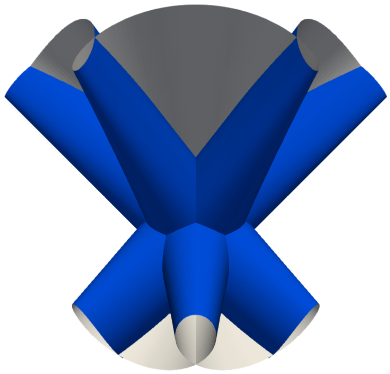
The condition on the average is satisfied, we have $\overline{\gamma}(x) = \partial_1 f_0(x)$, and we have to choose $r(x)$ such that

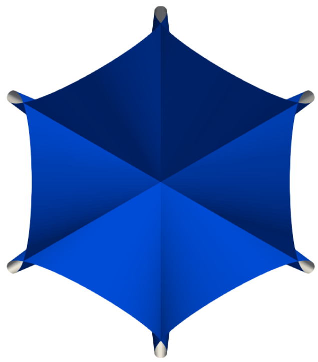
$$r(x) > \text{dist}(\partial_1 f_0(x), \partial_2 f_0(x)) = \sqrt{(2\pi x_2)^2 + 2}$$

This condition will allow to ensure the circle does not intersect the line spanned by $\partial_2 f_0(x)$. As $x_2 \in [-1, 1]$, for example we can choose

$$r(x) := 4\pi.$$

Corrugated map (1). By using the Corrugation Process with the previous parameters, we build from a cone the following surface

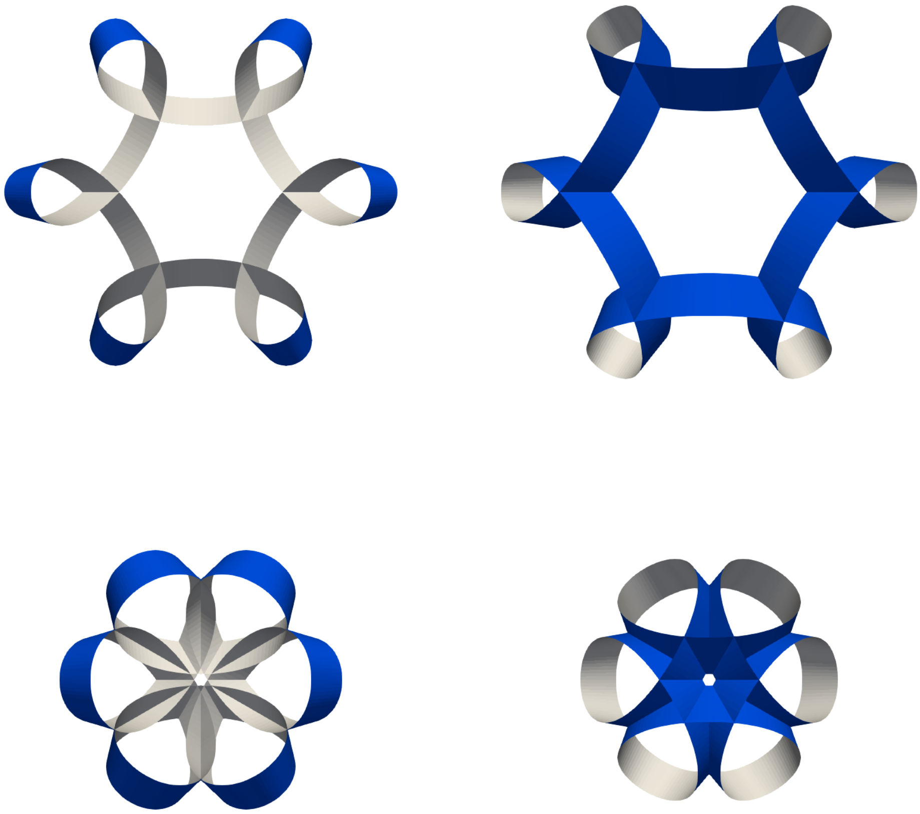


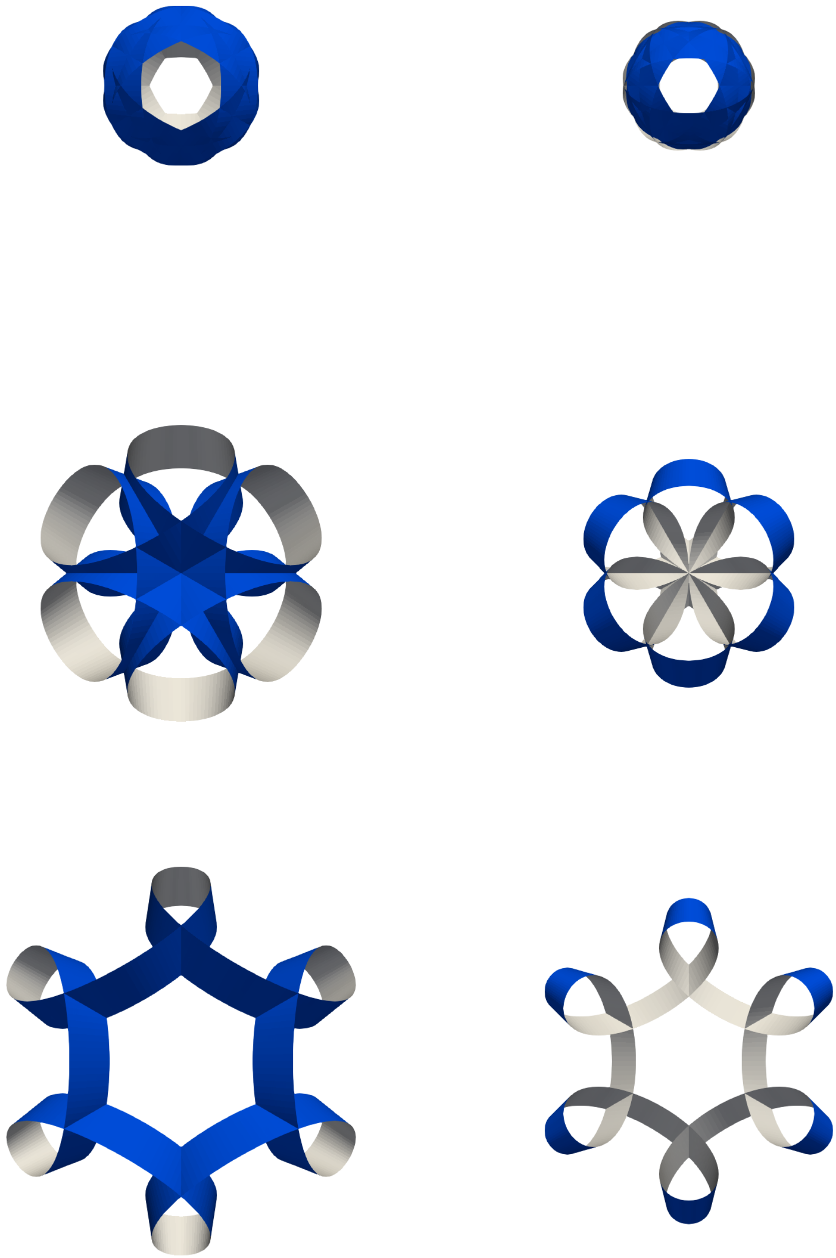


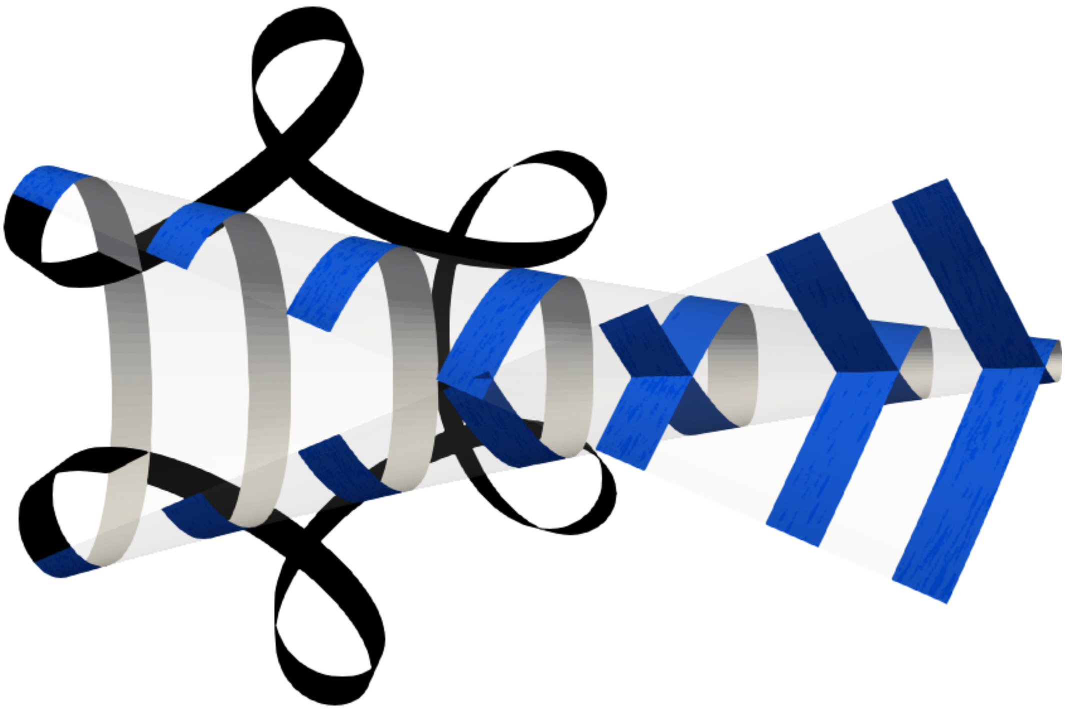
The corrugated map f_1 with $N = 6$.

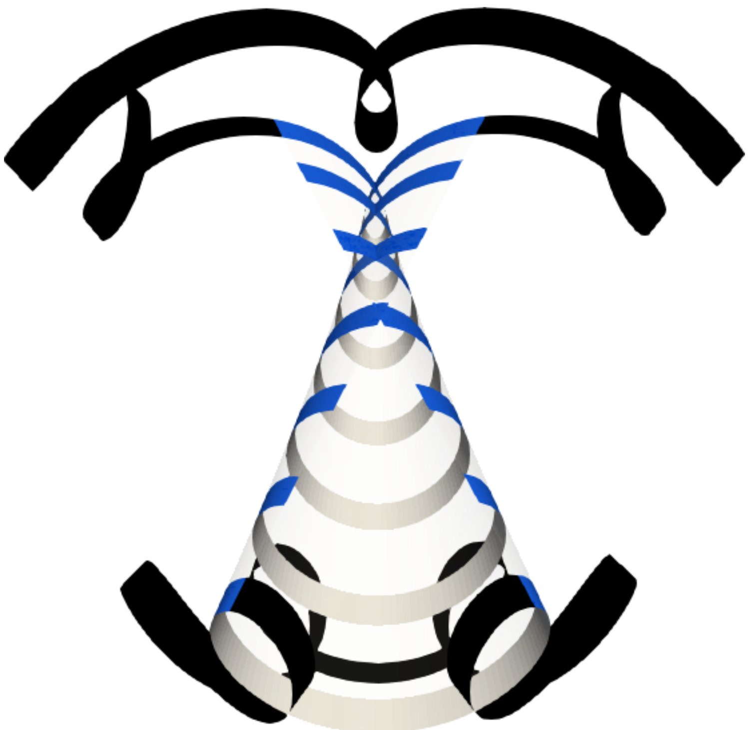
By looking the corrugated surface on the subsets $[0, 1] \times [-1, -1 + \frac{1}{6}]$, $[0, 1] \times [-1, -1 + \frac{3}{6}]$,... we obtain

View from the top | View from the bottom





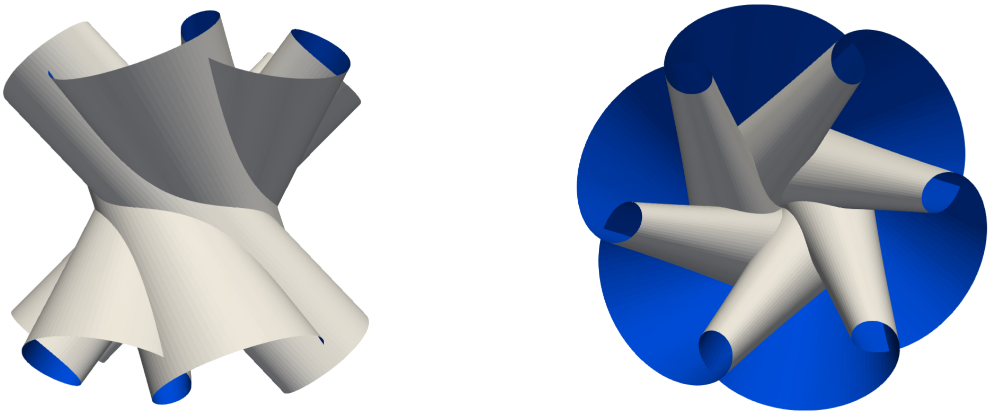




Loop family for immersions (2). By curiosity, we can switch the roles of $\frac{v_1(x)}{\|v_1(x)\|}$ and $\mathbf{n}(x)$, then we consider

$$\gamma : \quad (x,t) \quad \mapsto \quad \gamma(x,t) = r(x)\Big[\cos(2\pi t)\mathbf{n}(x) + \sin(2\pi t)\frac{v_1(x)}{\|v_1(x)\|}\Big] + \partial_1 f_0(x).$$

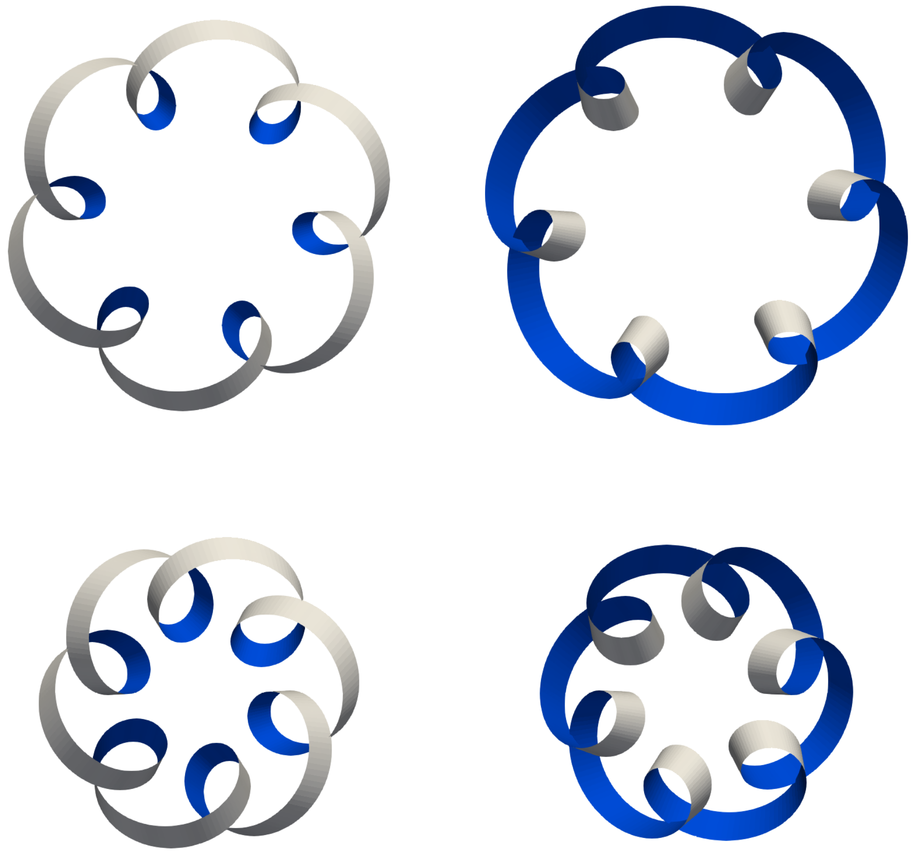
Corrugated map (2). With the previous parameters, we obtain



The corrugated map f_1 with $N = 6$.

Restricting f_1 to $[0,1]\times[-1,-1+\frac{1}{6}]$, $[0,1]\times[-1,-1+\frac{3}{6}]$, $[0,1]\times[-1,-1+\frac{5}{6}]$ we have

View from the top View from the bottom



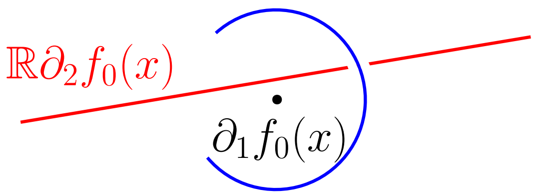
Loop family for immersions (3). Let us now consider an arc-shaped loop family

$$\gamma: \quad (x,t) \quad \mapsto \quad r(x)\Big[\cos(\alpha\cos(2\pi t))\frac{v_1(x)}{\|v_1(x)\|} + \sin(\alpha\cos(2\pi t))\mathbf{n}(x)\Big] + \partial_1 f_0(x).$$

In this case,

$$\begin{aligned}\overline{\gamma}(x) &= r(x)\int_{t=0}^{1}\cos(\alpha\cos(2\pi t))dt\frac{v_1(x)}{\|v_1(x)\|}\\ &= r(x)J_0(\alpha)\frac{v_1(x)}{\|v_1(x)\|} + \partial_1 f_0(x)\end{aligned}$$

where J_0 is the Bessel function of order 0. So for $\alpha \approx 2.4$ the first zero of J_0, the loop γ satisfies the average condition.



Loop family for immersions (4). Let us now consider an arc-shaped loop family

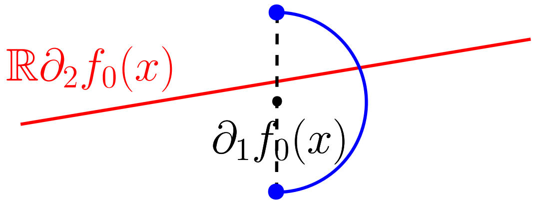
$$\gamma: \quad (x,t) \quad \mapsto \quad r(x)\Big[(\cos(g(t)) - \frac{8}{\pi}\epsilon)\frac{v_1(x)}{\|v_1(x)\|} + \sin(g(t))\mathbf{n}(x)\Big] + \partial_1 f_0(x).$$

with $g(t)$ a piecewise linear map given by

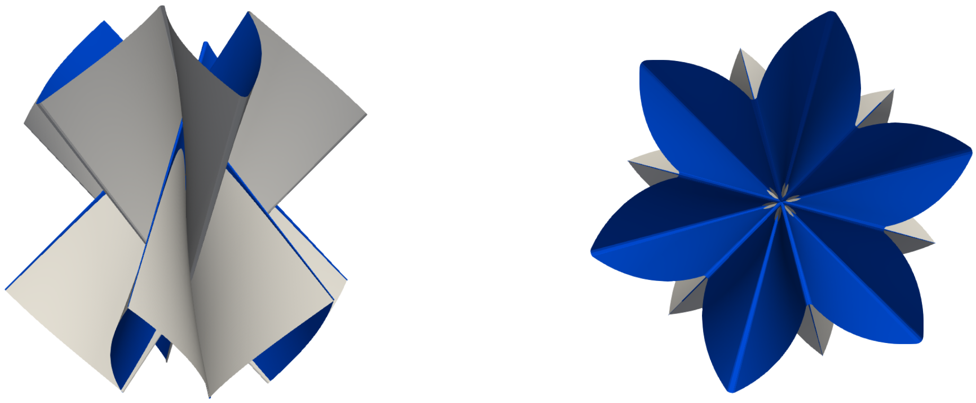
$$g(t) = \begin{cases} \frac{\pi}{2} & t \in [0, \frac{1}{4} - \epsilon] \\ -\frac{\pi}{2\epsilon}t + \frac{1}{4}\frac{\pi}{2\epsilon} & t \in [\frac{1}{4} - \epsilon, \frac{1}{4} + \epsilon] \\ -\frac{\pi}{2} & t \in [\frac{1}{4} + \epsilon, \frac{3}{4} + \epsilon] \\ \frac{\pi}{2\epsilon}t - \frac{3}{4}\frac{\pi}{2\epsilon} & t \in [\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon] \\ \frac{\pi}{2} & t \in [\frac{3}{4} - \epsilon, 1] \end{cases}$$

The average of the loop is

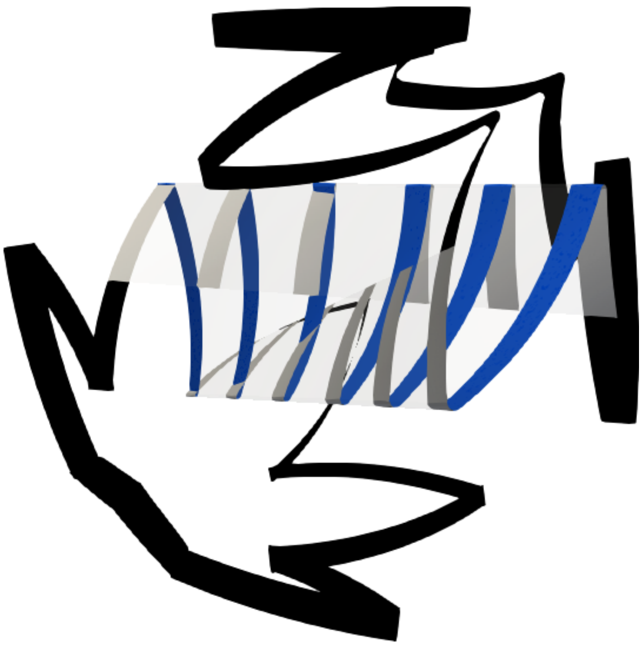
$$\overline{\gamma}(x) = \partial_1 f_0(x)$$



Corrugated map (4). With the previous parameters, we obtain



The corrugated map f_1 with $N = 6$.



Remark on these examples. Note that we have only modified the loop family without change the formal solution (here the vector field v_1). For example, we can choose a formal solution which equals to the 1-jet of f_0 except in a neighborhood of its singular point. This choice allows to modify only locally the surface (see my webpage or my PhD thesis [4] for such examples).

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