Convex Integration and C^1-fractal behavior

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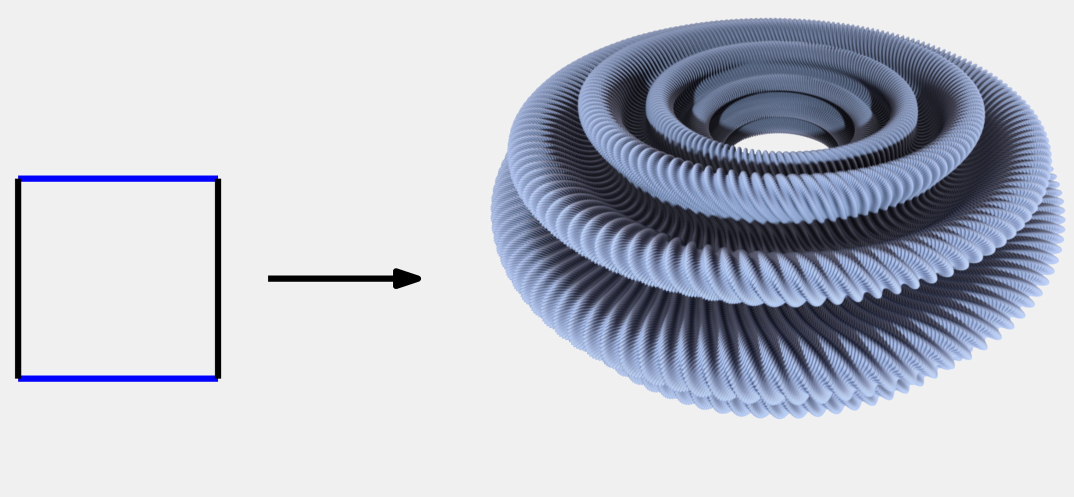
Images and unsuspected behavior

In 2012, a C^1-isometric embedding of a flat torus was built and visualized by a french team. The construction of such an embedding amounts to gluing the opposite sides of a square sheet of paper without folding it. The visualization reveals a C^1-fractal behavior: the differential of the embedding shows self-similarity properties. This construction is based on the Convex Integration of M. Gromov created in 70's and which allows to solve a large family of differential problems.



(image: Hevea Project)

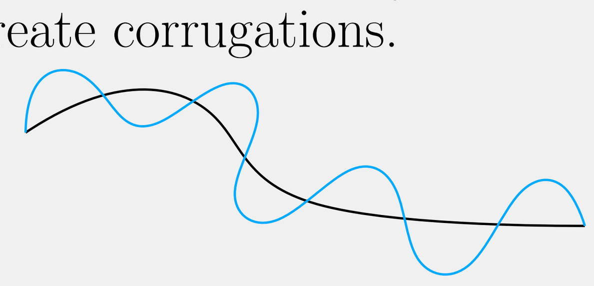
Nevertheless the C^1-fractal behavior observed for the flat torus does not follow from the Convex Integration and it is unclear if this behavior could be observed for more general differential relations. By simplifying the Convex Integration, we show that a class of differential relations, that we called "Kuiper relation", enables the appearance of self-similarity properties. We give an example of such an appearance for C^1-totally real isometric embeddings.

Convex Integration

The fundamental formula in Convex Integration is the following:

$$f_1(x) = f_0(x_1, \dots, 0, \dots, x_m) + \int_{t=0}^{x_i} \gamma(x_1, \dots, t, \dots, x_m, Nt)dt$$

where $f_0 : [0,1]^m \to \mathbb{R}^n$ and $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ such that for every $x \in [0,1]^m$ we have $\int_0^1 \gamma(x,t)dt = \partial_i f_0(x)$. The effect of this formula is to create corrugations.



one corrugation

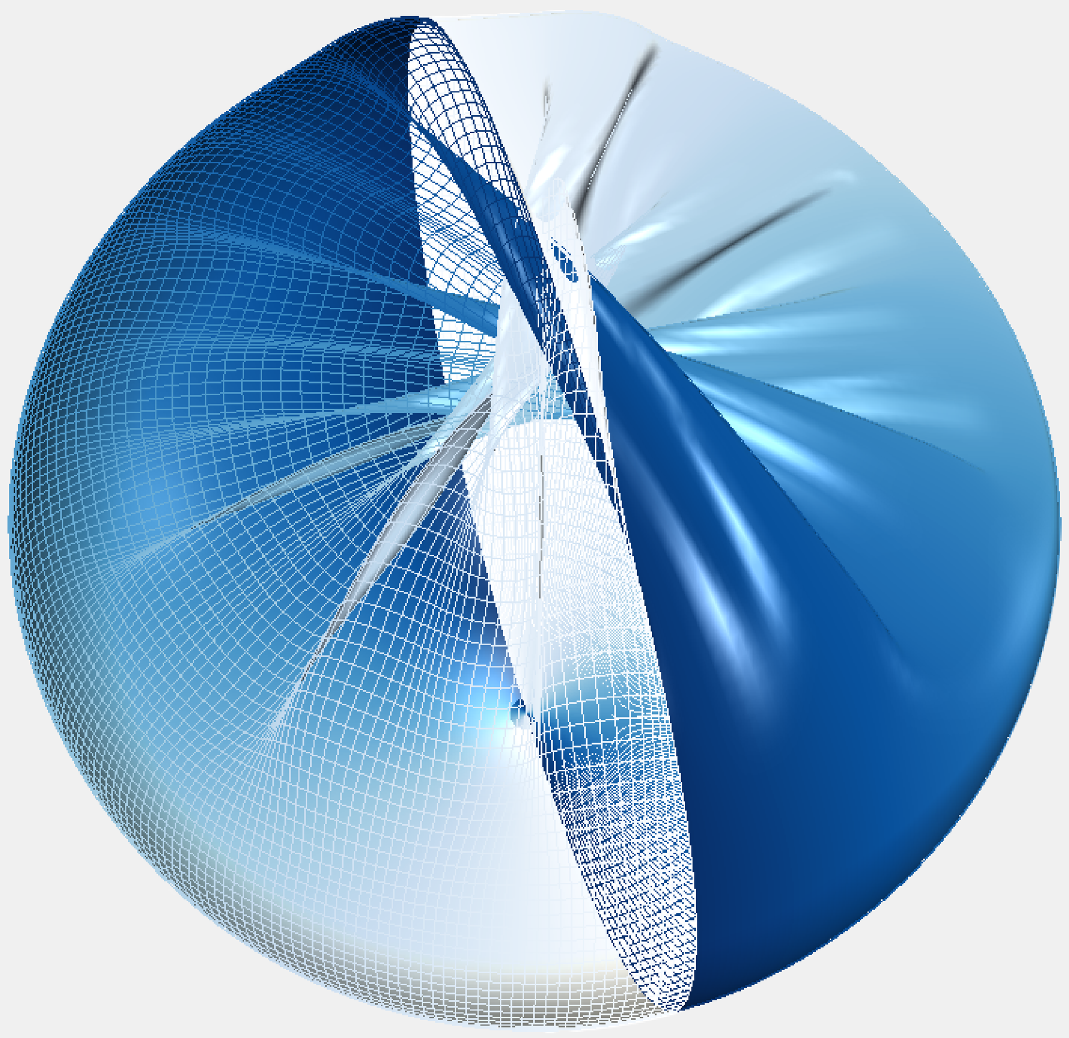
a stack of corrugations

Corrugation Process

We replace in the Convex Integration Theory the fundamental formula by a new one that we called Corrugation Process:

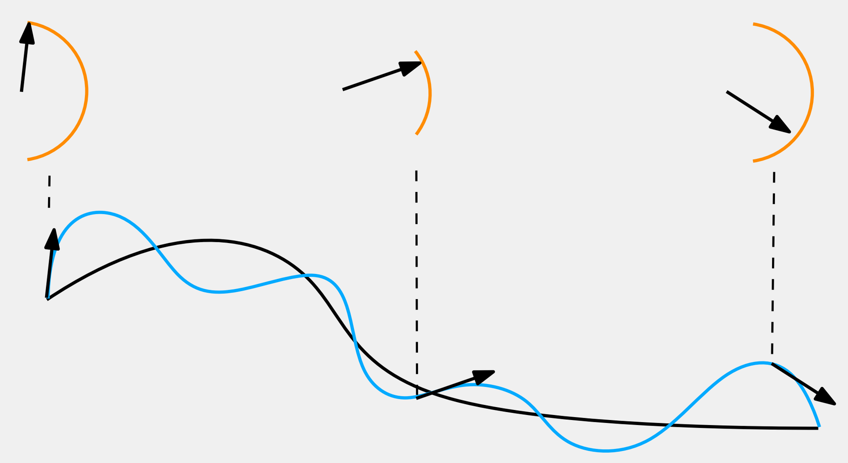
$$f_1(x) = f_0(x) + \frac{1}{N} \int_{t=0}^{Nx_i} \gamma(x,t) - \overline{\gamma}(x)dt$$

This formula allows a significant simplification of the Convex Integration. For example, this formula allow to build a new immersion of the real projective plane $\mathbb{R}P^2$ with a simple analytical expression:



Kuiper relations

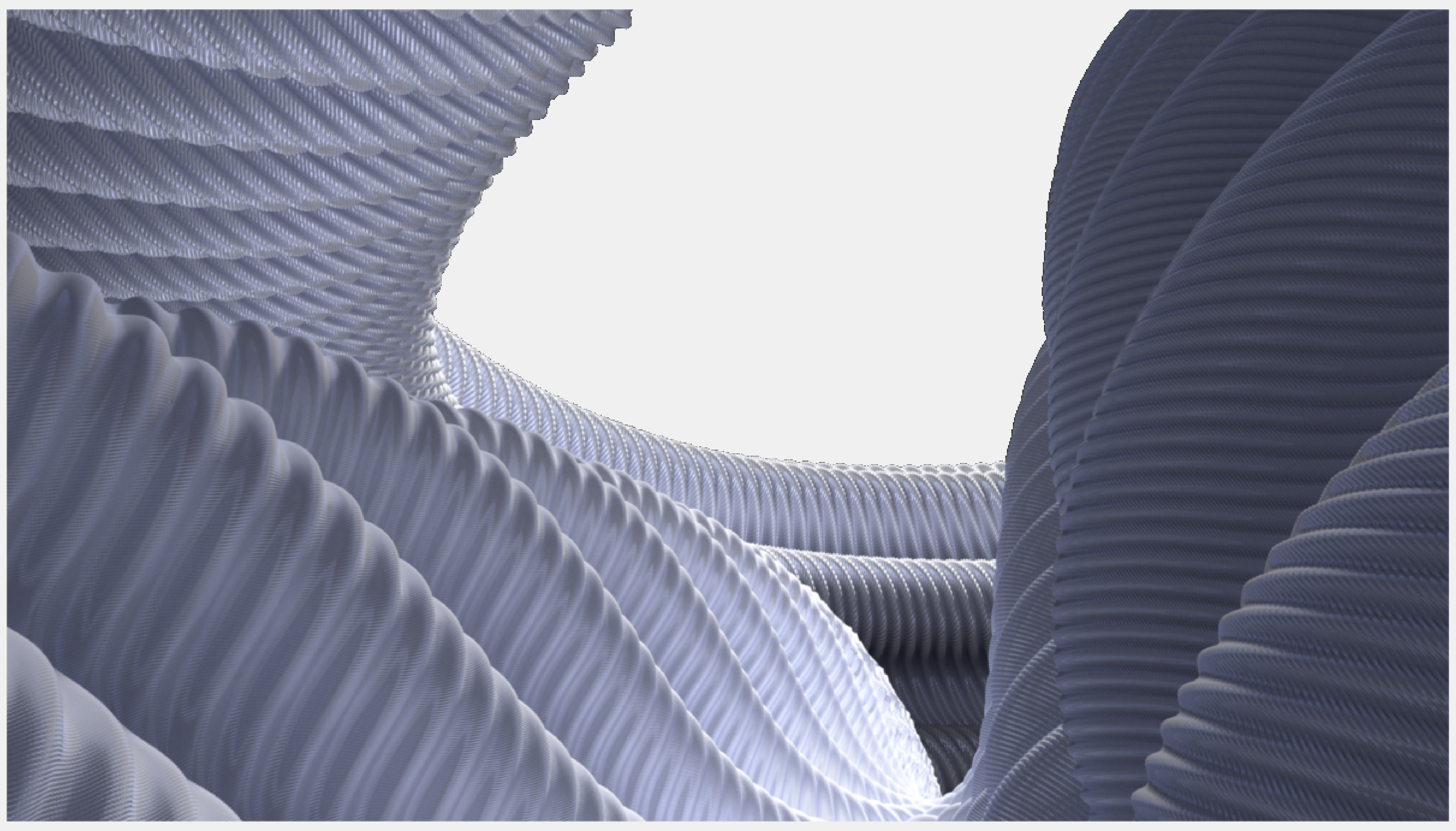
One of the key ingredients of the Convex Integration is the choice of the loop family γ. We are interested in differential relations where we can choose loops which share the same shape, for instance an arc of circle:



We call such relation: Kuiper relation. For example, the relation of immersions, the relation of ϵ-isomtries and the relation of totally real maps are Kuiper relations.

C^1-fractality

We explore specifically the totally real relation and we show that the combination of the Corrugation Process and the Kuiper property allows tht appearence of a C^1-fractal behavior. Precisely we state that the Maslov component of its Gauss map is similar to a Weierstarss function. In particular, the graph of this function is known to have a Hausdorff dimension greater than 1.



Waves in the foreground and in the background of the flat torus