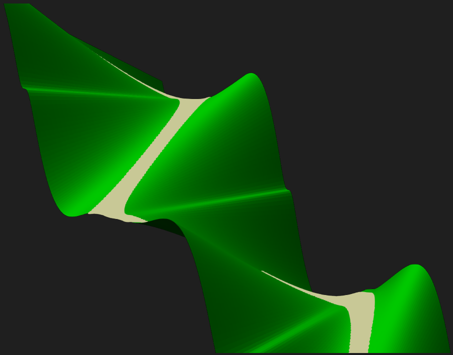
Holonomic Approximation through Convex Integration



(mountain path)

Mélanie Theillière, University of Luxembourg
joint work with Patrick Massot, Université Paris-Saclay

Introduction

M. Gromov

Yesterday, we had an overview of Gromov's Convex Integration Theory:

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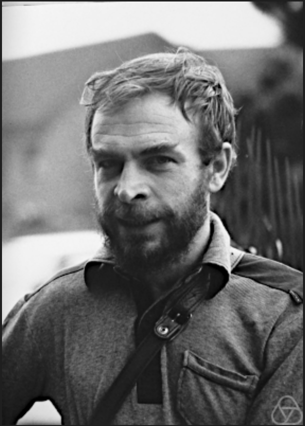

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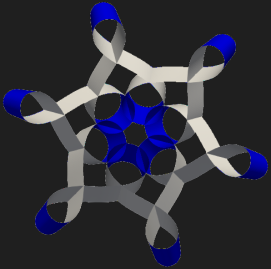
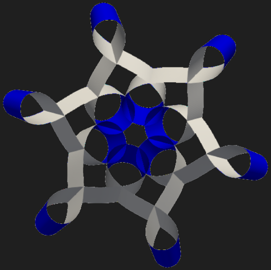
- we deformed singular surfaces to build immersions;
- we deformed surfaces with too short lengths to build isometric maps.

This idea of deformation is at the basis of the *h*-principle established by Gromov in the 1970's.



The idea of the *h*-principle is

- we start with an initial map f_0 and "false" derivatives (vector fields satisfying the differential constraint instead of the $\partial_i f_0$);
- if the problem satisfies some conditions, we deform the map f_0 to a solution f.



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Convex Integration is one of the main techniques to prove a *h*-principle.

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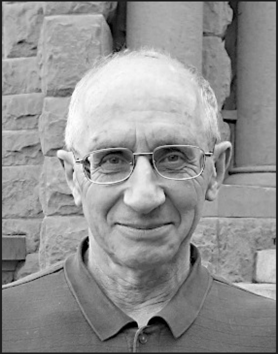

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N. Mishachev

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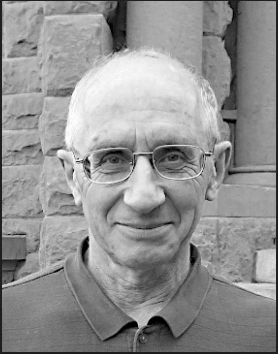

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In this talk, we start by presenting the *h*-principle and its proof using Convex Integration, then we proof the Holonomic Approximation for differential constraints of order 1 through Convex Integration.

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the 1-jet of f defined from $\mathbb{R}^q$ to the 1-jets space

$$\begin{aligned} J^1(\mathbb{R}^q, \mathbb{R}^n) &:= \{(p, y, L) \,|\, p \in \mathbb{R}^q,\ y \in \mathbb{R}^n,\ L \in \mathscr{L}(T_p\mathbb{R}^q, T_y\mathbb{R}^n)\} \\ &\simeq \mathbb{R}^q \times \mathbb{R}^n \times (\mathbb{R}^n)^q \end{aligned}$$

Let $\sigma : p \mapsto (p, f(p), L_p)$ be a section of the 1-jet space. We now can consider

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For example, the subset $\mathscr{I} := \{(p, y, L) \mid \text{the rank of } L \text{ is maximal}\}$ is the relation of immersions.

I - The h-principle with the Convex Integration

Finding a section σ of image in $\mathscr{R}$, ie

$$p \longmapsto \sigma(p) = (p, f(p), L_p) \in \mathscr{R}$$

is easier than finding a 1-jet $j^1 F$ (or a map F) of image in $\mathscr{R}$, ie

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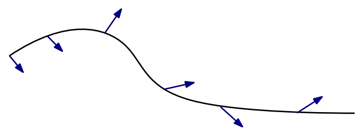
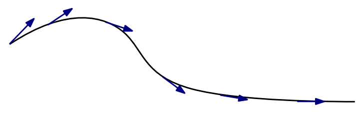
In dimension $q = 1$, we have $\sigma =$ (points, a curve, a vector field):



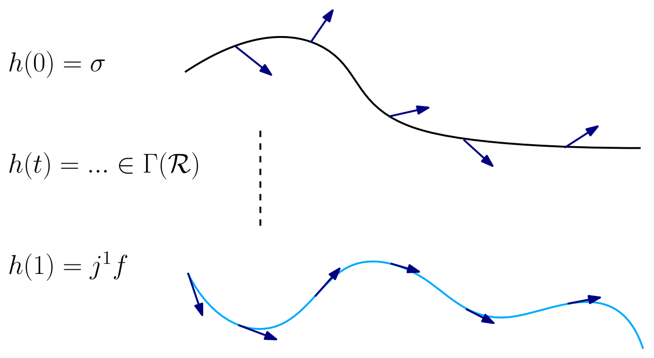
a formal solution,



a holonomic solution.

The property of the *h*-principle

Let $\mathscr{R}$ be a relation. If each formal solution σ can be homotopically deformed, in the space of sections of $\mathscr{R}$, to a holonomic solution j^1F, then the relation $\mathscr{R}$ satisfies the *h*-principle.



The idea of Convex Integration is to consider a formal solution and to change partial derivatives of f_0 one by one

$$\begin{array}{lcllllll} \sigma_0(p) & = & (p, & f_0(p), & L_1, & L_2, & \dots, & L_q) \in \mathscr{R} \\ \sigma_1(p) & = & (p, & f_1(p), & \partial_1 f_1(p), & L_2, & \dots, & L_q) \in \mathscr{R} \\ \sigma_2(p) & = & (p, & f_2(p), & \partial_1 f_2(p), & \partial_2 f_2(p), & \dots, & L_q) \in \mathscr{R} \\ & & & \dots & & & & \end{array}$$

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 & & & \dots & & & &
\end{array}
$$

to a holonomic solution

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\sigma_q(p) = (p, \; f_q(p), \; \partial_1 f_q(p), \; \partial_2 f_q(p), \; \dots, \; \partial_q f_q(p)) \in \mathscr{R}
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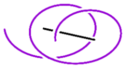
Nash's formula (codim 2, isometric case)

$$f_j(t) := f_{j-1}(t) + \frac{1}{N} h \left[\Gamma_1(Nt)\mathbf{n}_1(t) + \Gamma_2(Nt)\mathbf{n}_2(t)\right]$$



with $\Gamma_1(\cdot) = \cos(\cdot)$, $\Gamma_2(\cdot) = \sin(\cdot)$, h a parameter of the problem, $\mathbf{n}_1$, $\mathbf{n}_2$ two normal vectors and $N \in \mathbb{N}$.

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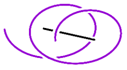

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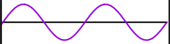
Kuiper's formula (codim 1, isometric case)

$$f_j(t) := f_{j-1}(t) + \frac{1}{N} h \left[\Gamma_1(Nt)\mathbf{t}(t) + \Gamma_2(Nt)\mathbf{n}(t)\right]$$



with $\Gamma_1(Nt) = \frac{-a^2 \sin(2Nt)}{8}$, $\Gamma_2(Nt) = a \sin(Nt - \frac{a^2 \sin(2Nt)}{8})$, h and a parameters of the problem, $\mathbf{t}$ a tangent vector and $\mathbf{n}$ a normal vector.

Gromov's formula (Convex Integration Theory in 1D)

$$f_j(t) := f_{j-1}(0) + \int_{s=0}^{t} \gamma_s(Ns)ds$$

with a loop family $(\gamma_t)_t$ et $N \in \mathbb{N}$.

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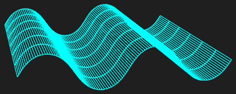
Corrugation Process (T. 2020)

$$f_j(p) := f_{j-1}(p) + \frac{1}{N}\int_{s=0}^{Np_j} \left(\gamma_p(s) - \overline{\gamma}_p\right)ds$$



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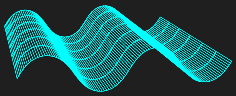

If, for every $p \in \mathbb{R}^q$, we have $\overline{\gamma}_p = \partial_j f_{j-1}(p)$, then these formulas satisfy:

- $\partial_j f_j(p) = \gamma_p(Np_j) + O(1/N)$
- $f_j(p) = f_{j-1}(p) + O(1/N)$
- $\partial_i f_j(p) = \partial_i f_{j-1}(p) + O(1/N)$, for every $i \neq j$

With these formulas we build f_j and

$$\begin{array}{rlllllllll}\sigma_{j-1} : p \mapsto (& p, & f_{j-1}, & \partial_1 f_{j-1}, & \dots, & \partial_{j-1} f_{j-1}, & L_j, & L_{j+1}, & \dots &) \\ & & \downarrow & \downarrow & & \downarrow & & & & \\ p \mapsto (& p, & f_j, & \partial_1 f_j, & \dots, & \partial_{j-1} f_j, & L_j, & L_{j+1}, & \dots &)\end{array}$$

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are close up to $O(1/N)$. We now set

$$\sigma_j : p \mapsto (p, f_j, \partial_1 f_j, \dots, \partial_{j-1} f_j, \partial_j f_j(p) = \gamma_p(Np_j), L_{j+1}, \dots)$$

so we have to find loops $(\gamma_p)_p$ such that $\sigma_j(p) \in \mathscr{R}$ for every p.

Definition

Let $\sigma = (p, f, L_1, \ldots, L_q)$ of image in $\mathscr{R}$ and $j \in \{1, \ldots, q\}$. We define the slice of $\mathscr{R}$ in the direction ∂_j over σ as

$$\mathscr{R}_{j,\sigma} := \{w \in \mathbb{R}^n \quad | \quad (p, f(p), L_1, \ldots, L_{j-1}, w, L_{j+1}, \ldots, L_q) \in \mathscr{R}\}.$$

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If we are not using coordinates, we can also define the slice fora section $\sigma = (p, f, L)$ and a hyperplane $H \subset T_p\mathbb{R}^q$ setting

$$\mathscr{R}_{H,\sigma} := \{\widetilde{L} \in \mathscr{L}(T_p\mathbb{R}^q, T_{f(p)}\mathbb{R}^n) \quad | \quad \widetilde{L}|_H = L|_H \text{ et } (p, f(p), \widetilde{L}) \in \mathscr{R}\}$$

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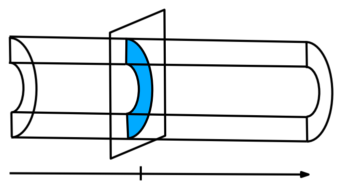
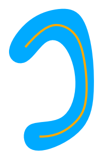
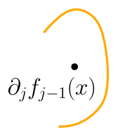


For a slice, if there exists a loop $t \mapsto \gamma(t)$ such that

- the image of γ is in the slice



- the average of γ equals to the derivative we want to modify



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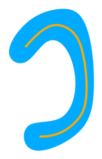

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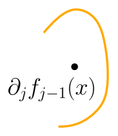

then we can build f_j from f_{j-1}.

In particular, if the convex hull of each path-component of the slice is the entire fiber, then the loop γ always exists. Such a relation is called ample.

The slice of the relation of immersions for surfaces. Let σ be a section whose image $\sigma = (p, f_0, L_1, L_2)$ is in $\mathscr{I}$.

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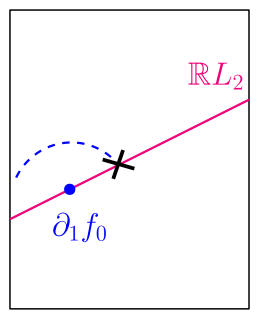

codimension 0

I - The h-principle with the Convex Integration

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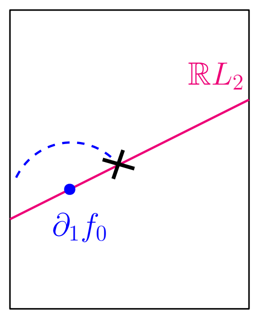

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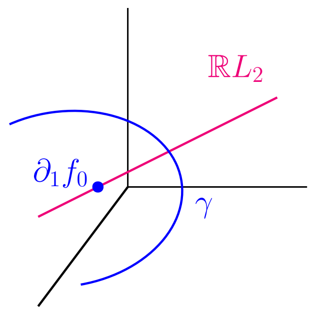

codimension 1 (or more)

Convex Integration Theorem (ample case)

Let $\mathscr{R}$ be an open and ample relation. Any formal solution σ can be deformed (homotopically in the space of sections in $\mathscr{R}$) to a holonomic solution $j^1 f_q$.

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The relation of immersions in codimension ≥ 1 is ample.

The relation of ϵ-isometric maps is not ample! Nevertheless the initial map is assumed to be short so belongs to the convex hull of the slice.

So these two relations satisfy the *h*-principle.

The question of Holonomic Approximation

Let σ be a section. The aim of Holonomic Approximation is to directly find a holonomic section j^1F close to σ:

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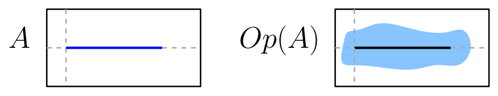
Note that, over a point p, $\sigma(p)$ can be approximated by the jet of a Taylor polynomial map, while, over a submanifold, the problem is usually unsolvable.

II - Holonomic Approximation through Convex Integration

Let $A = [0,1]^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}$ and $p = (x,t) \in \mathbb{R}^m \times \mathbb{R}$.



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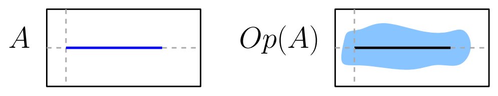
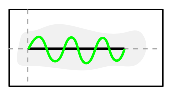


Holonomic Approximation theorem for order 1 and $A = [0,1]^m \times \{0\}$

Let $\sigma = ((x,t), f_0, L) : Op(A) \to J^1(Op(A), \mathbb{R}^n)$ be a section. For any $\epsilon > 0$, there exists

- a function $\delta : \mathbb{R}^m \to \mathbb{R}$ such that $\|\delta\| < \epsilon$, and we set

$$A_\delta := \{(x, \delta(x)) \,|\, x \in [0,1]^m\} =$$



- a map f_1 defined near A_δ such that $\|j^1 f_1 - \sigma\|_{C^0} < \epsilon$ on a sufficiently small open neighborhood of A_δ.

II - Holonomic Approximation through Convex Integration

For any subset A, we denote by $Op(A)$ an open neighborhood of A.

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Holonomic Approximation theorem (Eliashberg - Mishachev)

Let $r \in \mathbb{N}$. Let $A \subset \mathbb{R}^q$ be a polyhedron of positive codimension $k > 0$ and

$$\sigma : Op(A) \to J^r(Op(A), \mathbb{R}^n)$$

be a section. For every $\epsilon > 0$ there exists

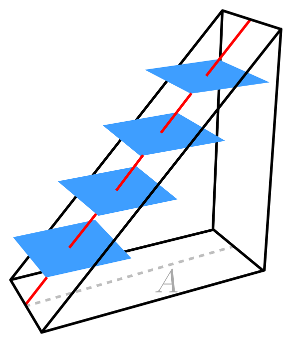
- a function $\delta : \mathbb{R}^{q-k} \to \mathbb{R}^k$ such that $\|\delta\| < \epsilon$, and we set

$$A_\delta := \{(x, \delta(x)) \quad | \quad (x, 0_k) \in A\}$$

- a holonomic section $j^r f_1 : Op(A_\delta) \to J^r(\mathbb{R}^q, \mathbb{R}^n)$ such that

$$\|j^r f_1 - \sigma|_{Op(A_\delta)}\|_{C^0} < \epsilon.$$

Example of the mountain path:



For $m = 1$ and $A = [0, 1] \times \{0\}$, let

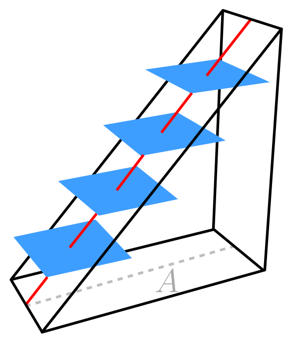
$$\sigma : Op(A) \to J^1(Op(A), \mathbb{R})$$

be the section given by

$$f_0 : (x, t) \mapsto x, \quad L_{(x,t)} = 0$$

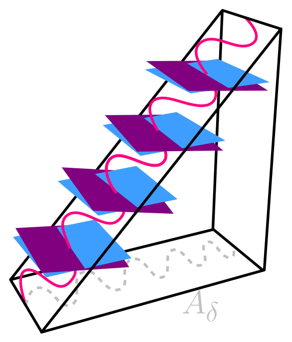
for every $(x, t) \in Op(A)$.

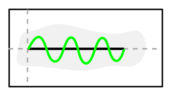
II - Holonomic Approximation through Convex Integration



Holonomic Approximation Theorem

$\dashrightarrow$



There exists a perturbation A_δ  of A and there exists a holonomic solution $j^1 f_1$ such that

$$\|j^1 f_1 - \sigma|_{Op(A_\delta)}\| < \epsilon.$$

Slice of the mountain path: let

$$\begin{array}{rccc} \sigma : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \\ & (x,t) & \longmapsto & ((x,t), f_0(x,t) = x, L_{(x,t)} = 0) \end{array}$$

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and the slice in the direction 1 is

$$\mathscr{R}_1 := \{w \in \mathbb{R} \quad | \quad \|w\| < \epsilon\}$$

which is not ample !

Theorem (Massot-T. 2021)

Every problem solvable by Holonomic Approximation for 1-order jets can be solved using Convex Integration.

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Every problem solvable by Holonomic Approximation for 1-order jets can be solved using Convex Integration.

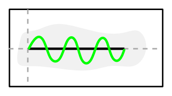
The proof splits in two parts:

- a rewriting of the Holonomic Approximation as a relation $\mathscr{R}_{ha}$;
- a proof that $\mathscr{R}_{ha}$ is open and ample.

What we're looking for:

From a section $\sigma = ((x,t), f_0, L) : Op(A) \to J^1(Op(A), \mathbb{R}^n)$, for every $\epsilon > 0$, we are looking for a function δ and a map f_1 such that

$$\|\delta\| < \epsilon, \quad \|j^1 f_1 - \sigma|_{Op(A_\delta)}\| < \epsilon$$

where A_δ is a perturbation  of A.

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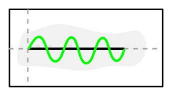
where A_δ is a perturbation  of A.

By writing $j^1 f_1$ over $A_\delta = \{(x, \delta(x))\}$ we have

$$j^1 f_1(x, \delta(x)) \;=\; (\,(x, \delta(x)),\; f_1(x, \delta(x)),\; (df_1)_{(x,\delta(x))}\,)$$

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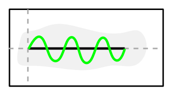
where A_δ is a perturbation  of A.

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and we would like to rewrite it under the form

$$j^1(\delta, w)(x) \;=\; (\ x,\ (\delta(x), w(x)),\ (d\delta_x, dw_x)\ )$$

Part I - The rewriting

Note that if we find $x \mapsto (\delta(x), w(x))$ then for every $(x, \delta(x)) \in A_\delta$, we have f_1 by setting

$$(x, \delta(x)) \longmapsto w(x) = f_1(x, \delta(x))$$

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now we have

$$\|\delta\| < \epsilon, \quad \|w - f(\cdot, \delta(\cdot))\| < \epsilon, \quad \|(dw \circ \pi_m - L)|_{TA_\delta}\| < \epsilon.$$

Lemma

Holonomic Approximation for 1-order jets can be rewritten as the differential relation

$$\mathscr{R}_{ha} := \left\{ (x,(y,w),(Y,W)) \;\middle|\; \begin{array}{l} \|y\| < \epsilon, \quad \|w - f(x,y)\| < \epsilon \\ \|(W \circ \pi_m - L_{(x,y)})|_{TA_y}\| < \epsilon \end{array} \right\}$$

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Observe that $\sigma \in J^1(\mathbb{R}^m \times \mathbb{R}, \mathbb{R}^n)$ but $\mathcal{R}_{ha} \subset J^1(\mathbb{R}^m, \mathbb{R} \times \mathbb{R}^n)$.

Part II - Ampleness

Now we are going to show the relation $\mathscr{R}_{ha}$ is (open) ample, ie slices of

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The final result of ampleness comes from applying the previous lemma n times, component to component on the variable belonging to $\mathbb{R}^n$.

Let $\mu = (x, (y_0, w_0), (Y_0, W_0))$ be a section of $\mathscr{R}_{ha}$ and H be a hyperplane of $T_x\mathbb{R}^m$.

By definition the slice is

$$\mathscr{R}_{ha,(H,\mu)} = \left\{ (Y, W) \in \mathscr{L}(\mathbb{R}^m, \mathbb{R} \times \mathbb{R}^n) \,\middle|\, \begin{array}{l} \|(W \circ \pi_m - L)|_{TA_y}\| < \epsilon \\ (Y, W)|_H = (Y_0, W_0)|_H \end{array} \right\}$$

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Let $(u, u') \in T_x\mathbb{R}^m$ with $u \in H$. From the second condition, we can set

$$Y(u, u') = Y_0 u + \alpha u', \quad W(u, u') = W_0 u + \beta u', \quad (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^n.$$

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Then

$$\mathscr{R}_{ha,(H,\mu)} = \left\{ (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^n \;\middle|\; \begin{array}{l} \|(W \circ \pi_m - L)|_{TA_y}\| < \epsilon \\ Y = (Y_0, \alpha), \quad W = (W_0, \beta) \end{array} \right\}$$

Developing and making suitable changes of variables $(\alpha, \beta) \leftrightarrow (a, b)$ and $(Y_0, W_0) \leftrightarrow (\widetilde{Y_0}, \widetilde{W_0})$ we obtain

$$\mathscr{R}_{ha,(H,\mu)} \simeq \left\{ \begin{array}{l} (a, b) \\ \in \mathbb{R} \times \mathbb{R}^n \end{array} \middle| \begin{array}{l} \forall (u, u') \in (\mathbb{R}^{m-1} \times \mathbb{R}) \setminus \{0\}, \\ \|u' b + \widetilde{W_0} u\|^2 < \epsilon^2 \left(u'^2 + \|u\|^2 + (a u' + \widetilde{Y_0} u)^2 \right) \end{array} \right\}$$

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which is the interior of a hyperbola, so ample.

Lemma

For $n = 1$, there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that the slice

$$\mathscr{R}_{ha,(H,\mu)} \simeq \left\{ (a, b) \in \mathbb{R} \times \mathbb{R} \;\middle|\; (b - c_1 a)^2 - c_2^2 a^2 < c_3^2 \right\}$$

is the interior of a hyperbola.

(but the proof is technically not straightforward)

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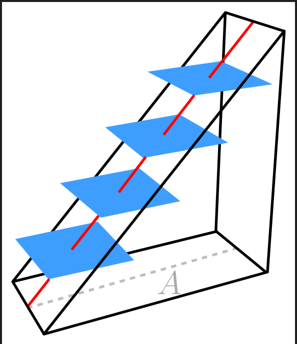
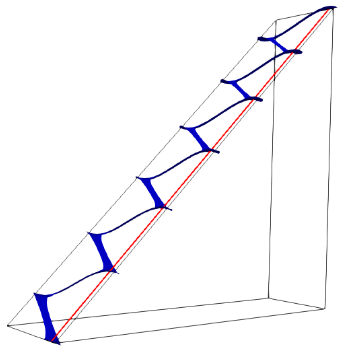
So the relation $\mathscr{R}_{ha}$ is ample, in particular we can solve the problem of the mountain path using Convex Integration.

II - Holonomic Approximation through Convex Integration



Convex
Integration

⇢



Thank you for your attention!