Computations of (non-holomorphic) automorphic forms on GL_2

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Maass waveforms on $PSL_2(\mathbb{Z})$.

Maass waveforms (cusp forms) are solutions to the following problem:

1. $(\Delta + \lambda)\phi(z) = 0$, $\lambda = \frac{1}{4} + R^2 > 0$,
2. $\phi(\gamma z) = \phi(z)$, $\forall \gamma \in PSL_2(\mathbb{Z})$,
3. $\int_{\Gamma\backslash\mathbb{H}} |\phi|^2 \, d\mu < \infty$, implies $\phi(z) \to 0$ as $y \to \infty$ (in our case).

Fourier expansion at ∞:

$$\phi(z) = \sum_{n \in \mathbb{Z}, n \neq 0} c_n \kappa_n(y) e(nx),$$

$$\kappa_n(y) = \sqrt{y} K_{iR}(2\pi |n| y) \sim \sqrt{\frac{1}{4|n|}} e^{-2\pi|n|y} \text{ as } |n| y \to \infty.$$

Truncation and Inversion

Let $\varepsilon > 0$, set $Y_{min} = \frac{\sqrt{3}}{2}$ and $Y_0 < Y_{min}$. Then $\exists M_0 = M(Y_0)$ s.t.

$$\phi(z) = \hat{\phi}(z) + [[\varepsilon]] = \sum_{|n| \le M_0} c_n \kappa_n(y) e(nx) + [[\varepsilon]], \forall y > Y_0.$$

(We will ignore the error $[[\varepsilon]]$ from now on). View $\hat{\phi}$ as a finite Fourier transform and invert $\hat{\phi}$ over a horocycle $z_m = x_m + iY = \frac{2m-1}{4Q} + iY$, $1 - Q \le m \le Q$ with $Q > M_0$ gives

$$c_n \kappa_n(Y) = \frac{1}{2Q} \sum_{m=1-Q}^{Q} \hat{\phi}(z_m) e(-nx_m).$$

Automorphy

Consider the standard (closed) f.d. $\mathcal{F}_0 = \left\{ z = x + iy \in \mathbb{H} \,\middle|\, |x| \leq \frac{1}{2}, |y| \geq 1 \right\}$. For $z \in \mathbb{H}$ let $z^* = Az \in \mathcal{F}_0$ denote the pullback of z to $\mathcal{F}_0$. Then

$$\phi(z_m) = \phi(z_m^*) . \qquad (*)$$

Hence

$$\begin{aligned} c(n)\kappa_n(Y) &= \frac{1}{2Q} \sum_{m=1-Q}^{Q} \hat{\phi}(z_m^*)\, e(-nx_m) \\ &= \sum_{|l| \leq M} V_{nl} c_l \qquad (**) \end{aligned}$$

Setting $\tilde{V}_{nl} = V_{nl} - \delta_{nl}\kappa_n(Y)$ we get the *homogeneous* system

$$\tilde{V}\vec{c} = 0 \qquad (***)$$

Normalize by e.g. $c_1 = 1$ (Hecke).

Locating Eigenvalues

- The previous system can be constructed and solved for arbitrary R and $Y < Y_{min}$, giving $\vec{c} = \vec{c}(Y, R)$.
- If $R^2 + \frac{1}{4}$ is in the discrete spectrum of Δ then the solution vectors $\vec{c} = \vec{c}(Y, R)$ are independent of $Y < Y_{min}$ (up to ε).
- Construct a functional of R : Let $\vec{c} = \vec{c}(Y_1, R)$ and $\vec{c}' = \vec{c}(Y_2, R)$ and set for example

$$h(R) = \varepsilon_2 (c_2 - c_2') + \varepsilon_3 (c_3 - c_3') + \varepsilon_4 (c_4 - c_4')$$

  here $\varepsilon_j = \pm 1$ is set so that h changes sign at the zeros.
- Then $h(R) = 0$ if R corresponds to an eigenvalue and we can use e.g. Newton's method to find eigenvalues.

Phase 2: get more than M_0 coefficients

If $Q > M(Y) > |n|$ then (**) is valid for n and

$$c_n = \frac{\sum_{|l| \le M_0} V_{nl} c_l}{\kappa_n(Y)} + \frac{[[\varepsilon]]}{\kappa_n(Y)}$$

We can use this to compute a large number of coefficients using the initial set. The main computational cost here comes from the fact that we not only need $Q > M(Y) > |n|$ but we also need to keep $\kappa_n(Y)$ reasonably large at the same time. This leads to a decreasing sequence of Y and increasing Q.

Successful generalizations

- $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{Z})$ a finite index subgroup [4]
- General weight and multiplier system [5]
- Eisenstein Series (Helen Avelin) [1, 3].
- Green's functions (Helen Avelin) [2] (will not be discussed)
- Weak Maass forms and vector-valued Poincaré series
- The Picard group (Holger Then) See e.g. [6] (will not be discussed)

One needs: Fourier expansion (possibility to truncate), pullback algorithm and automorphy relation.

Γ a Fuchsian group with $\kappa \geq 1$ cusp(s)

Fourier series

ϕ has κ Fourier series expansions:

$$\phi_j(z) = \phi\left(\sigma_j^{-1} z\right) = \sum_{n \neq 0} c_j(n)\, \kappa_n(y)\, e(nx)$$

where σ_j is a cusp normalizing map for cusp nr. j.

Automorphy relation

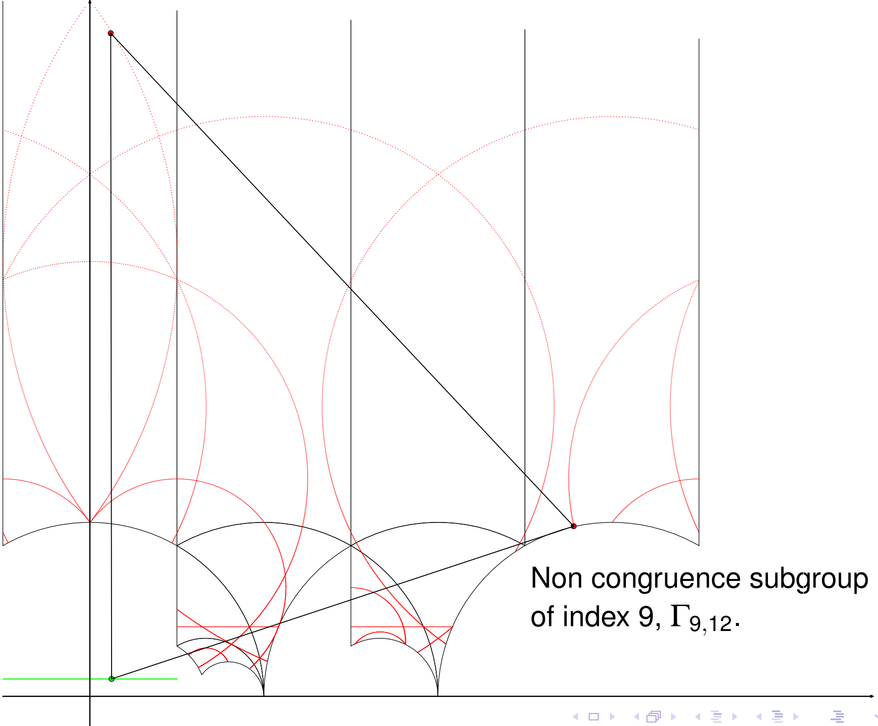
$$\phi_j(z_m) = \phi_{I(j,m)}\left(z^*_{m,j}\right)$$

where $I(j,m)$ and $z^*_{m,j}$ depends on both j and m.

Pullback Algorithm to $\Gamma \subseteq PSL_2(\mathbb{Z})$

1. Choose right coset representatives: $PSL_2(\mathbb{Z}) = \sqcup \Gamma V_j$
2. A f.d. for Γ is given by $\mathcal{F}_\Gamma = \cup V_j(\mathcal{F}_0)$
3. If $z \in \mathbb{H}$,
    1. $\tilde{z} = Tz \in \mathcal{F}_0$ is the pullback to $\mathcal{F}_0$.
    2. Find j s.t. $T^{-1} \in \Gamma V_j \Rightarrow V_j T \in \Gamma$
    3. $z^* = V_j \tilde{z} \in \mathcal{F}_\Gamma$ is a pullback to $\mathcal{F}_\Gamma$.

Important for the truncation to work is the existence of a minimal „invariant height" of $\mathcal{F}_\Gamma$, Y_0 i.e. if $Y < Y_0$ then $\Im z^*_{mj} > Y_0$ for all j and m. For $\Gamma_0(N)$ we have $Y_0 \geq \frac{\sqrt{3}}{2N}$.



Non congruence subgroup of index 9, $\Gamma_{9,12}$.

Characters, weights and multipliers

Let $\Gamma = \Gamma_0(N)$, $k \in \mathbb{R}$ and v be a multiplier system (or character) on Γ of weight k and $(\Delta_k + \lambda)\phi = 0$, $\Delta_k = \Delta - iyk\frac{\partial}{\partial x}$.

Fourier series

$$\phi_j(z) = \sum_{n_j \neq 0} \frac{c_j(n)}{\sqrt{|n_j|}} W_{\mathrm{sign}(n_j)\frac{k}{2}, iR}(4\pi |n_j| y)\, e(n_j x)$$

where $n_j = n + \alpha_j$, $\alpha_j \in [0,1)$ and $W_{\mu,iR}(x)$ is the Whittaker W-function.

Automorphy relation

$$\phi_j(z_m) = j_\gamma(z_{mj}^*)^k v(\gamma)\, \phi_{I(j,m)}(z_{mj}^*)$$

if $z_{mj}^* = \gamma^{-1} z_m$ and $j_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(z) = e^{i\mathrm{Arg}(cz+d)}$.

Holomorphic Automorphic Functions

Correspondence with Maass waveforms

$$\begin{aligned} F(z) &= \sum_{n\geq 0} c_n e(nz) \in \mathcal{M}_k(\Gamma), k>0 \\ &\Rightarrow \\ f(z) &= y^{\frac{k}{2}} F(z) \in Maass(\Gamma, k, v=1, \lambda) \end{aligned}$$

with $\lambda = \frac{k}{2}\left(1-\frac{k}{2}\right)$. Hence we can use our algorithm to compute the Fourier coefficients $d_n = n^{\frac{1-k}{2}} c_n$ of

$$f(z) = \sum_n d_n \kappa_n(y) e(nx),$$

with $\kappa_n(y) = \sqrt{y}(ny)^{\frac{k-1}{2}} e^{-2\pi ny}$.

Non-holomorphic Eisenstein Series for $PSL_2(\mathbb{Z})$

Fourier series

$$
\begin{aligned}
E(z;s) &= y^s + \varphi(s)\, y^{1-s} + \sum_{n\neq 0} c_n(s)\, K_{s-\frac{1}{2}}(2\pi|n|y)\, e(nx) \\
&= A(y) + \sum_{n} c_n(s)\, \kappa_n(y)\, e(nx)
\end{aligned}
$$

where $A(y) = y^s$, $c_0(s) = \varphi(s)$ and $\kappa_n(y) = K_{s-\frac{1}{2}}(2\pi|n|y)$ for $n \neq 0$ and $\kappa_0(y) = y^{1-s}$.

By Fourier inversion we get:

$$
\frac{1}{2Q} \sum_{m=1-Q}^{Q} E(z_m;s)\, e(-nx_m) = \begin{cases} A(Y) + c_0\kappa_0(Y), & n = 0, \\ c_n(s)\, \kappa_n(Y), & n \neq 0 \end{cases}
$$

Inhomogeneous system

Similar steps as above lead to an *inhomogeneous* system:

$$0 = \tilde{V}\vec{c} + \vec{\tilde{W}}(Y)$$

where $V = \left(\tilde{V}_{nl}\right)$ is essentially the same as before and

$$\tilde{W}_n = \frac{1}{2Q} \sum_{m=1-Q}^{Q} A(y_m^*) e(-nx_m) - \delta_{n0} A(Y).$$

Note that if $s(1-s)$ belongs to the discrete spectrum of Δ the corresponding Maass waveform solves the homogeneous system and some extra care has to be taken to avoid multiple solutions.

This algorithm was first implemented by Helen Avelin for Hecke triangle groups and for deformations of $\Gamma_0(5)$.

Harmonic Weak Maass forms

Definition (Harmonic weak Maass form)

f is a Harmonic weak Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ for $\Gamma = \Gamma_0(N)$ (or $\widetilde{\Gamma}$) and representation $\rho : \Gamma \to \mathbb{C}^h$ if

1. $\tilde{\Delta}_k f = 0$ where $\tilde{\Delta}_k = \Delta - iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \Delta_k + ky\frac{\partial}{\partial y}$,
2. $$f(z) = f_{|\rho,k}\gamma = J_\gamma(z)^{-2k}\rho(\gamma)^{-1} f(\gamma z), \forall \gamma \in \Gamma$$
   where $J_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(z) = \sqrt{cz+d} = |cz+d|^{\frac{1}{2}} e^{i\frac{1}{2}\mathrm{Arg}(cz+d)}$
3. There exists a polynomial $P_f = \sum_{n \le 0} c_n^+ e(nz)$ such that
   $$f(z) - P_f(z) = O\left(e^{-\varepsilon y}\right), \text{ as } y \to \infty.$$

Fourier Series

We have

$$f(z) = \sum_{n \gg -\infty} c^{+}(n) e(nz) + \sum_{n<0} c^{-}(n) \Gamma(1-k, 4\pi |n| y) e(nz)$$
$$= P_f(z) + \sum_{n>0} c^{+}(n) \kappa_n^{+}(y) e(nx) + \sum_{n<0} c^{-}(n) \kappa_n^{-}(y) e(nx)$$

where $\kappa_n^{+}(y) = e^{-2\pi n y}$ and $\kappa_n^{-}(y) = \kappa_n^{+}(y) \Gamma(1-k, 4\pi |n| y)$.

It is known that f is determined by P_f and we the space of weak Maass forms is spanned by Poincaré series.

The Weil representation

Let $q(x) = Nx^2$, $L = \langle \mathbb{Z}, q \rangle$, L' the dual lattice and $L'/L \cong \left\{0, \frac{1}{2N}, \cdots, \frac{2N-1}{2N}\right\}$. Let $\mathfrak{e}_\beta$ be the standard basis of $\mathbb{C}[L'/L]$, $\mathfrak{e}_\beta(x) = e^{2\pi i x}\mathfrak{e}_\beta$. Let $\rho_L : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}[L'/L]$ be the *Weil representation* corresponding to L, i.e. if $T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ and $-I = \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$ then

$$
\begin{aligned}
\rho_L(T)\mathfrak{e}_\gamma &= e(q(\gamma))\mathfrak{e}_\gamma \\
\rho_L(S)\mathfrak{e}_\gamma &= \frac{1}{\sqrt{2Ni}} \sum_{\delta \in L'/L} e(-(\gamma,\delta))\mathfrak{e}_\delta \\
\rho_L(-I)\mathfrak{e}_\gamma &= -i\mathfrak{e}_{-\gamma}
\end{aligned}
$$

Non-holomorphic Poincaré series

Definition (Non-holomorphic Poincaré series)

Let $\beta \in L'/L$ and $m_\beta \in \mathbb{Z} + q(\gamma) < 0$. Then

$$F^L_{\beta,m}(z,s) = \frac{1}{2\Gamma(2s)} \sum_{M \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \left[\mathcal{M}_s(4\pi|m|y)\,\mathfrak{e}_\beta(mx)\right]|_{\rho_L,k} M$$

here $\mathcal{M}$ is related to the Whittaker M-function

$$\begin{aligned} F^L_{\beta,m}(\gamma z,s) &= J_\gamma(z)^{2k}\rho_L(\gamma)F^L_{\beta,m}(z,s) \text{ and} \\ \Delta_k F^L_{\beta,m} &= \left(\left(s(1-s)+\frac{k}{2}\left(\frac{k}{2}-1\right)\right)\right)F^L_{\beta,m}. \end{aligned}$$

Harmonic Poincaré series

If $s = 1 - \frac{k}{2}$ then $\Delta_k F^L_{\beta,m} = 0$ and $(n \in \mathbb{Z} + q(\gamma))$

the Fourier expansion

$$F^L_{\beta,m}(z) = \mathfrak{e}_\beta(mz) + \mathfrak{e}_{-\beta}(mz) + \sum_{\gamma\in L'/L}\sum_{n\geq 0} c^+(\gamma,n)\,\mathfrak{e}_\gamma(nz) + \sum_{\gamma\in L'/L}\sum_{n<0} c^-(\gamma,n)\,\kappa^-_n(y)\,\mathfrak{e}_\gamma(nx)$$

Automorphy relation

$$F^L_{\beta,m}(\gamma z)\,\mathfrak{e}_\gamma = J_\gamma(z)^{2k}\sum_{\delta\in L'/L}\rho_L(\gamma)_{\gamma\delta}\,F^L_{\beta,m}(z)\,\mathfrak{e}_\delta.$$

$\kappa_n(y)$ decreases as $y|n| \to \infty$ so „the algorithm" applies and as for the Eisenstein series we get an inhomogeneous system:

$$\tilde{V}\vec{c} + \vec{\tilde{W}} = 0.$$

Examples of Coefficient Bounds

1. $\phi \Rightarrow c_n = O\left(n^{\frac{2}{5}+\varepsilon}\right)$
2. $E(z;s) \Rightarrow c_n(s) = O\left(|n|^{\frac{1}{2}+\left|\Re s-\frac{1}{2}\right|+\varepsilon}\right)$, $n \neq 0$ and $c_0(s) = \varphi(s)$ is unbounded (has poles)
3. $F^L_{\beta,m} \Rightarrow c^+(n) = O\left(\frac{1}{\sqrt{nN}}e^{4\pi\sqrt{|mn|}}\right)$ and $c^-(n) = O(1)$.

Some numerical aspects on the Linear systems

- Maass forms: Stable system after subtracting $\kappa_n(Y)$
- Eisenstein series: s_0 is a pole of φ the system is clearly unstable around s_0 but since $\varphi(1-s)\varphi(s) = 1$ and $E(z;s) = \varphi(s)E(z;1-s)$ we can always study the region where $\varphi(s) = 0$ instead.
- In Poincaré series computation things are much worse. The exponential growth in one direction makes the system unbalanced but we can scale the positive coefficients by the exact asymptotic.
- However we still need to use at least so many digits precision that $c^+(n)$ can be represented with an error of ε so the precision must increase with n.

Explict formulas

$$
\begin{aligned}
c^{+}(\gamma,n) &= 2\pi\left|\frac{n}{m}\right|^{\frac{k-1}{2}}\sum_{c\neq 0}H_c(\beta,m,\gamma,n)\,I_{1-k}\left(\frac{4\pi}{|c|}\sqrt{|mn|}\right),\, n>0\\
c^{+}(\gamma,0) &= \frac{(2\pi)^{2-k}|m|^{1-k}}{\Gamma(2-k)}\sum_{c\neq 0}|c|^{k-1}H_c(\beta,m,\gamma,0)\\
c^{-}(\gamma,n) &= \frac{-1}{\Gamma(1-k)}\delta_{mn}\left(\delta_{\beta,n}+\delta_{-\beta,\gamma}\right)\\
&\quad+\frac{2\pi}{\Gamma(1-k)}\left|\frac{n}{m}\right|^{\frac{k-1}{2}}\sum_{c\neq 0}H_c(\beta,m,\gamma,n)\,J_{1-k}\left(\frac{4\pi}{|c|}\sqrt{|mn|}\right)
\end{aligned}
$$

$H_c(\beta,\gamma,n)$ is a ρ_L-twisted Kloosterman sum. These formulas have terrible convergence!

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