Thickness and Relative Hyperbolicity for Graphs of Multicurves

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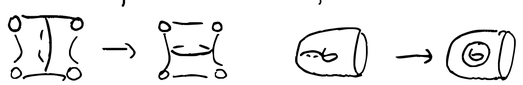
The pants graph, $\mathcal{P}(S)$

- vertices: homotopy classes of pants decompositions of S
- edges:



Theorem (Brock-Farb, Behrstock-Druţu-Mosher, Brock-Masur)
Classification of when:
- $\mathcal{P}(S)$ δ-hyperbolic
- $\mathcal{P}(S)$ relatively hyperbolic and not hyperbolic
- $\mathcal{P}(S)$ a thick metric space

This is a trichotomy.

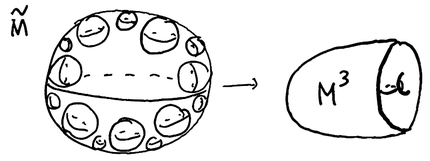
Relative hyperbolicity

A generalisation of δ-hyperbolicity.

Examples

Tree of flats

$\pi_1(M^3)$ hyperbolic 3-manifold with torus boundary



References. Russell, Vokes, *Thickness and relative hyperbolicity for graphs of multicurves*, J. Topol. **15** (2022); Behrstock, Druţu, Mosher, *Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity*, Math. Annal. **344** (2009); Brock, Masur, *Coarse and synthetic Weil–Petersson geometry*, Geom. Topol. **12** (2008)

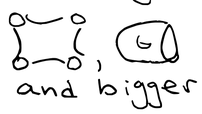
Thick metric spaces

- Thick of order 0: generalisation of product of infinite diameter spaces
- Thick of order 1: thick of order 0 spaces chained together with infinite diameter intersections ... and so on

Proposition (Behrstock-Druţu-Mosher)
Thick metric spaces are not relatively hyperbolic

Witnesses

Let $\mathcal{G}(S)$ be a graph whose vertices represent multicurves in S.
A witness for $\mathcal{G}(S)$ is a subsurface that every vertex of $\mathcal{G}(S)$ must intersect.

Graph	Vertices	Witnesses
$\mathcal{C}(S)$	curves	S only
$\mathcal{P}(S)$	pants decompositions	 and bigger
$\mathrm{Sep}(S)$	separating curves	e.g. 

Hierarchical graphs of multicurves

- Every vertex represents a multicurve in S
- $\mathcal{G}(S)$ is connected
- There is a natural action $\mathrm{MCG}(S) \curvearrowright \mathcal{G}(S)$
- The action $\mathrm{MCG}(S) \curvearrowright \mathcal{G}(S)$ is cocompact
- No witness for $\mathcal{G}(S)$ is an annulus.

Examples: $\mathcal{C}(S)$, $\mathcal{P}(S)$, $\mathrm{Sep}(S)$...

Main Theorem

Let $\mathcal{G}(S)$ be a hierarchical graph of multicurves

Do there exist witnesses w, z for $\mathcal{G}(S)$ so that w, z are disjoint?

NO → $\mathcal{G}(S)$ is δ-hyperbolic

e.g. $\mathcal{P}(S_{0,5})$ 

$\mathrm{Sep}(S_{g,3})$ $(g \geq 1)$ 

YES ↓

Do there exist witnesses w, z for $\mathcal{G}(S)$ so that
- **$S \setminus w$, $S \setminus z$ are connected,**
- **w, z are disjoint,**
- **$w \cup z$ is not all of S?**

NO → $\mathcal{G}(S)$ is relatively hyperbolic

e.g. $\mathcal{P}(S_{0,6})$ 

$\mathrm{Sep}(S_g)$ $(g \geq 3)$ 

YES → $\mathcal{G}(S)$ is a thick metric space

e.g. $\mathcal{P}(S_{0,7})$ 

$\mathrm{Sep}(S_{g,1})$ $(g \geq 3)$