Cohomological mod ℓ modular forms over $Q(i)$

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Goal

To investigate the expected correspondence between mod ℓ modular forms and mod ℓ Galois representations over imaginary quadratic fields.

Serre's Conjecture

In 1987, Serre made the following conjecture in [7]. Suppose

$$\rho : G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_\ell)$$

is continuous, irreducible and odd. Then ρ is "modular", i.e., ρ arises from a normalized cuspidal eigenform f in the sense that

1. $tr(\rho(\mathrm{Frob}_p)) =$ eigenvalue of f at p; and
2. $\det(\rho(\mathrm{Frob}_p)) = \varepsilon(p)p^{k-1}$

for all $p \nmid N\ell$, where N, k and ε are the level, weight and character (respectively) of f.

Serre then gives a refinement of this conjecture in which he specifies the minimal level $N(\rho)$ and weight $k(\rho)$ of f (each depending only on ρ) and such that $\ell \nmid N(\rho)$ and $k(\rho) \geq 2$.

Generalization to number fields

Buzzard, Diamond and Jarvis give an extension of Serre's conjecture to totally real fields in their forthcoming paper [3]. They provide an explicit recipe detailing for which weights V one expects to find a mod ℓ Galois representation. Given a representation ρ, they define, for each prime $\mathfrak{p}$ dividing ℓ, a set of representations $W_{\mathfrak{p}}(\rho)$ depending only on ρ restricted to the inertia group at $\mathfrak{p}$. They then set

$$W(\rho) = \{\otimes_{\bar{\mathbb{F}}_\ell} V_{\mathfrak{p}} \mid V_{\mathfrak{p}} \in W_{\mathfrak{p}}(\rho)\}.$$

They conjecture that if ρ is modular, then

$$W(\rho) = \{V \mid \rho \text{ is modular of weight } V\}.$$

In [5], Dembélé, Diamond and Roberts provide computational evidence for this conjecture.

I am investigating what happens in the imaginary quadratic case. I hope to provide computational evidence for an analogous relationship in this situation. To that end, I have written code to compute cohomological mod ℓ modular forms over $\mathbb{Q}(i)$.

Modular Forms

We define a *mod ℓ modular form for $K = \mathbb{Q}(i)$ of level $\mathfrak{n}$ and Serre weight V* to be a non-zero cohomology class $v \in H^2(\Gamma_0(\mathfrak{n}), V)$, which is a simultaneous eigenvector for all the Hecke operators $T_{\mathfrak{q}}$.

Borel-Serre duality [2] gives an isomorphism

$$H^2(\Gamma_0(\mathfrak{n}), V) \xrightarrow{\sim} H_0(\Gamma_0(\mathfrak{n}), St \otimes V),$$

where St denotes the Steinberg module. My code is actually computing simultaneous eigenvectors in the *homology* group $H_0(\Gamma_0(\mathfrak{n}), St \otimes V)$.

Steinberg module and modular symbols

Following Ash in [1], we define the Steinberg module in terms of universal minimal modular symbols. Consider the set of formal $\bar{\mathbb{F}}_\ell$-linear sums of symbols $[v] = [v_1, v_2]$ where the v_i are unimodular vectors in $\mathcal{O}_K^2$. Mod out by the $\bar{\mathbb{F}}_\ell$-module generated by the following elements:

1. $[v_2, v_1] + [v_1, v_2]$;
2. $[v] = [v_1, v_2]$ whenever $\det(v) = 0$; and
3. $[v_1, v_3] - [v_1, v_2] - [v_2, v_3]$,

where the v_i again run over all unimodular vectors in $\mathcal{O}_K^2$. We take this quotient module as the definition of the Steinberg module St.

Computation of modular forms

With the Steinberg module written in terms of modular symbols, we can use methods of Cremona [4] et al. to compute our mod ℓ modular forms.

Manin symbols provide a computationally friendly description of modular symbols. In order to see the equivalence between these spaces, we follow the algebraic approach used by Wiese [8], adjusting some particular arguments to adapt the strategy to the $K = \mathbb{Q}(i)$ case.

My code uses these Manin symbols to compute the homology space $H_0(\Gamma_0(\mathfrak{n}), St \otimes V)$. The action of the Hecke operators on this space is then computed by converting back to modular symbols, computing the action of $T_{\mathfrak{q}}$ and then converting back to Manin symbols via the usual continued fraction convergents method.

Serre weights

For simplicity, we assume ℓ is inert and let $\mathfrak{p}$ be the prime above ℓ. We let $k_{\mathfrak{p}} = \mathcal{O}_K/\mathfrak{p}$ and $S_{\mathfrak{p}}$ be the set of embeddings $k_{\mathfrak{p}} \hookrightarrow \bar{\mathbb{F}}_\ell$. A *Serre Weight* is an irreducible $\bar{\mathbb{F}}_\ell$-representation of

$$G = GL_2(\mathcal{O}_K/\ell\mathcal{O}_K).$$

Such representations are of the form

$$V_{\vec{a},\vec{b}} = \bigotimes_{\tau \in S_{\mathfrak{p}}} \left(\det{}^{a_\tau} \otimes_{k_{\mathfrak{p}}} \mathrm{Sym}^{b_\tau} k_{\mathfrak{p}}^2\right) \otimes_\tau \bar{\mathbb{F}}_\ell,$$

where each of the a_τ and b_τ are integers and $0 \leq b_\tau \leq \ell - 1$. Furthermore, we may assume that $0 \leq a_\tau \leq \ell - 1$ for each $\tau \in S_{\mathfrak{p}}$ and (to guarantee the representations are inequivalent) that $a_\tau < \ell - 1$ for some τ.

In the code, we represent $\mathrm{Sym}^{b_\tau} k_{\mathfrak{p}}^2$ as the space of homogeneous polynomials of degree b_τ in two variables with coefficients in $k_{\mathfrak{p}}$, equipped with the natural action of $\mathrm{GL}_2(\mathcal{O}_K)$.

Modular of weight V

We say that a continuous, irreducible representation

$$\rho : G_K \to \mathrm{GL}_2(\bar{\mathbb{F}}_\ell)$$

is *modular of Serre weight V* if $H^2(\Gamma_0(\mathfrak{n}), V)[\mathfrak{m}_\rho] \neq 0$, where

$$\mathfrak{m}_\rho = \langle T_{\mathfrak{q}} - tr(\rho(\mathrm{Frob}_{\mathfrak{q}})) \mid \mathfrak{q} \text{ prime}\rangle$$

is the maximal ideal of the Hecke algebra associated to ρ. So a representation ρ is modular of weight V if the corresponding system of eigenvalues shows up in our computed forms for that weight.

Implementation Specifications

The code is written in C, using the PARI [6] library.

The code computes mod ℓ modular forms over $\mathbb{Q}(i)$ for:

- ℓ a rational prime which is inert in $K = \mathbb{Q}(i)$
- congruence subgroup $\Gamma = \Gamma_0(\mathfrak{n})$ for level $\mathfrak{n} \subset \mathcal{O}_K$
- arbitrary "Serre weight" V

The code takes as input:

- level $\mathfrak{n}$
- weight $V_{\vec{a},\vec{b}}$ (or will optionally compute all weights for a given level)
- maximum prime $\mathfrak{q}$ for Hecke operators $T_{\mathfrak{q}}$

The code outputs the system(s) of eigenvalues for any simultaneous eigenvectors it finds.

Planned extensions of the code

- Support primes ℓ which split in K (currently the code will only compute for inert primes).
- Implement congruence subgroups $\Gamma_1(\mathfrak{n})$ and accompanying character.
- Allow for other class number one imaginary quadratic fields.
- Compute forms for imaginary quadratic fields with higher class number.
- Look for eigenvalues in larger extensions of $\mathbb{F}_\ell$.

References

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