Hecke operators and Hilbert modular forms

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Overview

Let F be a real quadratic field of class number 1 with ring of integers $\mathcal{O}$. Let Γ be a congruence subgroup of $\mathrm{GL}_2(\mathcal{O})$.

The cohomology group $H^3(\Gamma; \mathbb{C})$ contains the cuspidal cohomology corresponding to cuspidal Hilbert modular forms of parallel weight 2.

We describe a technique to compute the action of the Hecke operators on the cohomology $H^3(\Gamma; \mathbb{C})$, giving a way to compute the Hecke action on these Hilbert modular forms.

A motivating example

Let $\mathfrak{H}$ be the upper half-plane in the complex numbers

$$\mathfrak{H} = \{x + iy \mid y > 0\}.$$

Let $\Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ the subgroup of matrices that are upper triangular modulo N.

The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathfrak{H}$ via fractional linear transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Related geometric object

$Y_0(N) = \Gamma_0(N)\backslash\mathfrak{H}$ is a punctured Riemann surface.

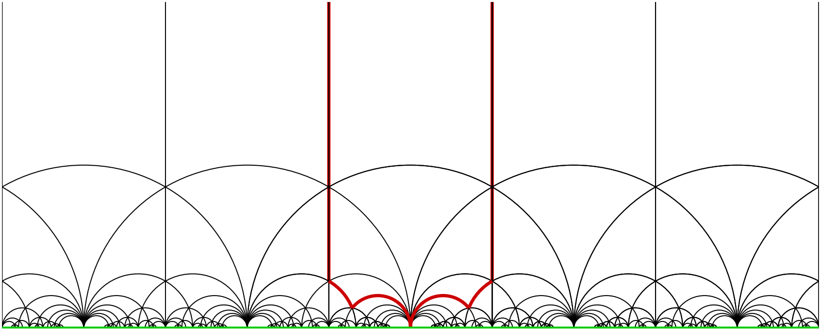

Figure: Tessellation of $\mathfrak{H}$ with $Y_0(5)$ shown in red, cusps shown in green.

Modular forms and cohomology

$$H^1(Y_0(N); \mathbb{C}) \simeq S_2(N) \oplus \overline{S_2(N)} \oplus \mathrm{Eis}_2(N).$$

We can study Hecke eigenvalues by understanding the action of Hecke operators on the cohomology.

Hecke operators

Let $g \in \mathrm{Comm}(\Gamma)$. Decompose the double coset $\Gamma g \Gamma$ into a finite disjoint union

$$\Gamma g \Gamma = \coprod_{s \in S} \Gamma s.$$

Then the Hecke correspondence associated to g carries a point $\Gamma x \in \Gamma \backslash X$ to the finite set of points $\{\Gamma s x\}_{s \in S}$. The Hecke operator associated to g is the induced map T_g on cohomology.

Modular symbols

Modular symbols for $\mathrm{SL}_2(\mathbb{Z})$ (Manin 1972) can be defined as a pair of cusps $\{\alpha, \beta\}$, or the geodesic joining them, viewed as a homology class in $H_1(X_0(N))$.

Hecke operators act on the space of modular symbols.

There is a group of *unimodular symbols* that is finite modulo $\mathrm{SL}_2(\mathbb{Z})$ and a reduction algorithm for writing a general modular symbol as a linear combination of unimodular symbols.

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$$\{0, 12/5\} = \{0, \infty\} + \{\infty, 2\} + \{2, 5/2\} + \{5/2, 12/5\}.$$

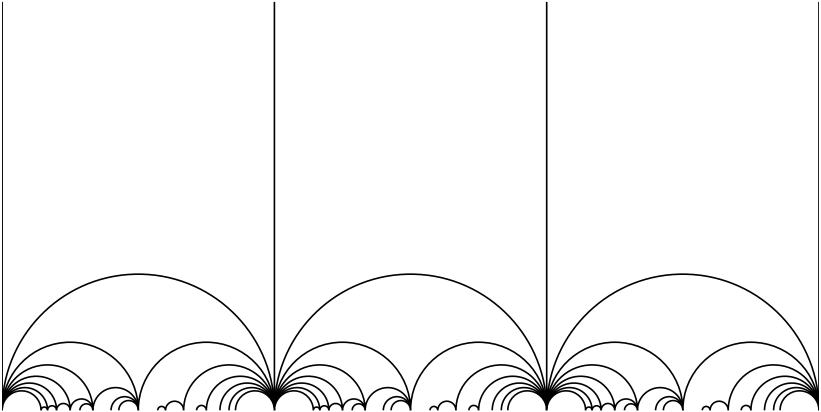

Figure: Unimodular symbols

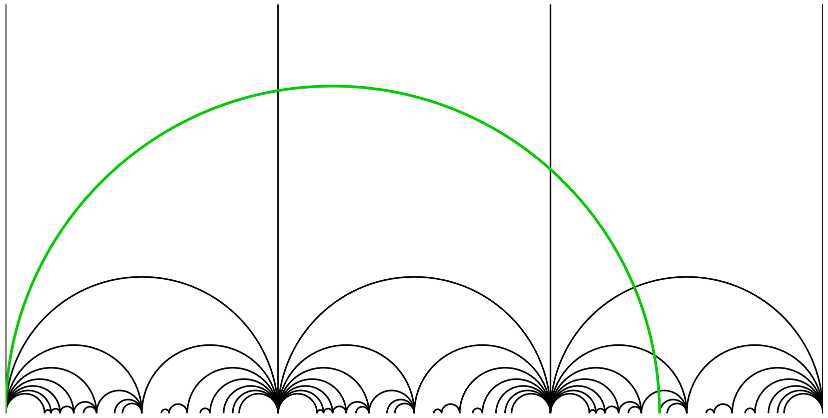

Figure: The modular symbol $\{0, 12/5\}$ is shown in green.

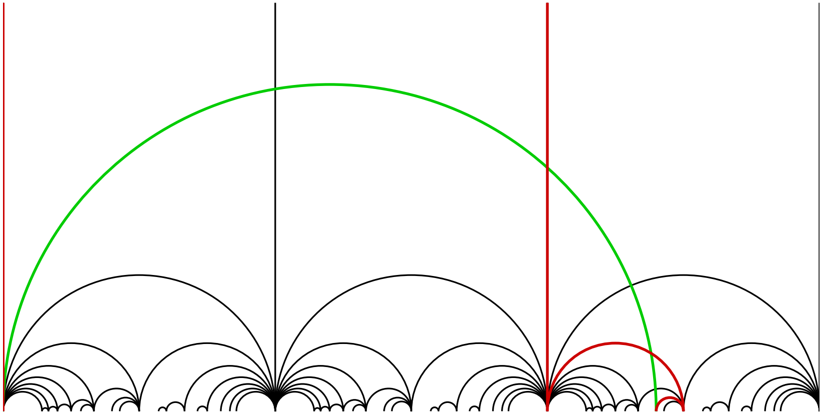

Figure: The reduction is shown in red.

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Summary of motivational example

- One wants to understand the action of Hecke operators on spaces of modular forms.
- There is a geometric object Y, attached to $G = \mathrm{SL}_2(\mathbb{R})$ and Γ whose cohomology "sees" modular forms.
- Compute Hecke operators on objects related to cohomology of Y. e.g. Modular symbols (Manin)

Hilbert modular forms over real quadratic fields

Let $F/\mathbb{Q}$ be a real quadratic field with ring of integers $\mathcal{O}$ and with class number 1. Let $\Gamma \subseteq \mathrm{GL}_2(\mathcal{O})^+$ be a congruence subgroup.

A holomorphic function $f : \mathfrak{H}^2 \to \mathbb{C}$ is a *Hilbert modular form* of weight $k = (k_1, k_2)$ if

$$f(\gamma \cdot z) = \left(\prod \det(\gamma_i)^{-k_i/2} (c_i z_i + d_i)^{k_i} \right) f(z)$$

for every $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$.

Hilbert modular forms and cohomology

Let $\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)$. The associated symmetric space is $X \simeq \mathfrak{H} \times \mathfrak{H} \times \mathbb{R}$. Then $Y = \Gamma \backslash X$ is a circle bundle over a Hilbert modular surface, possibly with orbifold singularities if Γ has torsion.

We will compute Hecke operators on Hilbert modular forms by computing the Hecke action on the corresponding cohomology groups $H^*(Y)$.

Related results: totally different technique

Socrates and Whitehouse (2005), Dembélé (2005, 2007), an Dembélé-Donnelly (2008) compute the Hecke action on Hilbert modular forms using the Jaquet-Langlands correspondence.

Sharbly complex

For our case, $\nu = 4$, but the cuspidal cohomology occurs in degrees 2 and 3. Modular symbols compute in degree ν, and hence will not see the cuspidal cohomology.

The sharbly complex $S_*(\Gamma)$, a homology complex with modular symbols in degree 0, provides the proper setting in which to study $H^*(Y; \mathcal{M})$ (Ash, Gunnells, Lee-Szczarba)[1]. There is a natural action of Hecke operators on $S_*(\Gamma)$.

[1] The name of this complex is due to Lee Rudolph, in honor of *On the homology and cohomology of congruence subgroups* by Lee and Szczarba.

Outline

1. The sharbly complex provides a model for the cohomology.
2. There is an analogue of the tessellation of $\mathfrak{H}$ by ideal triangles for X. It comes from viewing points in X as quadratic forms modulo homothety (Koecher, Ash). This gives rise to a notion of reduced sharblies.
3. Reduced 1-sharblies, which look like triples of cusps, will span the cohomology in degree 3.
4. There is a reduction algorithm (Gunnells-Y) which works in practice, to express a 1-sharbly as a linear combination of reduced 1-sharblies.

X as quadratic forms

Let $G = \mathbf{G}(\mathbb{R})$. The two real embeddings of F into $\mathbb{R}$ give rise to an isomorphism

$$G \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}).$$

Thinking of $\mathrm{GL}_2(\mathbb{R})/\operatorname{O}(2)$ as the cone C of positive definite quadratic forms via $g \operatorname{O}(2) \mapsto g\,{}^t g$, we get a map

$$G/KA_G \to (C \times C)/\mathbb{R}_{>0}.$$

Rational boundary components

For $v \in F^2$, let $R(v)$ be the ray $\mathbb{R}_{>0} \cdot v^t v \subset \overline{C \times C}$. Equivalence classes of these rays in $\overline{C \times C}$ correspond to the usual cusps of the Hilbert modular variety.

One has a decomposition of $\overline{C \times C}$ into Voronoĭ-cones which descends to a tessellation of X with vertices contained in $R(F^2)$.

Voronoĭ polyhedron

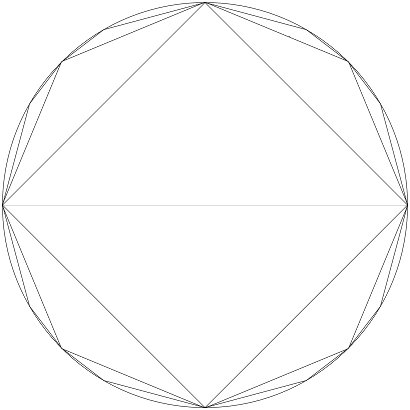


Figure: The facets of the Voronoĭ polyhedron for $\mathrm{SL}_2(\mathbb{Z})$.

Sharbly complex

Let S_k, $k \geq 0$, be the Γ-module A_k/C_k, where A_k is the set of formal $\mathbb{C}$-linear sums of symbols $[v] = [v_1, \cdots, v_{k+2}]$, where each v_i is in F^2, and C_k is the submodule generated by

1. $[v_{\sigma(1)}, \cdots, v_{\sigma(k+2)}] - \mathrm{sgn}(\sigma)[v_1, \cdots, v_{k+2}]$,
2. $[v, v_2, \cdots, v_{k+2}] - [w, v_2, \cdots v_{k+2}]$ if $R(v) = R(w)$, and
3. $[v]$, if v is *degenerate*, i.e., if $v_1, \cdots, v_{k+2}$ are contained in a hyperplane.

We define a boundary map $\partial \colon S_{k+1} \to S_k$ by

$$\partial[v_1, \cdots, v_{k+2}] = \sum_{i=1}^{k+2} (-1)^i [v_1, \cdots, \hat{v}_i, \cdots, v_{k+2}]. \qquad (1)$$

This makes S_* into a homological complex, called the *sharbly complex*.

Γ-coinvariants

The boundary map commutes with the action of Γ, and we let $S_*(\Gamma)$ be the homological complex of coinvariants. Specifically, $S_k(\Gamma)$ is the quotient of S_k by relations of the form $\gamma \cdot \mathbf{v} - \mathbf{v}$, where $\gamma \in \Gamma$ and $\mathbf{v} \in S_k$.

A theorem of Borel and Serre gives that

$$H^{4-k}(\Gamma; \mathbb{C}) \simeq H_k(S_*(\Gamma)).$$

Moreover, there is a natural action of the Hecke operators on $S_*(\Gamma)$.

Thus to compute $H^3(\Gamma; \mathbb{C})$, which will realize cuspidal Hilbert modular forms over F of weight $(2, 2)$, we work with 1-sharblies.

1-sharblies

We think of a 1-sharbly **v** as a triangle, with vertices labeled by the spanning vectors of **v**. The boundary 0-sharblies correspond to the edges of the triangle.

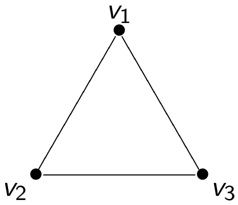


Figure: A 1-sharbly

1-sharbly chain in $S_1(\Gamma)$

Thus a 1-sharbly chain $\xi = \sum a(\mathbf{v})\mathbf{v}$ can be thought of a collection of triangles with vertices labeled by rays in $\overline{C \times C}$. If ξ becomes a cycle in $S_1(\Gamma)$, then its boundary must vanish modulo Γ.

Voronoĭ reduced sharblies

Definition

A k-sharbly $[v_1, \cdots, v_{k+2}]$ is *Voronoĭ reduced* if its spanning vectors $\{R(v_i)\}$ are a subset of the vertices of a Voronoĭ cone.

Main step in reduction algorithm

We must take a general 1-sharbly cycle ξ and to modify it by subtracting an appropriate coboundary to obtain a homologous cycle ξ' that is closer to being Voronoĭ reduced.

By iterating this process, we eventually obtain a cycle that lies in our finite-dimensional subspace $S_1^{\mathrm{red}}(\Gamma)$.

Unfortunately, we are unable to prove that at each step the output cycle ξ' is better than the input cycle ξ, in other words that it is somehow "more reduced." However, in practice this always works.

The reduction algorithm

Definition

Given a 0-sharbly $\mathbf{v}$, the *size* Size($\mathbf{v}$) *of* $\mathbf{v}$ is given by the absolute value of the norm determinant of the 2×2 matrix formed by spanning vectors for $\mathbf{v}$.

The basic strategy

Voronoĭ reduced 1-sharblies tend to have edges of small size. Thus our first goal is to systematically replace all the 1-sharblies in a cycle with edges of large size with 1-sharblies having smaller size edges.

Reducing points

Definition

Let **v** be a non-reduced 0-sharbly with spanning vectors $\{x, y\}$. Then $u \in \mathcal{O}^2 \smallsetminus \{0\}$ is a *reducing point for* **v** if the following hold:

1. $R(u) \neq R(x), R(y)$.
2. $R(u)$ is a vertex of the unique Voronoĭ cone σ (not necessarily top-dimensional) containing the ray $R(x+y)$.
3. If $x = ty$ for some $t \in F^\times$, then u is in the span of x.
4. Of the vertices of σ, the point u minimizes the sum of the sizes of the 0-sharblies $[x, u]$ and $[u, y]$.

Reducing points (ctd.)

Given a non-Voronoĭ reduced 0-sharbly $\mathbf{v} = [x, y]$ and a reducing point u, we apply the relation

$$[x, y] = [x, u] + [u, y]$$

in the hopes that the two new 0-sharblies created are closer to being Voronoĭ reduced.

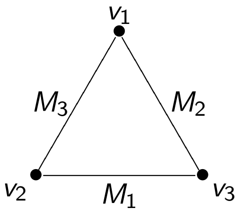
Remark

Note that choosing u uses the geometry of the Voronoĭ decomposition instead of (a variation of) the continued fraction algorithms of (Manin, Cremona, Ash-Rudolph).

Γ-invariance

The reduction algorithm proceeds by picking reducing points for non-Voronoĭ reduced edges. We make sure that this is done Γ-equivariantly; in other words that if two edges $\mathbf{v}$, $\mathbf{v}'$ satisfy $\gamma \cdot \mathbf{v} = \mathbf{v}'$, then if we choose u for $\mathbf{v}$ we need to make sure that we choose γu for $\mathbf{v}'$.

We achieve this by attaching a lift matrix to each edge, and making sure that the choice of reducing point for $\mathbf{v}$ only depends on the lift matrix M that labels $\mathbf{v}$.



Algorithm

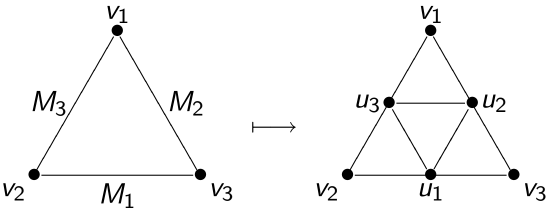
Let T be a non-degenerate 1-sharbly. The method of subdividing depends on the number of edges of T that are Voronoĭ reduced.

Remark

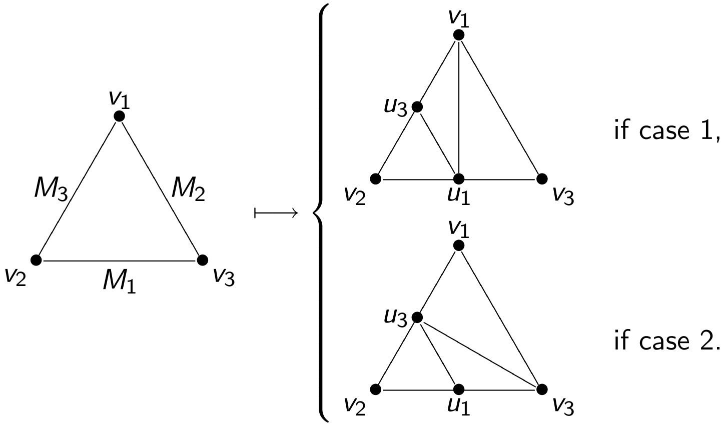
The reduction algorithm can be viewed as a two stage process.

- If T is "far" from being Voronoĭ reduced, one tries to replace T by a sum of 1-sharblies that are more reduced in that the edges have smaller size.
- If T is "close" to being Voronoĭ reduced, then one must use the geometry of the Voronoĭ cones more heavily.

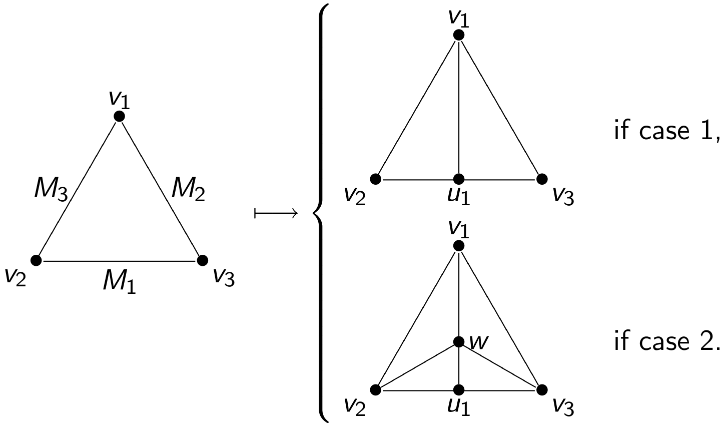
(I) Three non-reduced edges



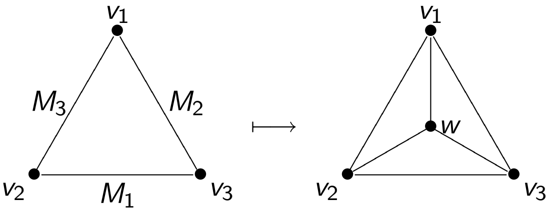
(II) Two non-reduced edges.



(III) One non-reduced edge.



(IV) All edges Voronoĭ reduced



Comment

First, we emphasize that the reducing point u of works in practice to shrink the size of a 0-sharbly $\mathbf{v}$, but we have no proof that it will do so. The difficulty is that the reducing point is chosen using the geometry of the Voronoĭ polyhedron Π and not the size of $\mathbf{v}$ directly. Moreover, our experience with examples shows that this use of the structure of Π is essential.

Testing

Let ξ be a 1-sharbly cycle. The reduction algorithm is a local computation that tries to reduce each triangle contributing to ξ. While the algorithm does not use the fact that ξ is a cycle, by using the lift data for the edges, each step of the algorithm maintains this property.

Example:

Let $F = \mathbb{Q}(\sqrt{5})$ and let $\mathcal{O}$ be the ring of integers of F.

level norm	31	41	49	61	71
rank of H^3	4	4	4	5	4

Table: The cohomology of $\mathrm{GL}_2(\mathcal{O})$.

The action of the Hecke operator $T_{\mathfrak{p}}$ on H^3 is diagonalizable. The eigenspaces consist of a 3-dimensional piece with corresponding eigenvalue $N(\mathfrak{p}) + 1$ coming from Eisenstein series and the remaining part coming from parallel weight $(2, 2)$ Hilbert cusp forms.

Example (ctd.)

level norm	31	41	49	61	71
rank of H^3	4	4	4	5	4

Table: The cohomology of $\mathrm{GL}_2(\mathcal{O})$.

The eigenvalues for $T_{\mathfrak{p}}$ were computed for small values of $\mathfrak{p}$ and match the data of Dembélé's tables. The eigenvalues are in $\mathbb{Q}$ for norm level $31, 41, 49, 71$ and in $\mathbb{Q}(\sqrt{5})$ for norm level 61, explaining the 2-dimensional contribution of the cusp forms to the cohomology.

Implementation Details

Voronoĭ polyhedron

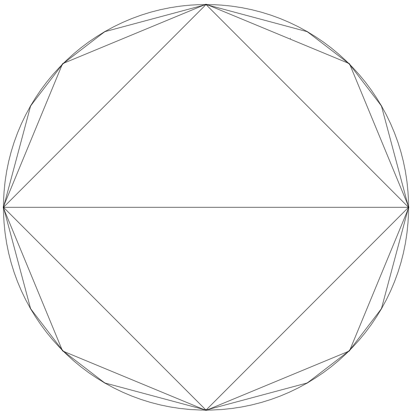


Figure: The facets of the Voronoĭ polyhedron for $\mathrm{SL}_2(\mathbb{Z})$.

Steps

1. Compute Voronoĭ polyhedron data (depends on the field $F = \mathbb{Q}(\sqrt{d})$).
2. Set level and compute cohomology.
3. Compute Hecke action.

Construct Voronoĭ polyhedron

1. Facets correspond to perfect binary quadratic forms over F.
2. Once an initial perfect form is found, a complete set of $\mathrm{GL}_2(O)$-class representatives can be computed algorithmically.
3. Work of Kitaoka guarantees a perfect form of the form $f(x, y) = \alpha(x^2 - xy + y^2)$.

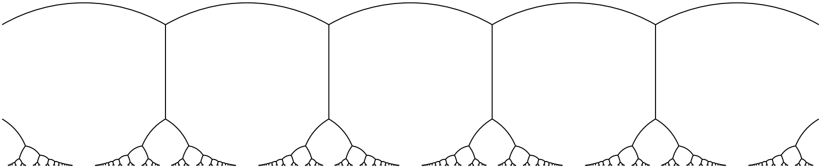


Figure: The well-rounded binary quadratic forms.

Set level and compute cohomology

To compute the $\Gamma_0(N)$-classes of Voronoĭ-face σ, we want to compute

$$\begin{aligned}\Gamma_0(N)\backslash \mathrm{GL}_2(\mathcal{O})/\mathrm{Stab}(\sigma) &= \Gamma_0(N)\backslash(\sigma\text{-type faces}) \\ &= \mathbb{P}^1(\mathcal{O}/N\mathcal{O})/\mathrm{Stab}(\sigma).\end{aligned}$$

Translate between projecive orbits and sharblies.

Compute Hecke action

1. Fix a basis for $H^3(\Gamma_0(N))$.
2. Translate each basis vector to 1-sharbly cycle.
3. Act by Hecke operator.
4. Reduce.
5. Translate back to cohomology.

Current limitations of implementation

1. Very slow.
2. Voronoĭ polyhedron must be simplicial in low degrees.
3. Voronoĭ polyhedron grows complicated very quickly as $F = \mathbb{Q}(\sqrt{d})$ varies. E.g. for $d = 46$, there are 4306 top dimensional cones.