Holonomic Approximation through Convex Integration

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Introduction

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The goal of this talk is to bring some new perspective on this topic by proving the Holonomic Approximation Theorem for 1-order jets using Convex Integration.

To prove it, we will show that Holonomic Approximation for 1-order jets can be reduced to proving the *h*-principle for some specific relation. Then we prove this relation is solvable using Convex Integration.

1-jet space and differential constraints

For a map $f : \mathbb{R}^q \to \mathbb{R}^n$, we denote by

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$$\begin{aligned} J^1(\mathbb{R}^q, \mathbb{R}^n) &:= \{(p, y, L) \mid p \in \mathbb{R}^q,\ y \in \mathbb{R}^n,\ L \in \mathscr{L}(T_p\mathbb{R}^q, T_y\mathbb{R}^n)\} \\ &\simeq \mathbb{R}^q \times \mathbb{R}^n \times (\mathbb{R}^n)^q \end{aligned}$$

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A differential constraint of order 1 is a relation

$$\phi(p, f(p), df_p) > 0 \quad (\text{or } = 0)$$

where $\phi : J^1(\mathbb{R}^q, \mathbb{R}^n) \xrightarrow{C^0} \mathbb{R}$.

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where $\phi : J^1(\mathbb{R}^q, \mathbb{R}^n) \xrightarrow{C^0} \mathbb{R}$. In coordinates,

$$\phi(p, f(p), \partial_1 f(p), \dots, \partial_q f(p)) > 0 \quad (\text{or } = 0).$$

1-jet space and differential constraints

Let

$$\begin{array}{rccc} \sigma : & \mathbb{R}^q & \longrightarrow & J^1(\mathbb{R}^q, \mathbb{R}^n) \\ & p & \longmapsto & (p, f(p), L_p) \end{array}$$

be a section of the bundle $J^1(\mathbb{R}^q, \mathbb{R}^n) \to \mathbb{R}^q$.

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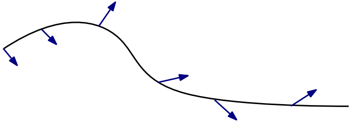
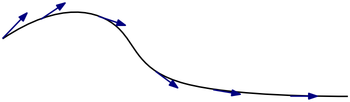


a section $\sigma \neq j^1 f$,



a holonomic section $\sigma = j^1 f$.

1-jet space and differential constraints

Finding a section $\sigma : p \mapsto (p, f(p), L_p)$ satisfying the differential constraint

$$\phi(\sigma(p)) = \phi(p, f(p), L_p) > 0$$

is easier than finding a holonomic section $j^1 F$ (or a map F) satisfying

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Question: From a section σ satisfying ϕ, can we find a holonomic section j^1F (homotopic to σ) satisfying ϕ?

- **Answer of the Convex Integration:** yes if the differential constraint has some convex properties.
- **Answer of the Holonomic Approximation:** yes if we have a positive codimension for the source space.

Convex Integration

To the differential constraint "$\phi((p, f(p), L_p)) > 0$" we associate the subset

$$\mathscr{R}_\phi := \{(p, y, L) \,|\, \phi((p, y, L)) > 0\} \subset J^1(\mathbb{R}^q, \mathbb{R}^n).$$

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We call differential relation any subset $\mathscr{R}$ of $J^1(\mathbb{R}^q, \mathbb{R}^n)$.

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Examples

- $\mathscr{R}_\phi$ is the relation associated to ϕ;
- $\mathscr{I} := \{(p, y, L) \mid \text{the rank of } L \text{ is maximal}\}$ is the relation of immersions.

Convex Integration

Let $\mathscr{R}$ be a relation and σ be a section written in coordinates $\sigma = (p, f, L_1, \ldots, L_q)$ where $L_k = L(e_k)$ for some basis $(e_1, \ldots, e_q)$.

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For $j \in \{1, \ldots, q\}$, σ is a j-solution of $\mathscr{R}$ if for any $p \in \mathbb{R}^q$

$$\sigma(p) = (p, f(p), \partial_1 f(p), \ldots, \partial_j f(p), L_{j+1}, \ldots, L_q) \in \mathscr{R}.$$

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Definition

If for any p we have $\sigma(p) = j^1 f(p) \in \mathscr{R}$, we say that σ is a holonomic solution of $\mathscr{R}$.

Convex Integration

Under some assumptions on the relation $\mathscr{R}$ (we'll see soon) and from a 0-solution $\sigma_0 = (p, f_0, L)$, Convex Integration allows to build a sequence

$$f_0 \to f_1 \to \cdots \to f_j \to \cdots \to f_q$$

such that, $\forall j$, the section

$$\sigma_j : p \mapsto (p, f_j(p), \partial_1 f_j(p), \ldots, \partial_j f_j(p), L_{j+1}, \ldots, L_q) \in \mathscr{R}.$$

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Applying iteratively Convex Integration, we build

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\sigma_0(p) & = & (p, & f_0(p), & L_1, & L_2, & \ldots, & L_q) & \in \mathscr{R} \\
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\sigma_q(p) & = & (p, & f_q(p), & \partial_1 f_q(p), & \partial_2 f_q(p), & \ldots, & \partial_q f_q(p)) & \in \mathscr{R}
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Eventually the section $\sigma_q = j^1 f_q$ is a holonomic solution of $\mathscr{R}$.

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To solve the step "$f_{j-1} \to f_j$", there was several formulas (given here in dimension 1, ie $p = t \in \mathbb{R}$, for the sake of clarity):

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Nash's formula (codim 2, isometric case)

$$f_1(t) := f_0(t) + \frac{1}{N} h \left[\Gamma_1(Nt)\mathbf{n}_1(t) + \Gamma_2(Nt)\mathbf{n}_2(t)\right]$$

with $\Gamma_1(Nt) = \cos(Nt)$, $\Gamma_2(Nt) = \sin(Nt)$, h a parameter depending on the problem, $\mathbf{n}_1$, $\mathbf{n}_2$ two unit normal vectors and $N \in \mathbb{N}$.

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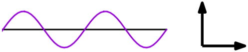
Kuiper's formula (codim 1, isometric case)

$$f_1(t) := f_0(t) + \frac{1}{N} h \left[\Gamma_1(Nt)\mathbf{t}(t) + \Gamma_2(Nt)\mathbf{n}(t)\right]$$

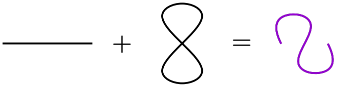


with $\Gamma_1(Nt) = \frac{-a^2 \sin(2Nt)}{8}$, $\Gamma_2(Nt) = a \sin(Nt - \frac{a^2 \sin(2Nt)}{8})$, h and a parameters depending on the problem, $\mathbf{t}$ a unit tangent vector and $\mathbf{n}$ a unit normal vector.

Convex Integration

Thurston's formula (Immersions Theory)

$$f_1(t) := f_0(t) + h\left[\Gamma_1(Nt) + i\Gamma_2(Nt)\right]$$



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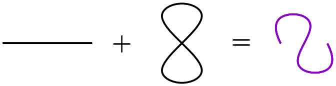

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$$f_1(t) := f_0(t) + \frac{1}{N}\left[\Gamma_1(t, Nt)\mathbf{t}(t) + \Gamma_2(t, Nt)\mathbf{n}(t)\right]$$

with $\Gamma_1(Nt) = \int_0^{Nt} h\cos(a\sin(2\pi s)) - 1ds$,
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Gromov's formula (Convex Integration Theory)

$$f_1(t) := f_0(0) + \int_{s=0}^{t} \gamma_s(Ns)ds$$

with a family of loops $(\gamma_t)_t$ and $N \in \mathbb{N}$.

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$$f_1(p) := f_0(p) + \frac{1}{N}\int_{s=0}^{Np_j} \Big(\gamma_p(s) - \overline{\gamma}_p\Big)ds$$

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- $\|f_1 - f_0\|_{C^0} = O(\frac{1}{N})$ and $\partial_j f_1(t) = \gamma_p(Np_j) + O(\frac{1}{N})$
- $\|\partial_i f_1 - \partial_i f_0\|_{C^0} = O(\frac{1}{N})$ where $i \neq j$.

Convex Integration

Using these formulas, we build a map f_j from f_{j-1}, and the sections

$$\begin{array}{rccccccccc}
\sigma_{j-1} : p \mapsto (& p, & f_{j-1}, & \partial_1 f_{j-1}, & \dots, & \partial_{j-1} f_{j-1}, & L_j, & L_{j+1}, & \dots &) \\
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are close if $O(1/N)$ is small enough. Now setting

$$\sigma_j : p \mapsto (p, f_j, \partial_1 f_j, \ldots, \partial_{j-1} f_j, \partial_j f_j(p) = \gamma_p(Np_j), L_{j+1}, \ldots)$$

we have to find $(\gamma_t)_t$ such that $\sigma_j(p) \in \mathscr{R}$ for any p.

Convex Integration

Definition

Let $\sigma = (p, f, L_1, \ldots, L_q)$ whose image lies in $\mathscr{R}$ and let $j \in \{1, \ldots, q\}$. We define the slice of $\mathscr{R}$ in the direction e_j over σ as

$$\mathscr{R}_{j,\sigma} := \{w \in \mathbb{R}^n \quad | \quad (p, y, L_1, \ldots, L_{j-1}, w, L_{j+1}, \ldots, L_q) \in \mathscr{R}\}.$$

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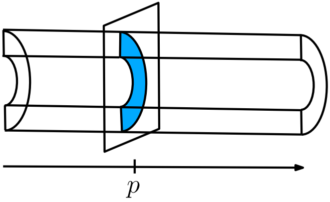
For example, for the relation

$$\mathscr{R} = \{(p, y, L_1, L_2) \quad | \quad (L_1)_{e_1'} \geq 0, \quad r - \epsilon \leq \|L_1\| \leq r + \epsilon\}$$

the slice in the direction e_1 over σ is



Rephrasing of the definition

Let $\sigma = (p, f, L) : M \to \mathscr{R} \subset J^1(M, N)$ and $H \subset T_pM$ be a hyperplane.
The slice of $\mathscr{R}$ for H over σ is

$$\mathscr{R}_{H,\sigma} := \{\widetilde{L} \in \mathscr{L}(T_pM, T_yN) \quad | \quad \widetilde{L}|_H = L|_H \text{ and } (p, y, \widetilde{L}) \in \mathscr{R}\}$$

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Remark. For $M = \mathbb{R}^q$, $N = \mathbb{R}^n$ and $H = Span(e_1, \ldots, \check{e}_j, \ldots, e_q)$, we have the previous definition. Indeed the condition $\widetilde{L}|_H = L|_H$ implies

$$\widetilde{L}(e_i) = L(e_i) = L_i \quad \forall e_i \neq e_j.$$

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Remark. For $M = \mathbb{R}^q$, $N = \mathbb{R}^n$ and $H = Span(e_1, \ldots, \check{e}_j, \ldots, e_q)$, we have the previous definition. Indeed the condition $\widetilde{L}|_H = L|_H$ implies

$$\widetilde{L}(e_i) = L(e_i) = L_i \quad \forall e_i \neq e_j.$$

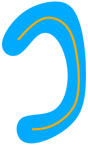
So

$$\mathscr{R}_{H,\sigma} \simeq \{\widetilde{L}_j \in \mathbb{R}^n \quad | \quad (p, y, L_1, \ldots, L_{j-1}, \widetilde{L}_j, L_{j+1}, \ldots, L_q) \in \mathscr{R}\}$$

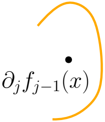
Convex Integration

For a slice, if there exists a loop $t \mapsto \gamma(t)$ such that

- the loop lies in the slice



- the average of the loop is the partial derivative to be modified



then Convex Integration builds a map f_j from f_{j-1}.

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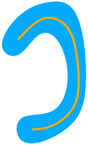

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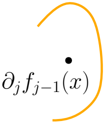

then Convex Integration builds a map f_j from f_{j-1}.

Definition

A relation $\mathscr{R}$ is ample if each slice satisfies one of these conditions:

- the slice is empty;
- the convex hull of each path-component of the slice is the entire fiber.

In the last case, the loop γ always exists!

The slice of the relation of immersions for surfaces. Let σ be a section whose image $\sigma(p) = (p, f_0(p), L_1, L_2)$ lies in $\mathscr{I}$.

Convex Integration

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$$\mathscr{I}_{1,\sigma} \quad := \quad \{w \in \mathbb{R}^n \,|\, (p, f_0(p), w, L_2) \in \mathscr{I}\} = \{w \in \mathbb{R}^n \,|\, w \notin \mathbb{R}L_2\}$$

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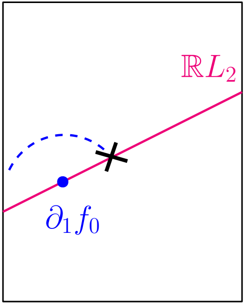

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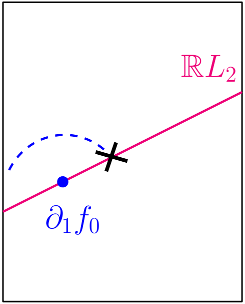

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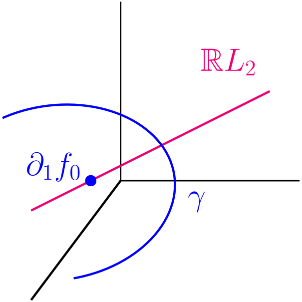

codimension 1 (or more)

Convex Integration

A famous result of Gromov's Convex Integration Theory is:

Theorem (Gromov)

Let $\mathscr{R}$ be an open ample relation. For any section $\sigma = (p, f_0, L)$ whose image belongs to $\mathscr{R}$, there exists a holonomic solution $j^1 f_q$ of $\mathscr{R}$.

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For example, the **relation of immersions in codimension** ≥ 1 **is ample**.

Note that the **relation of** ϵ**-isometric maps**, for $\epsilon > 0$, **is not ample!** The previous construction is possible if the map $f_0 : (M, g) \to (N, h)$ is *short*, ie satisfies

$$h - f_0^* g > 0$$

which is the key assumption of the Nash-Kuiper C^1-isometric Theorem.

Holonomic Approximation

Instead of looking for a holonomic section satisfying a differential constraint, the Holonomic Approximation aims to find a holonomic section j^1F which is close to a given section σ:

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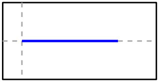
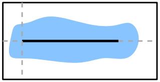
This approach **completely ignores the differential constraint!**

Note that, over a point, a section can be approximated by the 1-jet of a Taylor polynomial map, while, over a submanifold, the problem is usually unsolvable.

Holonomic Approximation

Let $A = [0,1]^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}$ and $p = (x,t) \in \mathbb{R}^m \times \mathbb{R}$.

A  $Op(A)$ 

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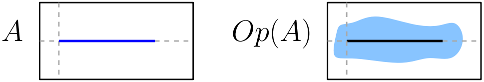
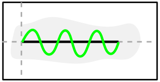


Holonomic Approximation Theorem for 1-jets and $A = [0,1]^m \times \{0\}$

Let $\sigma = ((x,t), f_0, L) : Op(A) \to J^1(\mathbb{R}^m \times \mathbb{R}, \mathbb{R}^n)$ be a section. For every $\epsilon > 0$, there exists

- a function $\delta : \mathbb{R}^m \to \mathbb{R}$ such that $\|\delta\| < \epsilon$, and we set

$$A_\delta := \{(x, \delta(x)) \,|\, x \in [0,1]^m\} =$$



- a map f_1 defined near A_δ such that $\|j^1 f_1 - \sigma\|_{C^0} < \epsilon$ on a sufficiently small open neighborhood of A_δ.

Holonomic Approximation

For any subset A, we denote by $Op(A)$ an open neighborhood of A.

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Holonomic Approximation Theorem (Eliashberg - Mishachev)

Let $r \in \mathbb{N}$. Let $A \subset \mathbb{R}^q$ be a polyhedron of positive codimension $k > 0$ and

$$\sigma : Op(A) \to J^r(\mathbb{R}^q, \mathbb{R}^n)$$

be a section. For every $\epsilon > 0$ there exists

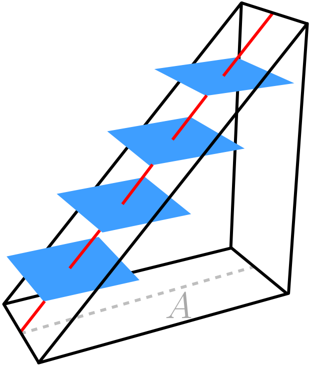
- a function $\delta : \mathbb{R}^{q-k} \to \mathbb{R}^k$ such that $\|\delta\| < \epsilon$, and we set

$$A_\delta := \{(x, \delta(x)) \quad | \quad (x, 0_k) \in A\}$$

- a holonomic section $j^r f_1 : Op(A_\delta) \to J^r(\mathbb{R}^q, \mathbb{R}^n)$ such that

$$\|j^r f_1 - \sigma|_{Op(A_\delta)}\|_{C^0} < \epsilon.$$

Example of the Mountain Path



For $m = 1$ and $A = [0,1] \times \{0\}$, let

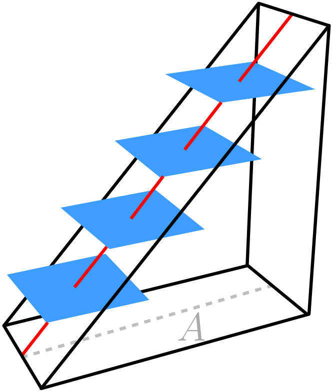
$$\sigma : Op(A) \to J^1(Op(A), \mathbb{R})$$

be the section given by

$$f_0 : (x,t) \mapsto x, \quad L_{(x,t)} = 0$$

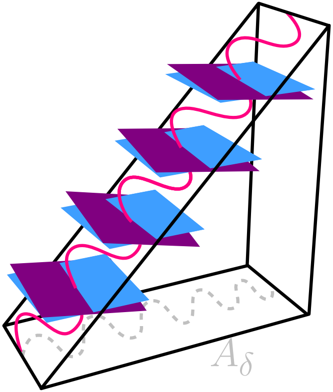
for every $(x,t) \in Op(A)$.

Example of the Mountain Path

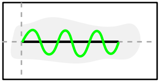


Holonomic
Approximation
Theorem

$\dashrightarrow$



Holonomic Approximation: there exists a perturbation A_δ of A and there exists a holonomic solution $j^1 f_1$ such that



$$\|j^1 f_1 - \sigma|_{Op(A_\delta)}\| < \epsilon.$$

Slice of the Mountain Path

Let

$$\begin{array}{rccc} \sigma : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \\ & (x,t) & \longmapsto & ((x,t), f_0(x,t) = x, L_{(x,t)} = 0) \end{array}$$

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the associated relation is

$$\mathscr{R} := \{((x,t), y, v_1, v_2) \quad | \quad \|y - x\| < \epsilon, \quad \|v_i\| < \epsilon\}$$

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and the slice in the direction 1 is

$$\mathscr{R}_1 := \{w \in \mathbb{R} \quad | \quad \|w\| < \epsilon\}$$

which is not ample !

Holonomic Approximation through Convex Integration

Theorem (Massot-T. 2021)

Every problem solvable by Holonomic Approximation for 1-order jets can be solved using Convex Integration.

Holonomic Approximation through Convex Integration

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The proof splits in two parts:

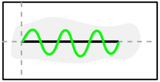
- a rewriting of the Holonomic Approximation as a relation $\mathscr{R}_{ha}$;
- a proof that $\mathscr{R}_{ha}$ is open and ample.

Part I - The rewriting

Our aim

From a section $\sigma = ((x,t), f_0, L) : Op(A) \to J^1(Op(A), \mathbb{R}^n)$, for any $\epsilon > 0$, we are looking for a function δ and a map f_1 such that

$$\|\delta\| < \epsilon, \quad \|j^1 f_1 - \sigma|_{Op(A_\delta)}\| < \epsilon$$

where A_δ is a perturbation  of A.

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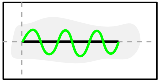
where A_δ is a perturbation  of A.

Writing $j^1 f_1$ over $A_\delta = \{(x, \delta(x))\}$ we have

$$j^1 f_1(x, \delta(x)) \;=\; (\, (x, \delta(x)),\; f_1(x, \delta(x)),\; (df_1)_{(x,\delta(x))}\,)$$

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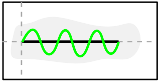
where A_δ is a perturbation  of A.

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and we would like to rewrite it under the form

$$j^1(\delta, w)(x) \;=\; (\,x,\; (\delta(x), w(x)),\; (d\delta_x, dw_x)\,)$$

Part I - The rewriting

Note that if we find $x \mapsto (\delta(x), w(x))$ then we can find f_1 on A_δ setting

$$(x, \delta(x)) \longmapsto w(x) = f_1(x, \delta(x))$$

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So from the previous condition

$$\|\delta\| < \epsilon, \quad \|j^1 f_1 - \sigma|_{A_\delta}\| < \epsilon \; (\Rightarrow \|f_1 - f\| < \epsilon, \; \|df_1 - L\| < \epsilon$$

now we have

$$\|\delta\| < \epsilon, \quad \|w - f(\cdot, \delta(\cdot))\| < \epsilon, \quad \|(dw \circ \pi_m - L)|_{TA_\delta}\| < \epsilon.$$

Lemma

Holonomic Approximation for 1-order jets can be rewritten as the differential relation

$$\mathscr{R}_{ha} := \left\{ (x,(y,w),(Y,W)) \;\middle|\; \begin{array}{l} \|y\| < \epsilon, \quad \|w - f(x,y)\| < \epsilon \\ \|(W \circ \pi_m - L_{(x,y)})|_{TA_y}\| < \epsilon \end{array} \right\}$$

Remark. Observe that $\mathscr{R}_{ha} \subset J^1(\mathbb{R}^m, \mathbb{R} \times \mathbb{R}^n)$ while $\sigma \in J^1(\mathbb{R}^m \times \mathbb{R}, \mathbb{R}^n)$.

Part II - Ampleness

Now we are going to show the realtion $\mathscr{R}_{ha}$ is (open) ample, ie slices of

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For $n = 1$, the slice $\mathscr{R}_{ha,(H,\mu)}$ is the interior of a hyperbola, so ample.

Then, it is enough to apply this lemma n times to show that $\mathscr{R}_{ha}$ is ample.

Part II - Ampleness

Let $\mu = (x, (y_0, w_0), (Y_0, W_0))$ be a section of $\mathscr{R}_{ha}$ and let H be a hyperplane of $T_x\mathbb{R}^m$.

By definition the slice is

$$\mathscr{R}_{ha,(H,\mu)} = \left\{ (Y, W) \in \mathscr{L}(\mathbb{R}^m, \mathbb{R} \times \mathbb{R}^n) \,\middle|\, \begin{array}{l} \|(W \circ \pi_m - L)|_{TA_y}\| < \epsilon \\ (Y, W)|_H = (Y_0, W_0)|_H \end{array} \right\}$$

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Let $(u, u') \in T_x\mathbb{R}^m$ with $u \in H$. From the second condition, we can set

$$Y(u, u') = Y_0 u + \alpha u', \quad W(u, u') = W_0 u + \beta u', \quad (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^n.$$

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Then

$$\mathscr{R}_{ha,(H,\mu)} = \left\{ (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^n \;\middle|\; \begin{array}{l} \|(W \circ \pi_m - L)|_{TA_y}\| < \epsilon \\ Y = (Y_0, \alpha), \quad W = (W_0, \beta) \end{array} \right\}$$

Developing and making suitable changes of variables $(\alpha, \beta) \leftrightarrow (a, b)$ and $(Y_0, W_0) \leftrightarrow (\widetilde{Y_0}, \widetilde{W_0})$ we obtain

$$\mathscr{R}_{ha,(H,\mu)} \simeq \left\{ \begin{array}{l} (a,b) \\ \in \mathbb{R} \times \mathbb{R}^n \end{array} \middle| \begin{array}{l} \forall (u,u') \in (\mathbb{R}^{m-1} \times \mathbb{R}) \backslash \{0\}, \\ \|u'b + \widetilde{W_0} u\|^2 < \epsilon^2 \Big(u'^2 + \|u\|^2 + (au' + \widetilde{Y_0} u)^2 \Big) \end{array} \right\}$$

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Part II - Ampleness

Developing and making suitable changes of variables $(\alpha, \beta) \leftrightarrow (a, b)$ and $(Y_0, W_0) \leftrightarrow (\widetilde{Y_0}, \widetilde{W_0})$ we obtain

$$\mathscr{R}_{ha,(H,\mu)} \simeq \left\{ \begin{array}{l} (a,b) \\ \in \mathbb{R} \times \mathbb{R}^n \end{array} \middle| \begin{array}{l} \forall (u, u') \in (\mathbb{R}^{m-1} \times \mathbb{R}) \backslash \{0\}, \\ \|u'b + \widetilde{W_0} u\|^2 < \epsilon^2 \left(u'^2 + \|u\|^2 + (au' + \widetilde{Y_0} u)^2 \right) \end{array} \right\}$$

For $m = 1$, we have

$$\begin{aligned} \mathscr{R}_{ha,(H,\mu)} &\simeq \left\{ \begin{array}{l} (a,b) \\ \in \mathbb{R} \times \mathbb{R}^n \end{array} \middle| \begin{array}{l} \forall u' \in \mathbb{R} \backslash \{0\}, \\ \|u'b\|^2 < \epsilon^2 \left(u'^2 + (au')^2 \right) \end{array} \right\} \\ &\simeq \left\{ (a,b) \;\middle|\; \|b\|^2 - \epsilon^2 a^2 < \epsilon^2 \right\} \end{aligned}$$

which is the interior of a hyperbola, so ample.

Lemma

For $n = 1$, there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that the slice

$$\mathscr{R}_{ha,(H,\mu)} \simeq \left\{ (a,b) \in \mathbb{R} \times \mathbb{R} \;\middle|\; (b - c_1 a)^2 - c_2^2 a^2 < c_3^2 \right\}$$

is the interior of a hyperbola.

But the proof is technically not straightforward.

Lemma

For $n = 1$, there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that the slice

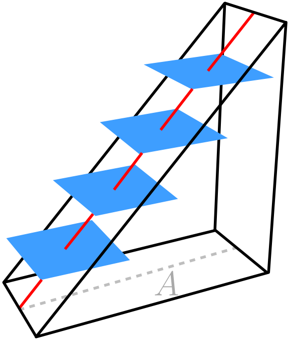
$$\mathscr{R}_{ha,(H,\mu)} \simeq \left\{(a,b) \in \mathbb{R} \times \mathbb{R} \;\middle|\; (b - c_1 a)^2 - c_2^2 a^2 < c_3^2 \right\}$$

is the interior of a hyperbola.

But the proof is technically not straightforward.

So the relation $\mathscr{R}_{ha}$ is ample, in particular we can solve the problem of the Mountain Path using Convex Integration.

The Mountain Path via Convex Integration



Convex Integration

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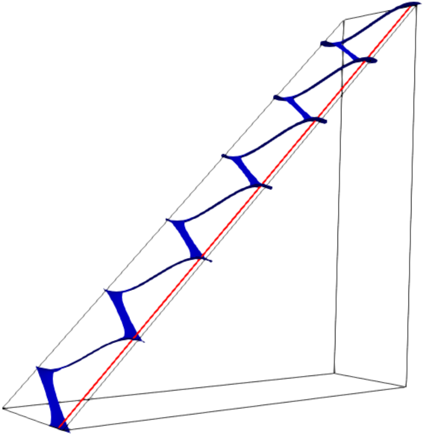

Thank you for your attention !