SIEGEL MODULAR FORMS, HECKE OPERATORS, AND L-SERIES

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Siegel introduced generalised theta series to study quadratic forms: Given an $m\times m$, symmetric, integral matrix Q and a lattice $L=\mathbb{Z}x_1\oplus\cdots\oplus\mathbb{Z}x_m$, Q defines a quadratic form on L via

$$x=\alpha_1x_1+\cdots+\alpha_mx_m\mapsto\ {}^t\underline{x}Q\underline{x}\text{ where }\underline{x}=\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\end{pmatrix}.$$

For $n\in\mathbb{Z}_+$, set

$$\theta^{(n)}(L;\tau)=\sum_{C\in\mathbb{Z}^{m,n}}\mathrm{e}\{{}^tCQC\tau\}$$

where $\tau=X+iY$, $X,Y\in\mathbb{R}^{n,n}_{\text{sym}}$, $Y>0$, and $\mathrm{e}\{*\}=\exp(\pi iTr(*))$. Presuming $Q>0$, this series is absolutely convergent and in fact defines an analytic function. For convenience, we assume Q is even integral on L (meaning $Q(L)\subseteq 2\mathbb{Z}$); then

$$\theta^{(n)}(L;\tau)=\sum_T r(L,T)\mathrm{e}\{T\tau\}$$

where T runs over $n\times n$, symmetric, even integral, positive semi-definite matrices, and

$$\begin{aligned}r(L,T)&=\#\{C\in\mathbb{Z}^{m,n}:\ {}^tCQC=T\ \}\\ &=\#\{(y_1,\dots,y_n):\ y_i\in L,\ Q\text{ restricted to}\\ &\qquad\qquad\text{to }\mathbb{Z}y_1\oplus\cdots\oplus\mathbb{Z}y_n\text{ is given by }T\ \}.\end{aligned}$$

By Poisson summation, there is a relation between $\theta^{(n)}(L;\tau)$ and $\theta^{(n)}(L^{\#};-\tau^{-1})$ where $L^{\#}\approx\mathbb{Z}^m$ has the quadratic form Q^{-1}. Consequently,

$$\theta^{(n)}(L;(A\tau+B)(C\tau+D)^{-1})=\chi(\det D)(\det(C\tau+D))^{m/2}\theta^{(n)}(L;\tau)$$

for all $\gamma=\begin{pmatrix}A&B\\ C&D\end{pmatrix}\in Sp_n(\mathbb{Z})$ with $N|C$, N the level of L (meaning N is the smallest positive integer so that NQ^{-1} is even integral), and χ is a quadratic

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX character modulo N. When m is odd, we need to interpret $(\det(C\tau+D))^{m/2}$ carefully. Instead of using a metaplectic group, we can follow Shimura (where n was 1) and take

$$(\det(C\tau+D))^{m/2} = \left(\frac{\theta^{(n)}(\gamma\circ\tau)}{\theta^{(n)}(\tau)}\right)^m$$

where

$$\theta^{(n)}(\tau) = \sum_{C\in\mathbb{Z}^{1,n}} \mathrm{e}\{{}^tCC\tau\}.$$

(When m is odd, we necessarily have $4|N$, and then $\left|\frac{\theta^{(n)}(\gamma\circ\tau)}{\theta^{(n)}(\tau)}\right| = |\det(C\tau+D)|^{1/2}$.) Then we find

$$\chi(d) = \begin{cases} (\mathrm{sgn} d)^k \left(\frac{(-1)^k \det Q}{|d|}\right) & \text{if } m=2k,\\ \left(\frac{(-1)^k 2\cdot\det Q}{|d|}\right) & \text{if } m=2k+1.\end{cases}$$

So $\theta^{(n)}(L;\tau)$ is our prototypical example of a Siegel modular form of degree n, weight $m/2$, level N, and character χ.

Slash operator: For convenience, with $\gamma = \begin{pmatrix} A & B\\ C & D\end{pmatrix} \in Sp_n(\mathbb{Z})$, we set

$$F|\gamma(\tau) = (\det(C\tau+D))^{-k}F(\gamma\circ\tau)$$

when F is a Siegel modular form of degree n and integral weight k, and

$$F|\gamma(\tau) = \left(\frac{\theta^{(n)}(\gamma\circ\tau)}{\theta^{(n)}(\tau)}\right)^{-k} F(\gamma\circ\tau)$$

when F is a Siegel modular form of degree n and half-integral weight $k/2$ (and $4|C$).

Fourier expansions: Given a Siegel modular form F of degree n and weight k,

$$F(\tau) = \sum_T c(T)\mathrm{e}\{T\tau\}$$

where T varies over $n\times n$, symmetric, even integral, positive semi-definite matrices. For any $G\in GL_n(\mathbb{Z})$, $\begin{pmatrix} G^{-1} & \\ & {}^tG\end{pmatrix} \in Sp_n(\mathbb{Z})$; thus

$$F(G^{-1}\tau\ {}^tG^{-1}) = \chi(\det G)(\det G)^k F(\tau)$$

and hence $c(T) = \chi(\det G)(\det G)^k c({}^tGTG)$. Thus we can write

$$F(\tau) = \sum_{\mathrm{cls}\Lambda} c(\Lambda)\mathrm{e}\{\Lambda\tau\}$$

where clsΛ varies over isometry classes of rank n lattices, each equipped with a positive semi-definite quadratic form (oriented when $\chi(-1)(-1)^k \neq 1$), and for T a matrix for the quadratic form on Λ,

$$\mathrm{e}\{\Lambda\tau\} = \sum_G \mathrm{e}\{{}^tGTG\tau\}$$

with $G \in O(T)\backslash GL_n(\mathbb{Z})$ if $\chi(-1)(-1)^k = 1$, and $G \in O^+(T)\backslash SL_n(\mathbb{Z})$ otherwise.

Hecke operators: For a prime p, we have Hecke operators $T(p)$, $T_j(p^2)$ ($1 \leq j \leq n$). (We know n of these are algebraically independent; the algebraic relation is given in [5].) To define the Hecke operators on Siegel modular forms of degree n, weight k, level N, and character χ, let

$$\underline{\delta} = \begin{cases} \begin{pmatrix} pI_n & \\ & I_n \end{pmatrix} & \text{for } T = T(p), \\ \begin{pmatrix} pI_j & & & \\ & I_{n-j} & & \\ & & \frac{1}{p}I_j & \\ & & & I_{n-j} \end{pmatrix} & \text{for } T = T_j(p^2), \end{cases}$$

$\Gamma = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) : \ N|C \right\}$, $\Gamma' = \underline{\delta}\Gamma\underline{\delta}^{-1}$; then (up to normalisation)

$$F|T = \sum_\gamma \overline{\chi}(\det D_\gamma) F|\underline{\delta}^{-1}\gamma$$

where γ runs over $(\Gamma' \cap \Gamma)\backslash\Gamma$. Also,

$$F|\underline{\delta}^{-1}(\tau) = \begin{cases} p^{-kn/2} F(\underline{\delta}^{-1} \circ \tau) & \text{if } T = T(p), \\ p^{-kj} F(\underline{\delta}^{-1} \circ \tau) & \text{if } T = T_j(p^2). \end{cases}$$

(Actually for half-integral weight, the matrices in the quotients have their "automorphy factors" attached to them.)

In work with Jim Hafner, we found coset representatives for the Hecke operators when the weight is integral. When evaluating the action of $T_j(p^2)$, we encountered incomplete character sum; to complete these, we replace $T_j(p^2)$ with $\widetilde{T}_j(p^2)$, a linear combination of $1, T_1(p^2), \dots, T_j(p^2)$.

Theorem. *(Hafner-W [5]) Say F is a Siegel modular form with degree n and integral weight k. The Λth coefficient of $F|\widetilde{T}_j(p^2)$ is*

$$\sum_{p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda} \chi(p^*)p^*\alpha_j(\Lambda, \Omega)c(\Omega)$$

where $\alpha_j(\Lambda,\Omega)$ is the number of codimension $n-j$ totally isotropic subspaces of $\Lambda\cap\Omega/p(\Lambda+\Omega)$.

The Λth coefficient of $F|T(p)$ is

$$\sum_{\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda}\chi(p^*)p^*c(\Omega^p)$$

where Ω^p denotes the lattice Ω equipped with the quadratic form attached to Ω multiplied by p. (This latter result was first proved by Maass, stated in the terms of coefficients attached to matrices.)

The coset representatives we found to give the action of the Hecke operators are explicit with the exception of a choice of change of basis matrix; in [14], we make an explicit choice and describe matrices giving the action of each Hecke operator.

Also, the actual coset representatives are not upper block triangular, although these can be replaced by upper block triangular matrices. However, in the case of half-integral weight, this replacement doesn't just introduce a power of $\chi(p)$, but also a Legrendre symbol $\left(\frac{\det Y_1}{p}\right)$.

Theorem (W. [15]). *Say F is a Siegel modular form of degree n, half-integral weight $k/2$, level N, and character χ. Then $F|T(p)=0$. For $1\le j\le n$, the Λth coefficient of $F|T_j(p^2)$ is*

$$\sum_{p\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda}\chi(p^*)p^*\mathcal{G}_j(\Lambda,\Omega)c(\Omega)$$

where

$$\mathcal{G}_j(\Lambda,\Omega)=\sum_{T_1}\mathcal{G}_{T_1}(p),$$

T_1 varying over matrices for the quadratic forms on the codimension $n-j$ subspaces of $\Lambda\cap\Omega/p(\Lambda+\Omega)$, and $\mathcal{G}_{T_1}(p)=\sum_{Y_1\ (p)}\left(\frac{\det Y}{p}\right)e\{T_1Y_1/p\}$ is a generalised Gauss sum.

Having such explicit matrices giving the action of the Hecke operators is useful when studying Siegel theta series. When L is a lattice of even rank $m=2k$, I used these to show $\theta^{(n)}(L)|T$ is a linear combination of Siegel theta series attached to lattices closely related to L; consequently, setting

$$\theta^{(n)}(\mathrm{gen}L)=\sum_{\mathrm{cls}L'\in\mathrm{gen}L}\frac{1}{o(L')}\theta^{(n)}(L')$$

where $o(L')$ is the order of the orthogonal group of L', we have:

(1) For $j\le k$, $\theta^{(n)}(\mathrm{gen}L)|T'_j(p^2)=\lambda_j\theta^{(n)}(\mathrm{gen}L)$ where

$$\lambda_j=p^*\beta(n,j)(p^{k-1}+\chi(p))\cdots(p^{k-j}+\chi(p)),$$

$T'_j(p^2)$ is a linear combination of $1, T_1(p^2), \dots, T_j(p^2)$, and $\beta(n,j)$ is the number of j-dimensional subspaces of an n-dimensional space over $\mathbb{Z}/p\mathbb{Z}$;

(2) For $j > k$, $\theta^{(n)}(L)|T'_j(p^2) = 0$;

(3) If $\chi = 1$, $\theta^{(n)}(\mathrm{gen}L)|T(p) = (p^{k-1}+1)\cdots(p^{k-n}+1)\theta^{(n)}(\mathrm{gen}L)$;

(4) $\theta^{(n)}(\mathrm{gen}L)|T(p)^2 = \big((p^{k-1}+\chi(p))\cdots(p^{k-n}+\chi(p))\big)^2\theta^{(n)}(\mathrm{gen}L)$.

(Currently I am working to show that when m is odd and L is a lattice of rank m, $\theta^{(n)}(\mathrm{gen}L)$ is an eigenform for the $T_j(p^2)$.)

L-series: Given a Siegel modular form $F(\tau) = \sum_{\mathrm{cls}\Lambda} c(\Lambda)\mathrm{e}\{\Lambda\tau\}$, we define the Koecher-Maass series to be

$$D(F;s) = \sum_{\mathrm{cls}\Lambda>0} \frac{c(\Lambda)}{o(\Lambda)}(\ \mathrm{disc}\Lambda)^{-s}$$

where discΛ is the determinant of a matrix for the quadratic form on Λ. In the case $n = 2$, Jim Hafner and I introduced a 2-variable L-series that specialises to $D(F;s)$. Noticing that the Hecke operators do not distinguish between lattices in the same "family" (which is slightly larger than a genus), we defined average Fourier coefficients

$$a(\Lambda) = \sum_{\mathrm{cls}\Omega\in\mathrm{fam}\Lambda} \frac{c(\Omega)}{o(\Omega)}.$$

Then for F an eigenform for all the Hecke operators, we showed that $D(F;s)$ factors as a product of the standard L-series for F, the spinor L-series for F, and the series

$$\sum_{\Delta\ \text{max'l}} a(\Delta)(\ \mathrm{disc}\Delta)^{-s} N_\Delta$$

where N_Δ is a polynomial reflecting local structures of Δ. The standard and spinor L-functions for F are completely determined by the eigenvalues of F. This demonstrates that, a priori, the eigenvalues are nowhere near sufficient to determine the Fourier coefficients of F – sad, but true!

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