Jacquet-Langlands in weight 1

In this section we will prove the main theorem of our paper. Firstly, we introduce the concept of slopes of modular forms, which will be useful for us. Let f be a normalized eigenform; either a classical modular form or an overconvergent modular form. We define the *slope* of f to be the normalized p-valuation of the eigenvalue of U_p acting on f.

Theorem 1 (Coleman) *Let f be a classical modular eigenform of weight k. Then the normalized p-slope of f is less than or equal to $k-1$.*
Conversely, if f is a p-adic overconvergent modular form of weight k with normalized slope strictly less than $k-1$, then f is a classical modular form.

We see that there is a slight asymmetry in this result; if an overconvergent modular eigenform of weight k has normalized slope exactly $k-1$, then it can be either classical or non-classical. There are examples of both; we will see this in weight 1 in Section . The question of telling whether an overconvergent form of weight k and slope $k-1$ is classical or not is raised by Coleman, and is still open in general.
We now state the main theorem of this poster; that the standard Jacquet-Langlands correspondence can be extended to weight 1.

Theorem 2 *Let N be a positive odd integer and let i be either 0 or 1.*
If $f \in S_1^{2^i-new}(\Gamma_1(N) \cap \Gamma_0(2^{i+1}))$, then there exists an overconvergent automorphic form $f_A \in S_1^{D,\dagger}(U_1(N) \cdot G)$ with the same Hecke eigenvalues as f (if $i=0$, then there is no extra level structure at 2, and if $i=1$ then $G = 1+\mathfrak{m}$). Conversely, if f_A is an overconvergent automorphic form of weight 1, then there exists an overconvergent modular form f of weight 1 with the same Hecke eigenvalues as f_A.

We note that a version of this is true in more generality, for other subgroups of $D^\star$ of finite index, but we will not need this for the section on approximation eigenforms.

Approximating eigenforms

In this section we will give an account of how to actually find approximations to overconvergent automorphic eigenforms of weight 1, using PARI programs. We also indicate how this method can be generalized to find other forms.
This method is a development of the work of Gouvêa and Mazur, where they find overconvergent 5-adic modular eigenforms of weight 0 by iterating the action of the U_5 operator. This in turn builds on the work of Atkin and O'Brien which pioneered this technique for finding p-adic eigenforms for $p=13$.
On the modular side, we will consider the space of classical modular forms $S_1^{4-\mathrm{new}}(\Gamma_1(11) \cap \Gamma_0(4))$; this can be checked to be one-dimensional, and it is in fact generated by the η-product $f := \eta(q^2)\eta(q^{22})$, which is necessarily a Hecke eigenform. This has Fourier expansion at ∞ given by

$$f(q) = q\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{22n}) = q - q^3 - q^5 + q^{11} + q^{15} + O(q^{23});$$

in particular, it has 11-slope 0, which shows that it is in the interesting case left open by the theory of Coleman, where the slope is $k-1$. By the Ramanujan-Petersson Conjecture, the Fourier coefficients a_p of $f(q)$ satisfy $|a_p| < 2$.
We use code based on the work of Jacobs outlined above to find the slopes of the U_{11} operator acting on weight 1 overconvergent automorphic forms of level $U_0(11) \cdot (1+\mathfrak{m})$. This tells us that the lowest slopes are $0, 0, 1, 2, 2, 2$ with that multiplicity.
Because we know from background work and Theorem 2 that there should be two automorphic forms related to the classical modular form f appearing on the automorphic side, and from above we see that f has slope 0, we see that the two slope 0 automorphic forms are the ones that correspond to f, and all of the others do not correspond to classical forms.

The technical details

Let $\{h_i\}$ be the set of simultaneous eigenforms for $S_1^D(U_1(11) \cdot (1+\mathfrak{m}))$, so let g_1 be a random nonzero element of this space. We can think of g_1 as a triple of power series in z, because we know that an automorphic form is determined by its values on the tuple $\{d_i\}$. (In fact, to aid the calculations we can just choose a triple of polynomials, and make two of them zero).
We assume that we can write g_1 as a linear combination of these eigenforms:

$$g_1 = \alpha h_0 + \beta h_1 + \gamma h_2 + \cdots ,$$

where h_0 and h_1 are the two slope 0 forms. We now compute the action of U_{11}^N on g_1, for some large integer N. We see that $U_{11}^N(g_1)$ will be congruent to $\alpha h_0 + \beta h_1$ modulo p^N, because the action of U_{11}^N on the other h_i will include a multiplication by p^N at least. This means that we have an approximant to the sum of the eigenvectors h_0 and h_1.
We now choose a second random element g_2 and compute $U_{11}^N(g_2)$. This will also be congruent to a linear combination of h_0 and h_1 modulo p^N; with very high probability, these two linear combinations are linearly independent, and we can now use linear algebra to find h_0 and h_1 modulo p^N from them.
Finally, to find eigenforms for *all* of the Hecke operators, we consider the action of the W operator, the analogue of a diamond operator in the classical setting, which is defined to be

$$Wf = [U(1+i)U]f.$$

The action of this operator splits the 2-dimensional eigenspace for U_{11} into two one-dimensional eigenspaces. Basis elements for each of these eigenspaces are eigenforms for all of the Hecke operators T_ℓ (for ℓ a prime not equal to 2 or 11) and U_{11}.

Future developments

It would be interesting if one could find simultaneous eigenforms for U_p and for the other Hecke operators exactly (rather than approximately) by a similar process, given the eigenvalues. In this example, we know that such an eigenform has U_{11}-eigenvalue 1, so it satisfies an equation of the form

$$f(z) = 1 \cdot f(z) = (U_{11}f)(z) = \sum_{i=0}^{11} f(\gamma_i z) \cdot (c_i z + d_i)^{-1},$$

where the $\gamma_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ are a set of matrices that represent the Hecke operator U_{11}. We can write down similar recurrence relations for each of the Hecke operators. It would be interesting to be able to solve these explicitly.
This process can be performed in more generality, to find approximations to eigenforms of slope 0 and of higher slope. The higher-slope cases are more delicate; we cannot use the same techniques as above in general because we need to divide by powers of p, which will reduce the accuracy at which we are working. The same methods do work for slope 1, because we gain more accuracy by iterating the U_{11} operator than we lose by dividing by p.
For higher slopes, a generalization of the methods used by Loeffler would seem more appropriate. These use the properties of an inner product on the space of 5-adic overconvergent modular forms to compute spectral expansions of eigenfunctions for the U_5 operator.
We note here that the methods we have outlined will also work for higher weight forms; let k be a positive integer. We can find any automorphic forms of slope 0 using exactly this procedure; these will be classical automorphic forms, so they will be determined by a tuple of polynomials. After subtracting these out, we will be able to find forms of higher slope, and this will enable us to approximate overconvergent automorphic forms of weight k.

Telling classical from non-classical automorphic forms

It is well-known that p-adic modular forms of weight k and slope exactly $k-1$ can be either classical or non-classical. Examples of both can be exhibited; the form f from the previous section was classical, and we exhibit an example of a non-classical form below.
Using the same approximation techniques as above, we can find an approximation to an overconvergent automorphic form of slope 0, weight 1 and level $U_1(7)$ (with no level condition at 2). We know that such a form must exist because we can compute the slopes of U_7 acting on weight 1 overconvergent automorphic forms, and we find that these are $0, 1, 1, 2, 2, 2$.
This means that there is a unique form g (which aids the calculations) with eigenvalue $1 + 5 \cdot 7 + 4 \cdot 7^2 + 5 \cdot 7^3 + O(7^4)$. This is *not* a classical modular form, because we can check that there are no classical modular cuspforms of weight 1 and level $\Gamma_1(7) \cap \Gamma_0(2)$.
Using work on families of modular forms we can find classical automorphic forms of weight $1 + (p-1)p^n$, for any non-negative integer n, which lie in the same Hida family as our weight 1 form, whether or not it is classical; in fact, because the slope is 0, using the theorem of Coleman quoted above we can see that the only form in the family for which there is a doubt as to whether it is classical is the weight 1 form.
It would be interesting to have a method for telling an overconvergent automorphic form which comes from a classical modular form (via a generalized Jacquet-Langlands correspondence) from an overconvergent automorphic form which does *not* come from a classical modular form.
In some circumstances, we may know that a classical modular form will appear with multiplicity two on the automorphic side, but we would like to have a more intrinsic criterion. Also, we may not be given all of the automorphic forms; we would like a method that only requires us to consider one form.
If we are willing to invoke both a Jacquet-Langlands correspondence *and* the Ramanujan-Petersson conjecture on coefficients of classical modular forms, then we can show that certain automorphic forms are *not* coming from a classical modular form. The Ramanujan-Petersson conjecture tells us that, if $f(q) = \sum_{n \in \mathbb{N}} a_n q^n$ is the Fourier expansion of a normalized cuspidal modular eigenform of weight k, and ℓ is a prime, then

$$|a_\ell| \leq 2p^{\frac{k-1}{2}}.$$

This was conjectured by Ramanujan, generalized by Petersson, and proved by Deligne as a consequence of the Weil conjectures.
We can approximate the eigenvalues of the Hecke operators acting on automorphic forms to a high degree of accuracy using our computer programs, and if we can show that these eigenvalues are not algebraic numbers, then we are done. We can bound the maximum degree that these algebraic numbers can have by dimension considerations. We can use this criterion to show that the weight 1 form at level $U_1(7)$ is not classical; however, we are using several very high-level results to obtain this.

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