Computing modular forms[1] over imaginary quadratic fields

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[1] cusp forms of weight 2

Plan of the talk

- Goals and motivation

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- Cusp forms over imaginary quadratic fields

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Goals

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- To develop a better understanding of, and a more explicit description of, a Shimura-Taniyama-Weil type correspondence between elliptic curves and modular forms defined over number fields other than $\mathbb{Q}$;
- particularly over imaginary quadratic fields.

Motivating questions

Let K be a number field.

1. What do modular forms over K look like?
2. Can they be computed explicitly (say for given weight and level)?
3. Do they have associated L-series, with functional equations and interesting arithmetic properties, such as Fourier coefficients of number-theoretical interest or significant special values?
4. Do they have associated Galois representations?
5. What does it mean for an elliptic curve defined over K to be modular?

Motivating questions (contd.)

6. Are all elliptic curves over K modular?
7. Is it possible to construct modular elliptic curves over K explicitly?
8. If E is a modular elliptic curve over K, can we use the associated modular form to obtain arithmetic information about $E(K)$, in line with the Birch–Swinnerton-Dyer conjectures?

What is known over number fields

- Over $\mathbb{Q}$: well understood, with good answer to many of the questions.
- Over totally real fields: well-developed theory of Hilbert Modular Forms; some cases of modularity known.
- Over general number fields: theory formulated in terms of adelic functions (or representations), to deal efficiently with the infinite unit group and (possibly) nontrivial ideal class group.

Our approach: to specialize the general theory to the case of an imaginary quadratic field, in order to arrive at as concrete a description as possible of automorphic forms over these fields, amenable to explicit computations.

Selected history of the problem

- **1970s and 1980s:** Mennicke, Grunewald, Elstrodt and others carried out some explicit computations over $\mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3$, for split prime levels and weight 2. They produced tables of newforms and corresponding examples of curves.
- **1970s:** Kurčanov extended some of Manin's results over $\mathbb{Q}$, including the Manin-Drinfeld Theorem, to imaginary quadratic fields. The duality between spaces of cusp forms of weight 2 and the homology of quotients of hyperbolic 3-space is clearly set up, together with the definition of generalized modular symbols giving explicit generators for the homology.
- **From 1978:** I extended the work of Mennicke et al. to all five Euclidean fields ($d = 1, 2, 3, 7, 11$) and arbitrary levels, using more hyperbolic geometry and modular symbols. Tables of newforms and associated elliptic curves were produced using programs written in Algol68 (since rewritten in C++).

Selected history of the problem (contd.)

- **Late 1980s:** Elise Whitley extended this to the four other fields of class number 1 ($d = 19, 43, 67, 163$) and produced more tables, also using Algol68.
- **Early 1990s:** Whitley and I looked more closely at arithmetic aspects of the newform–curve correspondence, computing and comparing periods, analytic ranks and other data appearing in the Birch–Swinnerton-Dyer conjectures.
- **Early 1990s:** R. Taylor succeeded in constructing ℓ-adic Galois representations associated to automorphic forms over $K = \mathbb{Q}(\sqrt{-3})$. Explicit examples, computed with some help from me, showed that certain elliptic curves defined over K (and not over $\mathbb{Q}$) were modular in a weak sense.

Selected history of the problem (contd.)

- **Mid 1990s:** Jeremy Bygott extends the modular symbol methods used by me and Whitley to certain fields of higher class number, specifically class number 2, and produces tables of newforms over $\mathbb{Q}(\sqrt{-5})$, implemented in C++.
- **Early 2000s:** Mark Lingham extends the methods to certain fields of odd class number, and produces tables of newforms over $\mathbb{Q}(\sqrt{-23})$ and $\mathbb{Q}(\sqrt{-31})$ (class number 3), implemented in Magma.
- **2008:** Work starts to unify the above as much as possible, in order to have working programs maintained for posterity.

Hyperbolic 3-space

- $K = \mathbb{Q}(\sqrt{-d})$, $R = \mathbb{Z}_K$, $D_K = \mathrm{disc}(K)$, $w_K = \#R^\times$, $(\eta) = \mathfrak{d}_K$ (the different of K)
- $\mathbb{H}_3 = \mathbb{C} \times \mathbb{R}_+ = \{(z, t) \mid z \in \mathbb{C}, t > 0\}$
- $\mathrm{PSL}(2, \mathbb{C})$ acts on $\mathbb{H}_3$ via hyperbolic isometries:

  $$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : q \mapsto (\alpha q + \beta)(\gamma q + \delta)^{-1}$$

  where $q = z + tj$ is a quaternion.
- The Bianchi group $\Gamma = \mathrm{PSL}(2, R)$ acts discretely on $\mathbb{H}_3$, with a fundamental region of finite volume bounded by various geodesic planes, as do congruence subgroups $\Gamma_0(\mathfrak{n})$ for ideals $\mathfrak{n}$ of R.

Cusp forms of weight 2 over an imaginary quadratic field

Cusp forms of weight 2 for $\Gamma_0(\mathfrak{n})$:

$$F = (F_0, F_1, F_2) : \mathbb{H}_3 \to \mathbb{C}^3$$

such that $-F_0\frac{dz}{t} + F_1\frac{dt}{t} + F_2\frac{d\overline{z}}{t}$ is a $\Gamma_0(\mathfrak{n})$-invariant harmonic differential on $\mathbb{H}_3$, well-behaved at the cusps.
F has a Fourier expansion about the cusp ∞; when $h_K = 1$ this has the form

$$F(z,t) = \frac{16\pi^2 w_K}{|D_K|} \sum_{\mathfrak{a}} c(\mathfrak{a}) t^2 \mathbf{K}(4\pi|\eta|^{-1}|\alpha|t)\psi(\eta^{-1}\alpha z)$$

where $\psi(z) = \exp(2\pi i(z+\overline{z}))$, $\mathbf{K}(t) = (-\frac{1}{2}iK_1(t), K_0(t), \frac{1}{2}iK_1(t))$, and $\mathfrak{a} = \alpha R$ runs over all the integral ideals of R.

The correspondence between newforms and elliptic curves

- Consider newforms $F(z,t)$ for $\Gamma_0(\mathfrak{n})$ with $c(\mathfrak{a}) \in \mathbb{Z}$.
- These are eigenforms of a suitable Hecke algebra.
- Via Mellin transform we obtain an L-series which analytically continues a Dirichlet series with Euler product, all just as in the rational case:

$$\Lambda(F,s) = \int_0^\infty t^{2s-2} F_1(0,t)\,dt = \frac{w_K}{4}|D_K|^{s-1}(2\pi)^{-2s}\Gamma(s)^2 L(F,s)$$

where

$$L(F,s) = \sum_{\mathfrak{a}} c(\mathfrak{a}) N(\mathfrak{a})^{-s}.$$

- The Dirichlet series $L(F,s)$ thus has an analytic continuation, and (using a Fricke-type involution) we obtain a functional equation for $\Lambda(F,s)$.

The correspondence (contd.)

- Elliptic curves E defined over K of conductor $\mathfrak{n}$ have L-series of the same form

$$L(E,s) = \sum_{\mathfrak{a}} c(\mathfrak{a}) N(\mathfrak{a})^{-s} \qquad (\Re(s) > 3/2)$$

- The adjusted series $\Lambda(E,s)$, obtained by multiplying by the same gamma factors as for $\Lambda(F,s)$, is conjectured to have an analytic continuation to the whole complex plane and to satisfy a functional equation.
- Provided that $\Lambda(E,s)$ and sufficiently many of its twists do indeed satisfy functional equations, Weil showed that $L(E,s) = L(F,s)$ for a suitable modular form F.
- There are essentially no cases for which these functional equations are known to hold (only curves which are defined over $\mathbb{Q}$ or their twists).

Eichler-Shimura construction?

- There is no analogue of the Eichler-Shimura construction to construct an elliptic curve from a cusp form of weight 2 with rational Fourier coefficients via periods.
- The periods of F form a 1-dimensional lattice in $\mathbb{R}$, not a 2-dimensional lattice in $\mathbb{C}$.
- The generator $\Omega(F)$ of this lattice does seem to be related to the period lattice of an associated elliptic curve (when there is one)
- $\Omega(F)$ does not contain enough information to construct an elliptic curve.
- Such a construction (Mennicke's "pipe dream") would be very significant.
- There are cases in which a newform F has no associated elliptic curve E defined over K. Any general construction would need to allow for such exceptions.

Exceptional cases

- If E has CM by an order in the field K itself, then $L(E,s) = L(F,s)$ where F is an Eisenstein series.
- For then $h_K = 1$, E is defined (up to twist) over $\mathbb{Q}$, and the base-change lift from $\mathbb{Q}$ to K of a cusp form f is not a cusp form in this case, when f has CM in K itself.

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- Some newforms over K with rational coefficients are (up to twist) the result of base change from $\mathbb{Q}$ of classical cusp forms with "extra twist" which do not have rational coefficents, and in such cases there may be no corresponding elliptic curve.

A conjecture

Based on general philosophy, the situation over $\mathbb{Q}$, the computational evidence and the known exceptions, we can formulate the following:

Open Question: For each newform F for $\Gamma_0(\mathfrak{n})$ with rational Fourier coefficients, is there *either* an elliptic curve E defined over K with conductor $\mathfrak{n}$ such that $L(F,s) = L(E,s)$, *or* a quadratic twist ϵ such that $F \otimes \epsilon$ is the lift from $\mathbb{Q}$ to K of a modular form f over $\mathbb{Q}$?

Overview of the modular symbol method

- For each "level" $\mathfrak{n}$ (an integral ideal of R), the quotient space $X_0(\mathfrak{n}) = \Gamma_0(\mathfrak{n})\backslash\mathbb{H}_3$ is[2] a 3-manifold whose homology may be computed using modular symbols.
- When $h_K = 1$, the 1-homology $H(\mathfrak{n})$ is dual to the space $S_2(\mathfrak{n})$ of cusp forms of weight 2 for $\Gamma_0(\mathfrak{n})$.
- The duality respects the Hecke action.
- Hence, by computing $H(\mathfrak{n})$ and the Hecke action on it we can compute the dimension of $S_2(\mathfrak{n})$ and find (via Hecke eigenvalues) all newforms.
- When $h_K > 1$ this picture needs some modification.

[2] if $N(\mathfrak{n})$ is large enough so $\Gamma_0(\mathfrak{n})$ has no nontrivial elements of finite order

Overview of the modular symbol method (Contd.)

- To compute the homology of $X_0(\mathfrak{n})$ we tessellate $\mathbb{H}_3$ with hyperbolic polyhedra.
- Γ acts on this tessellation: its vertices (cusps), edges, faces and 3-cells (polyhedra).
- When $h_K = 1$ the action is transitive on the vertices (the cusps); in general there are h_K orbits.
- When K is Euclidean the action is transitive on the edges; in general there are several orbits. Each orbit leads to a "type" of modular symbol.
- In general there are several orbits on the faces; each orbit leads to a type of relation between the modular symbols.
- In general there are several orbits on the polyhedra.

Overview of the modular symbol method (Contd.)

For each field we do a precomputation to find the tessellation and the Γ-orbits on the vertices, edges and faces.

- Example: $K = \mathbb{Q}(\sqrt{-1})$: the tessellation is of of octahedra, the Γ-orbit of the one with vertices at $0, 1, i, 1+i, (1+i)/2$ and ∞, which consists of 12 copies of the fundamental region for Γ glued together.

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- Example: $K = \mathbb{Q}(\sqrt{-163})$: the tessellation has 25 Γ-orbits of 3-cells (11 tetrahedra, 8 triangular prisms, 3 square pyramids, 2 hexagonal caps and 1 cuboctahedron).

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- Example: $K = \mathbb{Q}(\sqrt{-163})$: the tessellation has 25 Γ-orbits of 3-cells (11 tetrahedra, 8 triangular prisms, 3 square pyramids, 2 hexagonal caps and 1 cuboctahedron).
- Modular symbols for this field are therefore rather complicated!

Summary of results I. Class number 1 fields

$K = \mathbb{Q}(\sqrt{-d})$ for $d \in \{1, 2, 3, 7, 11\}$: in my thesis (1981) I computed $S_2(\mathfrak{n})$ for levels $\mathfrak{n}$ with norms $N(\mathfrak{n})$ up to some bound; this was continued by Elise Whitley (1990) for $d \in \{19, 43, 67, 163\}$.

d	1	2	3	7	11	19	43	67	163
Bound on $N(\mathfrak{n})$	500	300	500	300	200	500	230	265	100
Number of newforms F	39	36	27	17	17	55	7	6	0

These progams were all in Algol 68. For the Euclidean fields I rewrote the program in C++: rerunning the above ($d \leq 11$) now takes under 5s on my laptop (demonstrations on request).

Higher class number: $h_K > 1$

- When $k_K > 1$, the adelic theory tells us that we should replace $\mathbb{H}_3$ by h_K disjoint copies; while Γ acts on the first, twisted forms of Γ act on the others.
- The homology of the quotient is in duality not with $S_2(\mathfrak{n})$ but with $\oplus_\chi S_2(\mathfrak{n}, \chi)$ where the sum is over all unramified Dirichlet characters χ.
- In the Hecke action, each operator $T_\mathfrak{p}$ permutes the components according to the class of $\mathfrak{p}$: the Hecke algebra is graded by the class group.
- In practice we can obtain all information from the action of just the principal Hecke operators $T_\mathfrak{a}$ (with $\mathfrak{a}$ principal), acting on the principal component.
- Hence we still only need to compute the homology of $X_0(\mathfrak{n})$.

$h_K = 2$

The case $h_K = 2$ was treated in Jeremy Bygott's thesis (1998), with examples computed for $K = \mathbb{Q}(\sqrt{-5})$.

- The homology of $X_0(\mathfrak{n})$ is only "half" the space we need.
- We only compute "half" the Hecke operators (the principal ones).
- Each system of eigenvalues for these extends in two ways to eigenvalue systems for the full Hecke algebra: these differ only in the signs of the eigenvalues of $T_\mathfrak{p}$ for non-principal $\mathfrak{p}$.
- Thus each eigenspace in $H^1(X_0(\mathfrak{n}))$ corresponds to two eigenforms which are quadratic twists of each other by the unramified quadratic character of K.

Results for $\mathbb{Q}(\sqrt{-5})$

Bygott's thesis contains a table of all newforms for $\mathbb{Q}(\sqrt{-5})$ of level $\mathfrak{n}$ with $N(\mathfrak{n}) \leq 135$.

For each, he gives the Hecke eigenvalues for small primes $\mathfrak{p}$.

Bygott's program is in C++ and is based on mine for the class number one fields, with extra code to handle ideals etc..

Demonstrations on request.

Odd vs. even class number

Bygott's thesis showed that the case of class number 2 could be handled nicely by using a larger group, containing Γ with index 2, to pick out the classes with trivial character directly. This case then becomes (as far as the hyperbolic geometry is concerned) rather like the case $h_K = 1$.

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Ongoing work will show how to handle arbitrary class groups, in every case obtaining full eigenvalue systems just from the action of principal Hecke operators on the "principal" homology of $X_0(\mathfrak{n})$.

$h_K = 3$

The case $h_K = 3$ was treated in Mark Lingham's thesis (2005), with examples computed for $K = \mathbb{Q}(\sqrt{-23})$ and $K = \mathbb{Q}(\sqrt{-31})$. He computed all levels of norm up to 200 in each case, finding many newforms with eigenvalues in fields of degree up to 3.
For many of the newforms with rational eigenvalues Lingham also found corresponding[3] elliptic curves.
Lingham also showed how to compute Atkin-Lehner and Fricke involutions, and degeneracy maps, even in the case where the ideals involved are not principal. His results apply to all odd class numbers, and ongoing work will extend them to the general case.

[3] in the sense that the conductor equals the level and the first several Euler factors of the L-series agree.

References (theses)

Theses (see my web page
`http://www.warwick.ac.uk/staff/J.E.Cremona/`):

- J. E. Cremona (Oxford 1981), “Modular symbols”.
- E. Whitley (Exeter 1990), ”Modular forms and elliptic curves over imaginary quadratic number fields of class number one”.
- J. S. Bygott (Exeter 1998), ”Modular forms and elliptic curves over imaginary quadratic number fields”.
- M. P. Lingham (Nottingham 2005), ”Modular forms and elliptic curves over imaginary quadratic fields”.

References (papers)

Papers:

- JEC, "Hyperbolic tessellations, modular symbols, and elliptic curves over complex quadratic fields", Compositio Mathematica 51 (1984), 275–323.
- JEC and E. Whitley, "Periods of cusp forms and elliptic curves over imaginary quadratic fields", Math. Comp. 62 (1994), 407–429.
- JEC, "Modular forms and Hecke operators over number fields", pre-preprint 2007.