# Computing automorphic forms

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(joint work with C. Bian and A. Strömbergsson)

## Setting

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$$

$$ds^2 = \frac{dx^2 + dy^2}{y^2},\; d\mathrm{vol} = \frac{dx\,dy}{y^2}$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),\; \gamma.z = \frac{az+b}{cz+d}$$

$$\Gamma \subset \mathrm{SL}(2,\mathbb{R}) \text{ discrete subgroup, } \mathrm{vol}(\Gamma\backslash\mathbb{H}) < \infty$$

$$\Delta = -\mathrm{div} \circ \mathrm{grad} = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

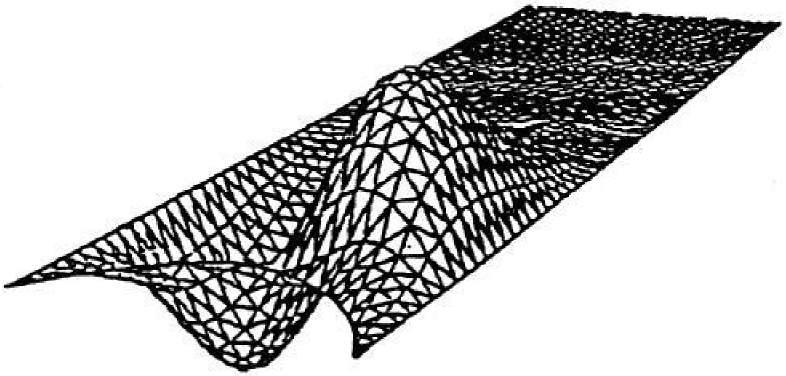
Problem: Compute the spectral resolution (i.e., the eigenfunctions) of $\Delta$ acting on $L^2(\Gamma\backslash\mathbb{H})$

## **Prototypical example:** $\Gamma = \mathsf{SL}(2,\mathbb{Z})$

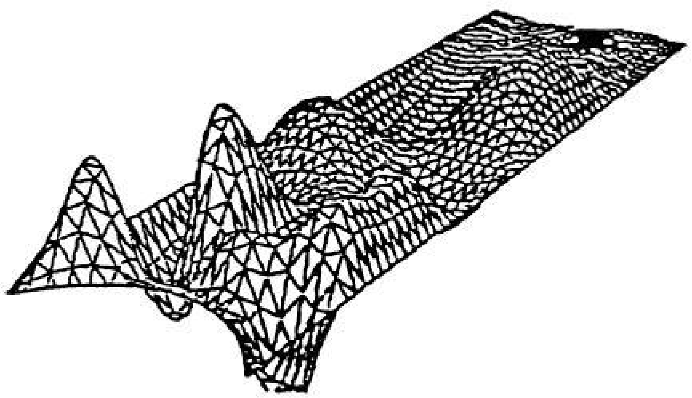
$$L^2(\Gamma\backslash\mathbb{H}) = \underbrace{\mathbb{C}}_{\substack{\text{constant}\\ \text{functions}}} \oplus \underbrace{L^2_{\text{Eisenstein}}(\Gamma\backslash\mathbb{H})}_{\substack{\text{continuous spectrum}\\ \text{spanned by}\\ \text{Eisenstein series}}} \oplus \underbrace{L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})}_{\substack{\text{discrete spectrum}\\ \text{spanned by}\\ \text{Maass forms}}}$$

$$f(z) = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a(n)\sqrt{y}K_{ir}(2\pi|n|y)e^{2\pi i n x}$$

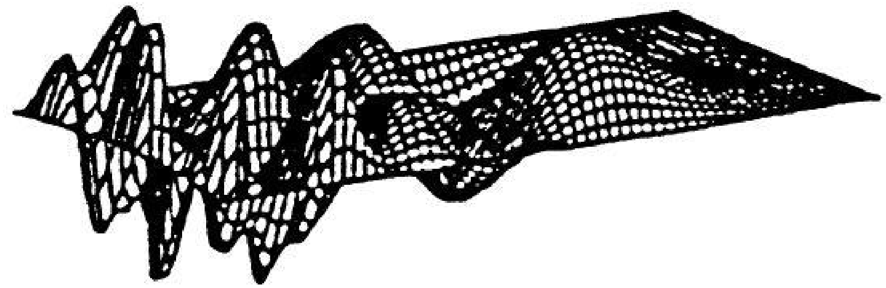
eigenvalue $\lambda = \frac{1}{4} + r^2$, $K_{ir} = K$-Bessel function



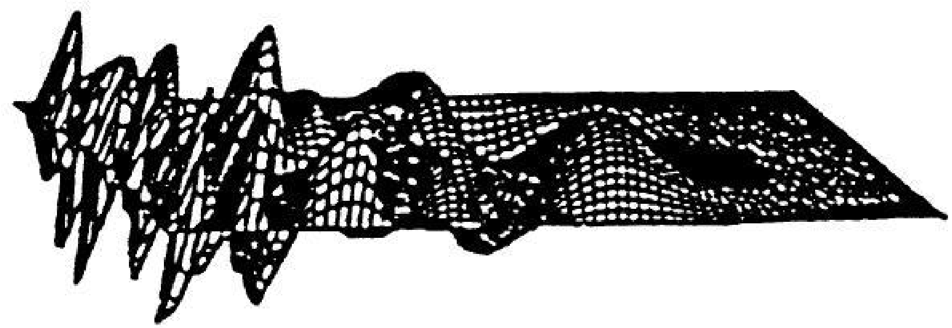
$\lambda_1 = 91.14\ldots$



$\lambda_{10} = 379.90\ldots$



$\lambda_{17} = 541.27\ldots$



$\lambda_{33} = 916.52\ldots$

P. Sarnak, "Spectra of Hyperbolic Surfaces", BAMS 40 No. 4, 2003

## “Computing Arithmetic Spectra”, March 9–14, American Institute of Mathematics

Question: *To what extent can one*

(A) *Compute the data (i.e., eigenvalues and Fourier coefficients) of automorphic forms, and*

(B) *prove theorems about them?*

Prototypical answer (again for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$):

(A) Some 50,000 eigenvalues computed approximately (6 or 7 decimal places) with heuristic justification of their correctness (work of H. Then based on an algorithm of D. Hejhal).

(B) First 2000 eigenvalues rigorously computed to high precision (more than 40 places). The eigenspaces are simple, and for each one the first several Fourier coefficients have been computed to high precision (joint work with A. Strömbergsson).

## Hejhal's algorithm

Suppose $f \in L^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H})$ is a Maass form with eigenvalue $\lambda = \frac{1}{4} + r^2$ and Fourier coefficients $a_n$. Then for $M$ large and $y$ bounded away from 0,

$$f(z) = \sum_{\substack{|n|\le M \\ n\ne 0}} a_n \sqrt{y} K_{ir}(2\pi|n|y) e(nx) + (\text{small}).$$

For a fixed $y$, $Q > M$ and $1 - Q \le j \le Q$, let

$$x_j = \frac{j - 1/2}{2Q} \qquad \text{and} \qquad z_j = x_j + iy,$$

and let $z_j^* = x_j^* + iy_j^*$ be its pullback to the fundamental domain. Then

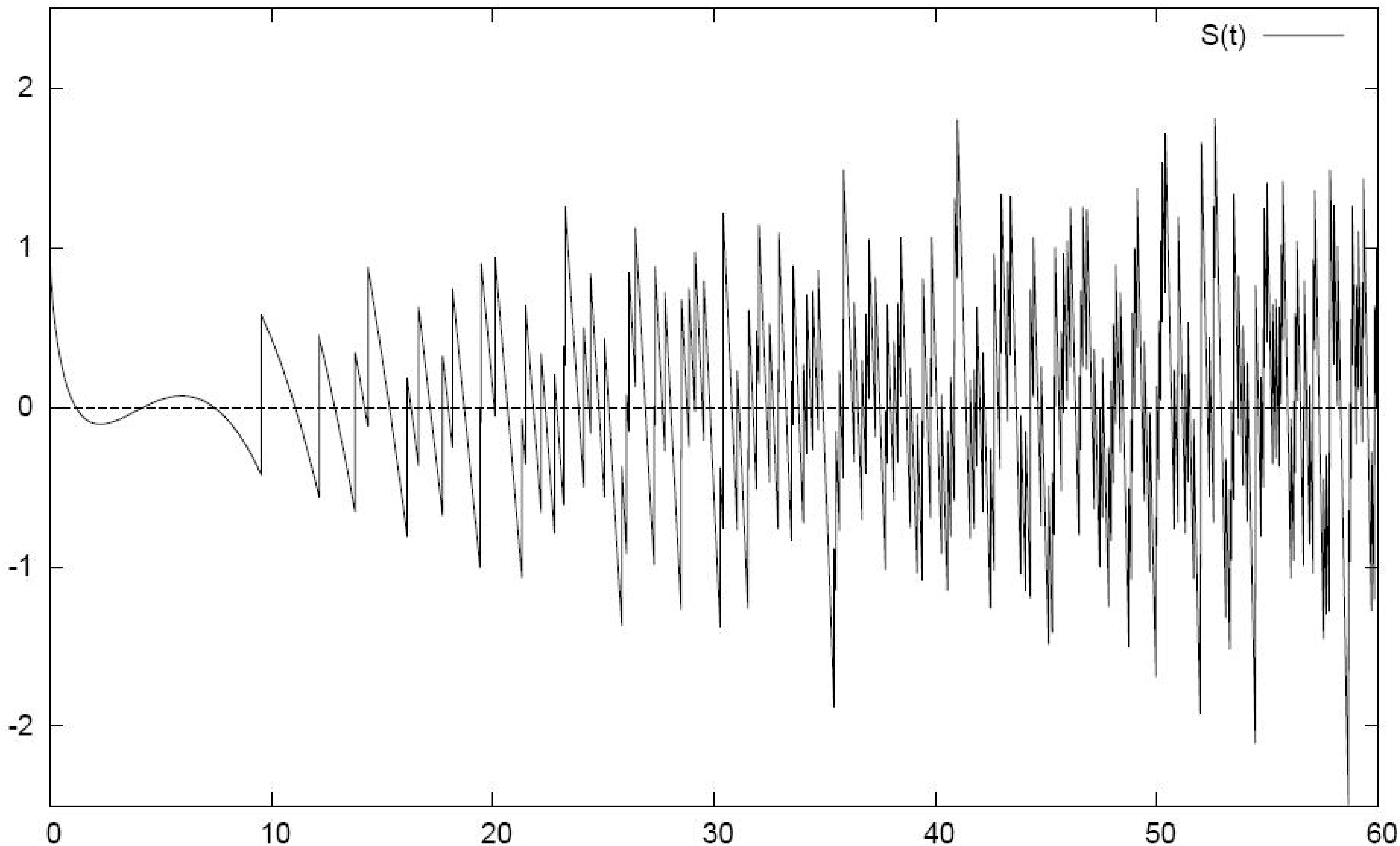
$$a_n \sqrt{y} K_{ir}(2\pi|n|y) \approx \frac{1}{2Q} \sum_{j=1-Q}^{Q} f(z_j) e(-nx_j).$$

$$a_n \sqrt{y} K_{ir}(2\pi |n| y) \approx \frac{1}{2Q} \sum_{j=1-Q}^{Q} f(z_j^*) e(-n x_j)$$

$$\approx \frac{1}{2Q} \sum_{j=1-Q}^{Q} \sum_{\substack{|m| \le M \\ m \neq 0}} a_m \sqrt{y_j^*} K_{ir}(2\pi |m| y_j^*) e(m x_j^*) e(-n x_j)$$

$$a_n \approx \sum_{\substack{|m| \le M \\ m \neq 0}} a_m \underbrace{\frac{\frac{1}{2Q} \sum_{j=1-Q}^{Q} \sqrt{y_j^*} K_{ir}(2\pi |m| y_j^*) e(m x_j^*) e(-n x_j)}{\sqrt{y} K_{ir}(2\pi |n| y)}}_{C(n,m)}$$

$$(C - I) \begin{pmatrix} a_{-M} \\ \vdots \\ a_M \end{pmatrix} \approx 0$$



S(t)
2
1
0
-1
-2
0
10
20
30
40
50
60

## Turing's method

**Theorem.** *Let $N(t)$ be the number of cuspidal eigenvalues $\lambda = \frac{1}{4} + r^2$ with $r \in [0, t]$. Define*

$$S(t) = N(t) - \left( \frac{t^2}{12} - \frac{2t}{\pi} \log \frac{t}{e\sqrt{\frac{\pi}{2}}} - \frac{131}{144} \right)$$

*and*

$$E(t) = \left(1 + \frac{6.59125}{\log t}\right)\left(\frac{\pi}{12 \log t}\right)^2.$$

*Then for $T > 1$,*

$$-2E(T) \leq \frac{1}{T} \int_0^T S(t)\, dt \leq E(T).$$

Moreover, we have:

**Theorem.** *There is an algorithm that, given* $\Lambda, D \geq 0$*, will compute all cuspidal eigenvalues* $\lambda \in [0, \Lambda]$ *on* $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$ *to within* $10^{-D}$ *in polynomial time in* $\Lambda, D$*.*

## Sample application

Question: *Given a large* $X > 0$*, how quickly can one determine the structure of the ideal groups of all real quadratic fields* $\mathbb{Q}(\sqrt{d})$*,* $0 < d < X$*?*

Under GRH, can be done in "essentially linear time" $O(X^{1+\varepsilon})$. Given a fast algorithm for computing eigenvalues and Fourier coefficients of Maass forms, one can remove the GRH assumption.

## Higher rank

$$\mathbb{H} \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \cong \mathrm{GL}(2,\mathbb{R})/\mathrm{O}(2,\mathbb{R}) \cdot \mathbb{R}^{\times}$$

$$z = x + iy \in \mathbb{H} \longmapsto \left(\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right)$$

Maass forms $\longleftrightarrow$ functions on $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$

"degree 3 hyperbolic space"

$$= \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R}) \cong \mathrm{GL}(3,\mathbb{R})/\mathrm{O}(3,\mathbb{R}) \cdot \mathbb{R}^{\times}$$

$$z = \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}$$

"degree 3 automorphic forms"

$$= \text{functions on } \mathrm{SL}(3,\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$$

## Fourier expansion

$$f(z) = \sum_{g\in\Gamma^2_\infty\backslash\Gamma^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a(n,m)}{nm} \underbrace{W_{u,v}\Big(\begin{pmatrix} nm & & \\ & m & \\ & & 1 \end{pmatrix} gz\Big)}_{\text{Jacquet's Whittaker function}},$$

$$\Gamma^2 = \left\{ \begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc = \pm 1 \right\},$$

$$\Gamma^2_\infty = \text{unipotent elements of } \Gamma^2$$

## $L$-functions

For $\chi$ a Dirichlet char. of conductor $q$ and parity $a \in \{0,1\}$, define

$$L(s, f\times\chi) = \sum_{n=1}^{\infty} a(n,1)\chi(n)n^{-s} \qquad (\Re(s) > 1)$$

$$= \prod_p \frac{1}{1 - a(p,1)\chi(p)p^{-s} + \overline{a(p,1)}\chi(p)^2 p^{-2s} - \chi(p)^3 p^{-3s}},$$

$$\gamma(s,u,v) = \Gamma_{\mathbb{R}}\Big(s - i\frac{2u+v}{3}\Big)\Gamma_{\mathbb{R}}\Big(s + i\frac{u-v}{3}\Big)\Gamma_{\mathbb{R}}\Big(s + i\frac{u+2v}{3}\Big),$$

$$\Lambda(s, f\times\chi) = \gamma(s+a,u,v)L(s,f\times\chi)$$

## Converse theorem

$f$ automorphic $\implies \Lambda(s, f \times \chi)$ continues to an entire function and satisfies a "functional equation":

$$(*) \quad \Lambda(s, f \times \chi) = \epsilon_\chi^3 q^{3(1/2-s)} \overline{\Lambda(1-\bar{s}, f \times \chi)}$$

$$\epsilon_\chi = \text{root number of } L(s, \chi)$$

In fact, these nice analytic properties *characterize* the degree 3 automorphic forms:

**Theorem** (Jacquet, Piatetski-Shapiro, Shalika). *Let $L(s, f \times \chi)$ be given by the Euler product on the previous slide, and suppose that for every $\chi$ the associated $\Lambda(s, f \times \chi)$ continues to an entire function of finite order satisfying (*). Then the Dirichlet coefficients $a(n, 1)$ are the Fourier coefficients of a degree 3 automorphic form.*

## Computing degree 3 automorphic forms

(joint work with Ce Bian)

- Treat Fourier coefficients $a(n,1)$ as unknowns
- Take Mellin transform of $\Lambda(s, f \times \chi)$:

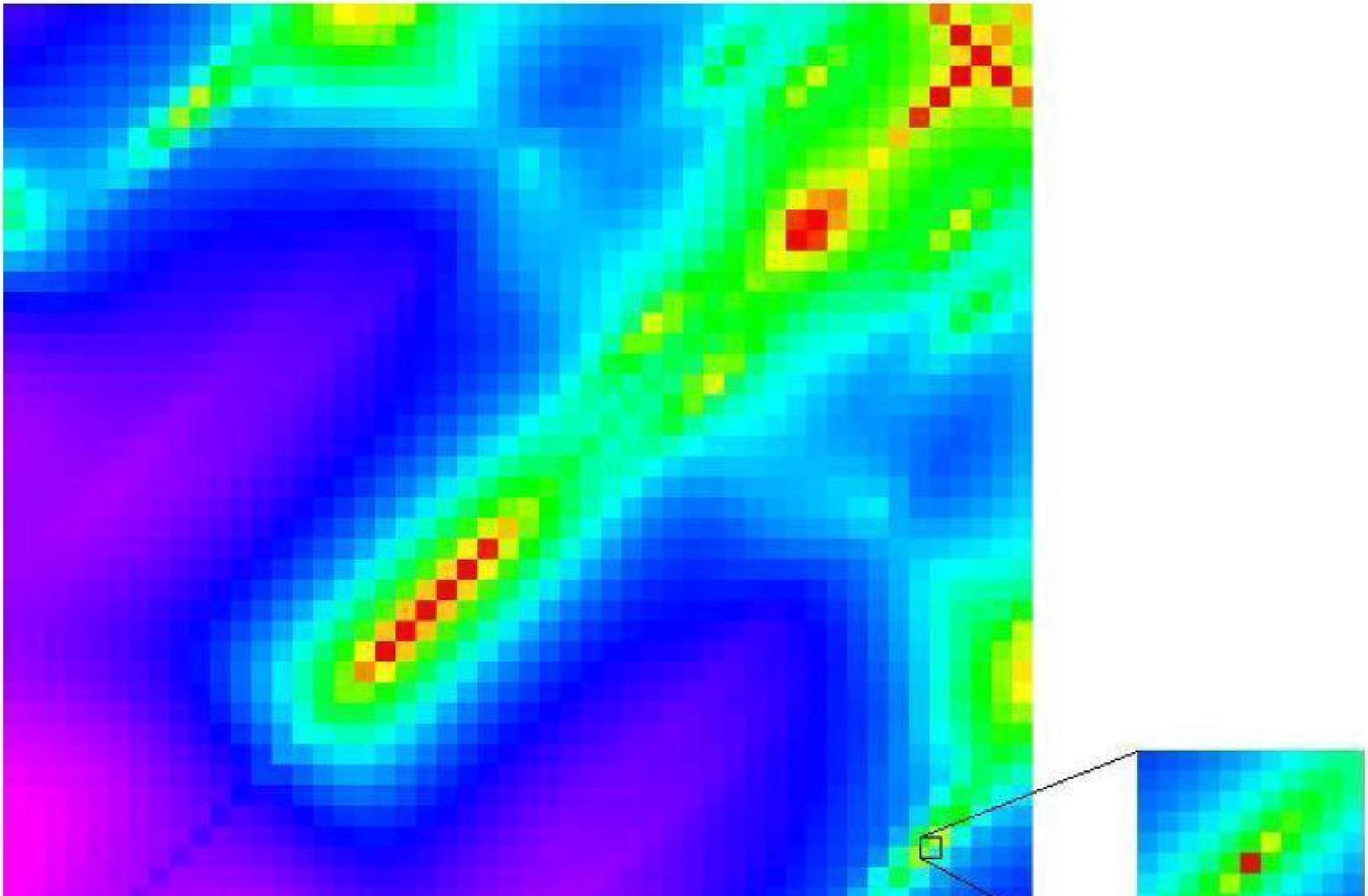
$$S(X, f \times \chi) = \frac{1}{\sqrt{X}} \sum_{n=1}^{\infty} a(n,1) F_{f\times\chi}(n/X),$$

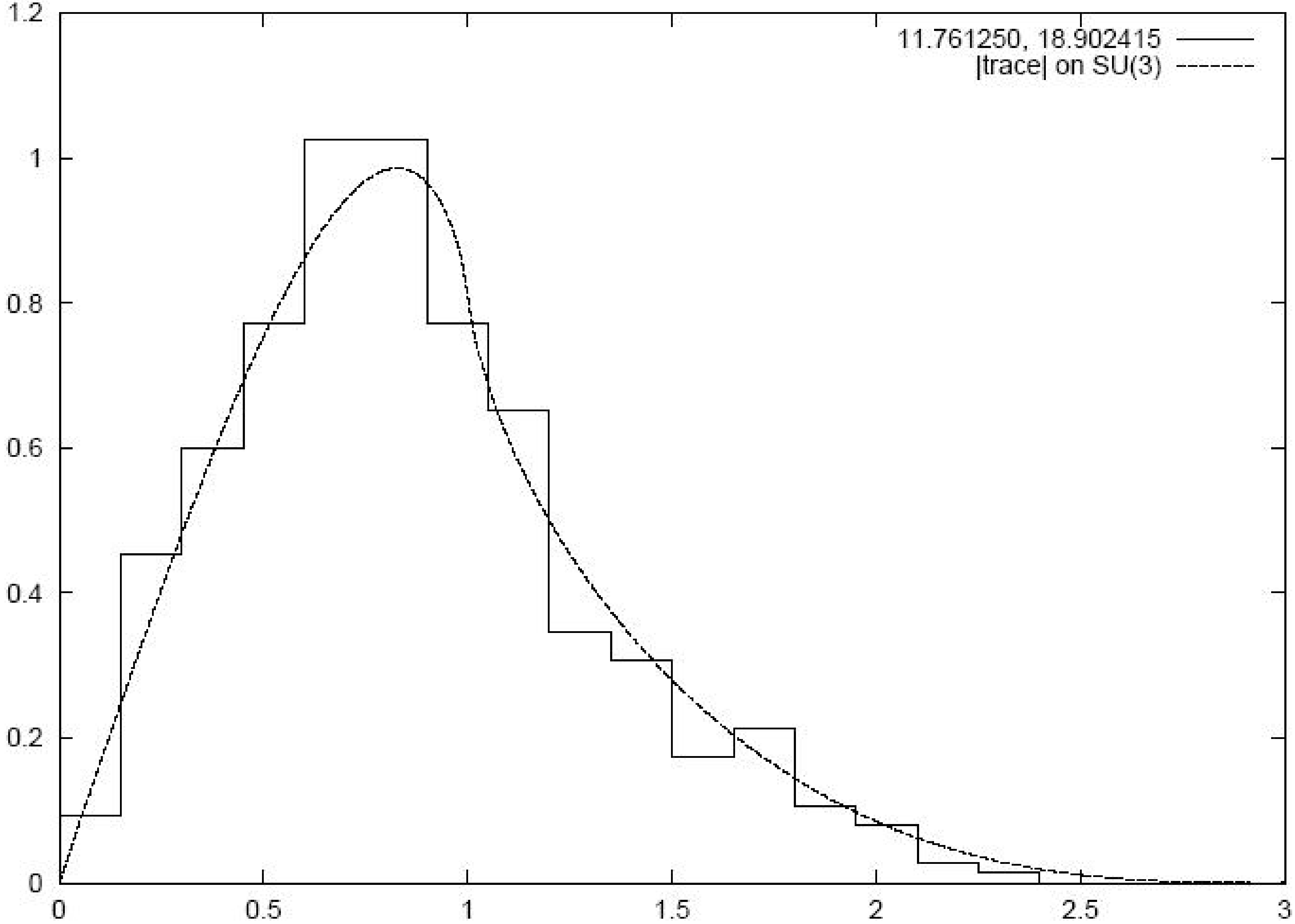
$$F_{f\times\chi}(y) = \frac{y^a}{2\pi i} \int_{\Re(s)=1} \gamma(s,u,v) y^{-s}\, ds$$

Functional equation for $\Lambda(s, f \times \chi) \iff$

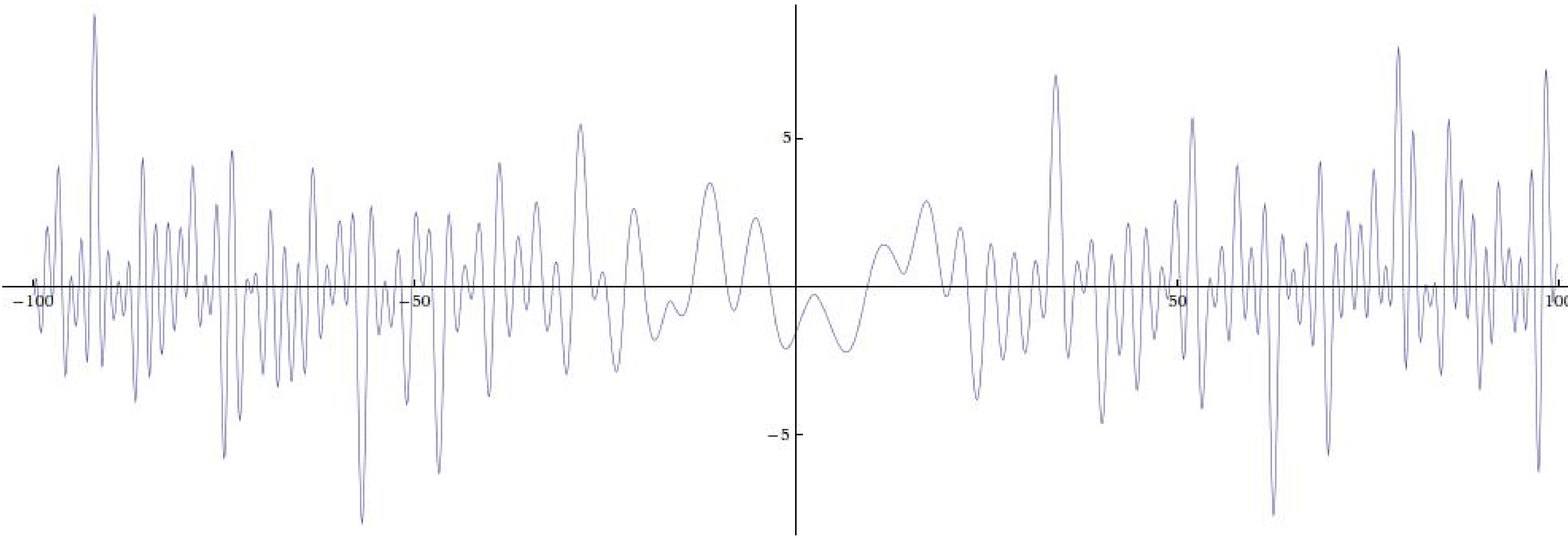
$$S(X, f \times \chi) = \epsilon_\chi^3 \overline{S(q^3/X, f \times \chi)}$$

- Choosing $X = q^{3/2}$, for each $\chi$ we get a linear equation in $\Re(a(n,1)), \Im(a(n,1))$ for $n$ up to $\approx q^{3/2}$
- There are about $\frac{18}{\pi^4} Q^2$ primitive $\chi$ of conductor $q \le Q$





11.761250, 18.902415
|trace| on SU(3)
1.2
1
0.8
0.6
0.4
0.2
0
0.5
1.5
2
2.5
3


5
−5
−100
−50
50
100